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A note on the $\mathcal{A}$-generators of the polynomial algebra of six variables and applications

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Dedicated to the anniversary of the 70th birthday of Prof. Vesselin S. Drensky

Abstract: Let $\mathcal{P}_n := H^*((\mathbb{R}P^\infty)^n) \cong \mathbb{Z}_2[x_1, x_2, \ldots, x_n]$ be the polynomial algebra of n generators $x_1, x_2, \ldots, x_n$ with the degree of each x_i being 1. We investigate the Peterson hit problem for the polynomial algebra $\mathcal{P}_n$, regarded as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. For $n > 4$, this problem remains unsolvable, even with the aid of computers in the case of $n = 5$.

In this article, we study the hit problem for the case $n = 6$ in degree $d_s = 6(2^s - 1) + 3.2^s$, with s an arbitrary nonnegative integer. By considering $\mathbb{Z}_2$ as a trivial $\mathcal{A}$-module, then the hit problem is equivalent to the problem of finding a basis of $\mathbb{Z}_2$-vector space $\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$. The main goal of the current article is to explicitly determine an admissible monomial basis of the $\mathbb{Z}_2$ vector space $\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$ for $n = 6$ in some degrees.

One of the most important applications of the hit problem is to investigate homomorphism introduced by Singer, which is a homomorphism

$$\varphi_n : \mathrm{Tor}^{\mathcal{A}}_{n,n+d}(\mathbb{Z}_2, \mathbb{Z}_2) \longrightarrow (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d^{GL(n;\mathbb{Z}_2)}$$

from the homology of the Steenrod algebra to the subspace of $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d$ consisting of all the $GL(n;\mathbb{Z}_2)$-invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $\mathrm{Tor}^{\mathcal{A}}_{n,n+d}(\mathbb{Z}_2, \mathbb{Z}_2)$. The behavior of the sixth Singer algebraic transfer in degree $d_s = 6(2^s - 1) + 3.2^s$ was also discussed at the end of this paper.

Key words: Polynomial algebra, hit problem, graded rings

1. Introduction

Throughout the paper, we denote a prime field with two elements by $\mathbb{Z}_2$. Let $\mathbb{R}\mathcal{P}^\infty$ be the infinite dimensional real projective space. Then, $H^*(\mathbb{R}\mathcal{P}^\infty) \cong \mathbb{Z}_2[x_1]$, and therefore, the mod-2 cohomology algebra of the direct product of n copies of $\mathbb{R}\mathcal{P}^\infty$ is isomorphic to the polynomial algebra $\mathbb{Z}_2[x_1, x_2, \ldots, x_n]$, regarded as an unstable $\mathcal{A}$-module on n generators $x_1, x_2, \ldots, x_n$, each of degree 1. In other words, based on the Künneth formula for cohomology, we have an isomorphism of $\mathbb{Z}_2$-algebras

$$\mathcal{P}_n := H^*\big((\mathbb{R}\mathcal{P}^\infty)^n\big) \cong \mathbb{Z}_2[x_1] \otimes_{\mathbb{Z}_2} \ldots \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[x_n] \cong \mathbb{Z}_2[x_1, x_2, \ldots, x_n],$$

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where $x_i \in H^1\big((\mathbb{R}\mathcal{P}^\infty)^n\big)$ for every i.

The $\mathcal{A}$-module structure of $\mathcal{P}_n$ is explicitly determined in Steenrod and Epstein [14] by the Cartan formula and the properties of the Steenrod operation.

Let g be a homogeneous polynomial of degree d in $\mathcal{P}_n$. Then, g is called "hit" if there is an equation in the form of a finite sum $g = \sum_{i\geqslant 0} Sq^{2^i}(g_i)$, where the degree of g_i is less than d. That means g belongs to $\mathcal{A}^+\mathcal{P}_n$, where $\mathcal{A}^+$ is an ideal of $\mathcal{A}$ generated by all Steenrod squares Sq^k with $k > 0$.

The *hit problem* in algebraic topology is to find a minimal generating set for $\mathcal{P}_n$, regarded as a module over the mod-2 Steenrod algebra $\mathcal{A}$. If we consider $\mathbb{Z}_2$ as a trivial $\mathcal{A}$-module, then the hit problem is equivalent to the problem of finding a basis of $\mathbb{Z}_2$-vector space:

$$\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n = \bigoplus_{d\geqslant 0} (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d \cong \mathcal{P}_n/\mathcal{A}^+\mathcal{P}_n.$$

Here, $(\mathcal{P}_n)_d$ is the subspace of $\mathcal{P}_n$ consisting of all the homogeneous polynomials of degree d in $\mathcal{P}_n$, and $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d$ is the subspace of $\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$ consisting of all the classes represented by the elements in $(\mathcal{P}_n)_d$.

In [7], Peterson conjectured that as a module over the Steenrod algebra $\mathcal{A}$, $\mathcal{P}_n$ is generated by monomials in degree d that satisfy the inequality $\alpha(d+n) \leqslant n$, where $\alpha(d)$ is the number of digits one in the binary expension of d, and proved it for $n \leqslant 2$, in general, it is proved by Wood [24]. This is an extremely useful tool for determining $\mathcal{A}$-generators for $\mathcal{P}_n$. After then, the hit problem was investigated by many authors (see Repka-Selick [11], Silverman [12], Nam [6], Sum [16], Sum-Tin [18], Tin [21, 23] and others).

Let r, s, t be nonnegative intergers. From the results in Wood [24], Kameko [3], and Sum [16], the hit problem is reduced to the case of degree d of the form $d = r(2^t - 1) + 2^t s$ such that $0 \leqslant \mu(s) < r \leqslant n$, where

$$\mu(d) = \min\{a \in \mathbb{Z} \ : \ \alpha(d + a) \leqslant a\}.$$

Now, the tensor product $\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$ was completely determined for $n \leqslant 4$, (see Peterson [7] for $n = 1$, and $n = 2$, see Kameko for $n = 3$ in his thesis [3], see Sum [16] for $n = 4$). For $n > 4$, this problem remains unsolvable, even with the aid of computers in the case of $n = 5$.

In the present paper, we study the hit problem for the case $n = 6$ in degree $d_s = 6(2^s - 1) + 3.2^s$, with s an arbitrary nonnegative integer. The main goal of the current paper is to explicitly determine an admissible monomial basis of the $\mathbb{Z}_2$-vector space $\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6$ in some degrees. The proofs of the main results will be presented in Section 3. In addition, the behavior of the sixth Singer algebraic transfer in degree $d_s = 6(2^s - 1) + 3.2^s$ was also discussed at the end of this article.

2. Preliminaries

First, we recall some necessary results in Kameko [3], and Sum [16], which will be used in the next section.

Notation 2.1 *We will denote by* $\mathbb{N}_n = \{1, 2, \ldots, n\}$ *and*

$$X_{\mathbb{J}} = X_{\{j_1, j_2, \ldots, j_s\}} = \prod_{j\in\mathbb{N}_n\setminus\mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_n,$$

In particular, $X_{\mathbb{N}_n} = 1$, $X_\emptyset = x_1 x_2 \ldots x_n$, $X_j = x_1 \ldots \hat{x}_j \ldots x_n$, $1 \leqslant j \leqslant n$, *and* $X_n \in \mathcal{P}_{n-1}$.

Let $\alpha_t(d)$ be the t-th coefficient in dyadic expansion of d. Then, $d = \sum_{t\geqslant 0}\alpha_t(d).2^t$ where $\alpha_t(d) \in \{0,1\}$. Let $x = x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n} \in \mathcal{P}_n$. Denote $\nu_j(x) = a_j, 1 \leqslant j \leqslant n$. Set

$$\mathbb{J}_t(x) = \{j \in \mathbb{N}_n : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geqslant 0$. Then, we have $x = \prod_{t\geqslant 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.2 *For a monomial x in $\mathcal{P}_n$, define two sequences associated with x by*

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_n(x)),$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant n}\alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)}$, $i \geqslant 1$.

The sequences $\omega(x)$ and $\sigma(x)$ are respectively called the weight vector and the exponent vector of x. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of nonnegative integers. The sequence ω is called the weight vector if $\omega_i = 0$ for $i \gg 0$.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order. For a weight vector ω, we define $\deg \omega = \sum_{i>0} 2^{i-1}\omega_i$. Denote by $\mathcal{P}_n(\omega)$ the subspace of $\mathcal{P}_n$ spanned by all monomials y such that $\deg y = \deg \omega$, $\omega(y) \leqslant \omega$, and by $\mathcal{P}_n^-(\omega)$ the subspace of $\mathcal{P}_n$ spanned by all monomials $y \in \mathcal{P}_n(\omega)$ such that $\omega(y) < \omega$.

Definition 2.3 *Let f, g be two polynomials of the same degree in $\mathcal{P}_n$, and ω a weight vector. We define the equivalence relations "$\equiv$" and "$\equiv_\omega$" on $\mathcal{P}_n$ by stating that*

(i) *$f \equiv g$ if and only if $f - g \in \mathcal{A}^+\mathcal{P}_n$.*

(ii) *$f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+\mathcal{P}_n + \mathcal{P}_n^-(\omega)$.*

It is very easy to check that the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Then, one has

$$Q\mathcal{P}_n(\omega) = \mathcal{P}_n(\omega)/((\mathcal{A}^+\mathcal{P}_n \cap \mathcal{P}_n(\omega)) + \mathcal{P}_n^-(\omega)).$$

For a polynomial $f \in \mathcal{P}_n$, we denote by $[f]$ the class in $\mathbb{Z}_2\otimes_{\mathcal{A}}\mathcal{P}_n$ represented by f. If ω is a weight vector, and $f \in \mathcal{P}_n(\omega)$, then denote by $[f]_\omega$ the class in $Q\mathcal{P}_n(\omega)$ represented by f.

Definition 2.4 *Let u and v be monomials of the same degree in $\mathcal{P}_n$. We say that $u < v$ if one of the following holds:*

(i) *$\omega(u) < \omega(v)$;*

(ii) *$\omega(u) = \omega(v)$, and $\sigma(u) < \sigma(v)$.*

Definition 2.5 *A monomial u is said to be inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_i < u$ for $i = 1, 2, \ldots, m$ and $u - \sum_{t=1}^m v_i \in \mathcal{A}^+\mathcal{P}_n$.*

We say u is admissible if it is not inadmissible. Clearly, the set of all the admissible monomials of degree d in $\mathcal{P}_n$ is a minimal set of $\mathcal{A}$-generators for $\mathcal{P}_n$ in degree d.

Definition 2.6 *Let u be a monomial in $\mathcal{P}_n$. We say that u is strictly inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_j < u$, for $j = 1, 2, \ldots, m$ and $u = \sum_{j=1}^{m} v_j + \sum_{i=1}^{2^s-1} Sq^i(f_i)$ with $s = \max\{k : \omega_k(u) > 0\}$, $f_i \in \mathcal{P}_n$.*

It follows immediately from the definitions 2.5 and 2.6 that if u is strictly inadmissible monomial, then it is inadmissible monomial. In general, the opposite is not true.

For instance, for each integer $r > 0$, the monomial $x = x_1 x_2^{2^r-1} x_3^{2^r-1} \ldots x_{n-1}^{2^r-1} x_n$ is inadmissible, but it is not strictly inadmissible.

Theorem 2.7 (Kameko [3], Sum [16]) *Let u, v, w be monomials in $\mathcal{P}_n$ such that $\omega_t(u) = 0$ for $t > k > 0$, $\omega_r(w) \neq 0$ and $\omega_t(w) = 0$ for $t > r > 0$. Then,*

(i) *uw^{2^k} is inadmissible if w is inadmissible.*

(ii) *wv^{2^r} is strictly inadmissible if w is strictly inadmissible.*

Singer showed in [13] that if $\mu(d) < n$, then there exists uniquely a minimal spike of degree d in $\mathcal{P}_n$. The spike monomial has the following definition.

Definition 2.8 *Let $z = x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}$ in $\mathcal{P}_n$. The monomial z is called a spike if $d_j = 2^{t_j} - 1$ for t_j a nonnegative integer, $j = 1, 2, \ldots, n$. Moreover, if $t_1 > t_2 > \ldots > t_{r-1} \geqslant t_r > 0$ and $t_j = 0$ for $j > r$, then z is called the minimal spike.*

The following is a Singer's criterion on the hit monomials in $\mathcal{P}_n$.

Theorem 2.9 (Singer [13]) *Suppose z is the minimal spike of degree d in $\mathcal{P}_n$, and $u \in (\mathcal{P}_n)_d$ such that $\mu(d) \leqslant n$. If $\omega(u) < \omega(z)$, then u is hit.*

We will denote by $\mathcal{P}_n^0$ and $\mathcal{P}_n^+$ the $\mathcal{A}$-submodules of $\mathcal{P}_n$ spanned all the monomials $x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}$ such that $d_1 \ldots d_n = 0$, and $d_1 \ldots d_n > 0$, respectively. It is easy to see that $\mathcal{P}_n^0$ and $\mathcal{P}_n^+$ are the $\mathcal{A}$-submodules of $\mathcal{P}_n$. Then, we have a direct summand decomposition of the $\mathbb{Z}_2$-vector spaces:

$$\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n = (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n^0) \oplus (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n^+).$$

From now on, let us denote by $\mathcal{K}_n(d)$ the set of all admissible monomials of degree d in $\mathcal{P}_n$. Denote by $|S|$ the cardinal of a set S.

3. The main results

In this section, we study the hit problem for the polynomial algebra of six variables in some degrees. For $s = 0$, then $d_0 = 6(2^0 - 1) + 3.2^0$. An easy computation proves that the following proposition, which is an immediate consequence of the result in [16].

Proposition 3.1 *The set $\{[x_i^3], [x_j x_k^2], [x_j x_k x_\ell] : 1 \leqslant i, j, k, \ell \leqslant 6,\ j < k < \ell\}$ is a basis of $\mathbb{Z}_2$-vector space $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^0-1)+3.2^0}$. This implies $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^0-1)+3.2^0}$ has dimension 41.*

For $s = 1$, then $d_1 = 6(2^1 - 1) + 3.2^1$. We explicitly determine an admissible monomial basis of the $\mathbb{Z}_2$-vector space $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{12}$ as follows:

First, we recall the definition of the Kameko's squaring operation

$$\widetilde{Sq}^0_* := (\widetilde{Sq}^0_*)_{(n;n+2d)} : (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_{n+2d} \longrightarrow (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d,$$

which is induced by an $\mathbb{Z}_2$-linear map $S_n : \mathcal{P}_n \to \mathcal{P}_n$, given by

$$S_n(x) = \begin{cases} y, & \text{if } x = \prod_{i=1}^{n} x_i y^2 \\ 0, & \text{otherwise} \end{cases}$$

for any monomial $x \in \mathcal{P}_n$ (see Kameko [3]).

Since $\mathcal{P}_n = \oplus_{d \geqslant 0} (\mathcal{P}_n)_d$ is the graded polynomial algebra, and Kameko's homomorphism $(\widetilde{Sq}^0_*)_{(6;12)}$ is a $\mathbb{Z}_2$-epimorphism, it follows that

$$(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{12} \cong (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^0_6)_{12} \bigoplus \big(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;12)} \cap (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^+_6)_{12}\big) \bigoplus \mathrm{Im}(\widetilde{Sq}^0_*)_{(6;12)}$$

Let $\mathscr{L}_{(n,t)} = \{I = (i_1, i_2, \ldots, i_t) : 1 \leqslant i_1 < \ldots < i_t \leqslant n\}$, $1 \leqslant t < n$. For $I \in \mathscr{L}_{(n,t)}$, we define the homomorphism $f_I : \mathcal{P}_t \to \mathcal{P}_n$ of algebras by substituting $f_I(x_\ell) = x_{i_\ell}$ with $1 \leqslant \ell \leqslant t$. Then, f_I is a monomorphism of $\mathcal{A}$-modules.

Using the result in Mothebe-Kaelo-Ramatebele [4], we have a direct summand decomposition of the $\mathbb{Z}_2$-vector subspaces:

$$\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^0_n = \bigoplus_{1 \leqslant t \leqslant n-1} \bigoplus_{I \in \mathscr{L}_{(n,t)}} (Qf_I(\mathcal{P}^+_t)),$$

where $Qf_I(\mathcal{P}^+_t) = \mathbb{Z}_2 \otimes_{\mathcal{A}} f_I(\mathcal{P}^+_t)$.

Hence, $\dim(Qf_I(\mathcal{P}^+_t))_d = \dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^+_t)_d$, and $|\mathscr{L}_{(n,t)}| = \binom{n}{t}$. Combining with the results in Wood [24], one gets

$$\dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^0_n)_d = \sum_{\mu(d) \leqslant t \leqslant n-1} \binom{n}{t} \dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^+_t)_d.$$

Since $\mu(12) = 4$, it follows that if $t < 4$ then the spaces $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^+_t)_{12}$ are trivial. Moreover, using the results in Phuc-Sum [8], and Sum[16], one gets

$$\dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^+_t)_{6(2^1-1)+3.2^1} = \begin{cases} 21, & \text{if } \ t = 4, \\ 85, & \text{if } \ t = 5. \end{cases}$$

From the above results, we obtain

$$\dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^0_6)_{6(2^1-1)+3.2^1} = \binom{6}{4}.21 + \binom{6}{5}.85 = 825.$$

On the other hand, consider the homomorphism $\mathcal{T}_t : \mathcal{P}_5 \to \mathcal{P}_6$, for $1 \leqslant t \leqslant 6$ by substituting:

$$\mathcal{T}_t(x_k) = \begin{cases} x_k, & \text{if } 1 \leqslant k \leqslant t-1, \\ x_{k+1}, & \text{if } t \leqslant k < 6. \end{cases}$$

It is easy to see that $\mathcal{T}_t$ is a homomorphism of $\mathcal{A}$-modules. Moreover, Phuc-Sum [8] showed that $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_5)_{6(2^1-1)+3.2^1}$ is an $\mathbb{Z}_2$-vector space of dimension 190 with a basis consisting of all the classes represented by the monomials a_j, $1 \leqslant j \leqslant 190$. Consequently, $|\mathcal{K}_5(6(2^1-1)+3.2^1)| = 190$.

Using the above result, an easy computation shows that

$$\Big|\bigcup_{t=1}^{6} \mathcal{T}_t(\mathcal{K}_5(6(2^1-1)+3.2^1))\Big| = 825,$$

and the set

$$\{b_i : b_i \in \bigcup_{t=1}^{6} \mathcal{T}_t(a_j), 1 \leqslant j \leqslant 190, 1 \leqslant i \leqslant 825\}$$

is a minimal set of generators for $\mathcal{A}$-modules $\mathcal{P}_6^0$ in degree twelve. More specifically, we obtain the following proposition.

Proposition 3.2 *The set* $\{[b_i] : 1 \leqslant i \leqslant 825\}$ *is a basis of* $\mathbb{Z}_2$*-vector space* $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^0)_{6(2^1-1)+3.2^1}$. *This implies* $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^0)_{6(2^1-1)+3.2^1}$ *has dimension* 825.

The following corollary is an immediate consequence of Proposition 3.1.

Corollary 3.3 *The space* $Im(\widetilde{Sq}^0_*)_{(6;12)}$ *is isomorphic to a subspace of* $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^1-1)+3.2^1}$ *generated by all the classes represented by the admissible monomials of the form* $\prod_{i=1}^{6} x_i u^2$, *for* $u \in \mathcal{K}_6(3)$. *Consequently,* $\dim(Im(\widetilde{Sq}^0_*)_{(6;12)}) = 41$.

Next, we explicitly determine the $\mathbb{Z}_2$-vector space $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;12)} \cap (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^+)_{12}$. We will denote by $Q\mathcal{P}_n^+(\omega) = Q\mathcal{P}_n(\omega) \cap (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n^+)$. Putting $\widetilde{\omega_1} := (4,4,0)$, and $\widetilde{\omega_2} := (4,2,1)$. Then, we have the following.

Theorem 3.4 *Suppose that* $[x]$ *does not belong to* $Im(\widetilde{Sq}^0_*)_{(6;12)}$ *such that* x *is an element of* $(\mathcal{K}_6(12) \cap \mathcal{P}_6^+)$, *then* $\omega(x) = \widetilde{\omega_t}$ *with* $t = 1, 2$. *Moreover, we have an isomorphism of the* $\mathcal{K}$*-vector spaces:*

$$\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;12)} \cap (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^+)_{12} \cong Q\mathcal{P}_6^+(\widetilde{\omega_1}) \oplus Q\mathcal{P}_6^+(\widetilde{\omega_2}).$$

Proof Assume $f \in (\mathcal{K}_6(12) \cap \mathcal{P}_6^+)$ such that $[f]$ does not belong to $\mathrm{Im}(\widetilde{Sq}^0_*)_{(6;12)}$. Since $(\widetilde{Sq}^0_*)_{(6;12)}$ is an $\mathbb{Z}_2$-epimorphism, it follows that $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;12)}([f]) = 0$. It is easy to check that $h = x_1^7 x_2^3 x_3 x_4$ is the minimal spike of degree twelve in $\mathcal{P}_6$ and $\omega(h) = (4,2,1)$. Since f is an admissible monomial, by Theorem 2.9, it shows that $\omega_1(f) \geqslant \omega_1(h) = 4$. Moreover, $\deg(f)$ is an even number, it implies either $\omega_1(f) = 6$ or $\omega_1(f) = 4$.

If $\omega_1(f) = 6$, then $f = \prod_{i=1}^{6} x_i g^2$, where $g \in \mathcal{K}_6(3)$. By Theorem 2.7, g is an admissible monomial. Thus, we have $(\widetilde{Sq}^0_*)_{(6;12)}([f]) = [g] \neq 0$. This contradicts the fact that $[f]$ belongs to $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;12)}$. Hence, $\omega_1(f) = 4$. From this, $f = x_ix_jx_kx_\ell u^2$, where $u \in \mathcal{K}_6(4)$, and $1 \leqslant i < j < k < \ell \leqslant 6$. By Theorem 2.7, u is also admissible monomial.

Note that if $d < n$ then $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d \equiv (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^0_n)_d$. Using the results in Kameko [3] and Sum [16], one has

$$\dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_4 = \sum_{2=\mu(4)\leqslant t\leqslant 5} \binom{6}{t} \dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^+_t)_4 = 85,$$

and the set $\big\{[x_ix_j^3], [x_ix_jx_kx_\ell], [x_kx_\ell x_t^2] : 1 \leqslant i,j,k,\ell,t \leqslant 6,\ \ell < t\big\}$ is a basis of $\mathbb{Z}_2$-vector space $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_4$.

Since $u \in \mathcal{K}_6(4)$, it follows that either $\omega(u) = (4,0)$ or $\omega(u) = (2,1)$. And therefore, either $\omega(f) = \widetilde{\omega_1}$ or $\omega(f) = \widetilde{\omega_2}$.

For a weight vector ω of degree d, we set $\mathcal{K}_n(\omega) := \mathcal{K}_n(d) \cap \mathcal{P}_n(\omega)$. Observe that $\mathcal{K}_n(d) = \bigcup_{\deg \omega = d} \mathcal{K}_n(\omega)$.

Denote

$$Q\mathcal{P}^{\omega}_n := \langle \{[u] \in \mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n : \omega(x) = \omega, \text{ and } u \text{ is admissible } \}\rangle.$$

It is easy to check that the map $Q\mathcal{P}_n(\omega) \longrightarrow Q\mathcal{P}^{\omega}_n$, $[u]_\omega \longrightarrow [u]$ is an isomorphism of $\mathbb{Z}_2$-vector spaces. Hence, we can identify the vector space $Q\mathcal{P}_n(\omega)$ with $Q\mathcal{P}^{\omega}_n \subset \mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$. Furthermore, we obtain

$$(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d = \bigoplus_{\deg \omega = d} Q\mathcal{P}^{\omega}_n \cong \bigoplus_{\deg \omega = d} Q\mathcal{P}_n(\omega).$$

From this, it follows that $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^+_6)_{12} \cong \bigoplus_{\deg \omega = 12} Q\mathcal{P}^+_6(\omega)$. Thus, we can deduce that

$$\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;12)} \cap (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^+_6)_{12} \cong Q\mathcal{P}^+_6(\widetilde{\omega_1}) \oplus Q\mathcal{P}^+_6(\widetilde{\omega_2}).$$

Therefore, the theorem is proved. □

Theorem 3.5 *Let $\mathcal{F}^+_n(\omega)$ be the set of all admissible monomials in $\mathcal{P}^+_n(\omega)$. Then, we have*

$$|\mathcal{F}^+_6(\widetilde{\omega_t})| = \begin{cases} 45, & \text{if } \ t = 1, \\ 90, & \text{if } \ t = 2. \end{cases}$$

Consequently, $\dim\big(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;12)} \cap (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}^+_6)_{12}\big) = 135.$

Proof We prove the above theorem by explicitly determining all admissible monomials in $\mathcal{P}^+_6(\widetilde{\omega_k})$ for $k = 1,2$. The proof is divided by 2 cases:

Case 1. Consider the weight vector $\omega = \widetilde{\omega_1}$. Assume that x is an admissible monomial in $\mathcal{P}_6$ such that $\omega(x) = \widetilde{\omega_1}$, then $x = x_ix_jx_kx_\ell.y^2$ with a suitable polynomial $y \in (\mathcal{P}_6)_4$, $1 \leqslant i < j < k < \ell \leqslant 6$.

We set $\mathcal{C}^1_6 := \{x_ix_jx_kx_\ell.y^2 : 1 \leqslant i < j < k < \ell \leqslant 6, \omega(y) = (4,0)\} \cap \mathcal{P}^+_6$. It is easy to check that $|\mathcal{C}^1_6| = 90$, and $\mathrm{Span}\{\mathcal{C}^1_6\} = \mathcal{P}^+_6(\widetilde{\omega_1})$. Moreover, using Theorem 2.7, it follows that if $x \in \mathcal{K}_6(12)$ such that $\omega(x) = (4,4)$, then $x \in \mathcal{C}^1_6$.

It is easy to check that every monomial $x_1^2x_ix_jx_\ell^2x_k^3x_t^3$ is an inadmissible (more precisely by Sq^1), where (i,j,ℓ,k,t) is an arbitrary permutation of $(2,3,4,5,6)$, and $x_1^2x_ix_jx_\ell^2x_k^3x_t^3 \in \mathcal{C}_6^1$. On the other hand, we have

$$x_1^3x_2^2x_jx_\ell x_k^2x_t^3 = Sq^1(x_1^3x_2x_jx_\ell x_k^2x_t^3) + \text{smaller than}.$$

From this, the monomials $x_1^3x_2^2x_jx_\ell x_k^2x_t^3$ are inadmissible, where (j,ℓ,k,t) is an arbitrary permutation of $(3,4,5,6)$. Similarly, $x_1^3x_2^3x_3^2x_\ell x_kx_t^2$ are inadmissible, with (ℓ,k,t) a permutation of $(4,5,6)$.

From the above results, it shows that $\mathcal{P}_6^+(\widetilde{\omega_1})$ is generated by 45 elements c_i, with $1 \leqslant i \leqslant 45$ as follows:

1. $x_1^3x_2^3x_3^1x_4^1x_5^2x_6^2$	2. $x_1^3x_2^1x_3^3x_4^1x_5^2x_6^2$	3. $x_1^3x_2^1x_3^1x_4^3x_5^2x_6^2$	4. $x_1^1x_2^3x_3^3x_4^1x_5^2x_6^2$
5. $x_1^1x_2^3x_3^1x_4^3x_5^2x_6^2$	6. $x_1^1x_2^1x_3^3x_4^3x_5^2x_6^2$	7. $x_1^3x_2^3x_3^1x_4^2x_5^1x_6^2$	8. $x_1^3x_2^1x_3^3x_4^2x_5^1x_6^2$
9. $x_1^3x_2^1x_3^1x_4^2x_5^3x_6^2$	10. $x_1^1x_2^3x_3^3x_4^2x_5^1x_6^2$	11. $x_1^1x_2^3x_3^1x_4^2x_5^3x_6^2$	12. $x_1^1x_2^1x_3^3x_4^2x_5^3x_6^2$
13. $x_1^3x_2^3x_3^1x_4^2x_5^2x_6^1$	14. $x_1^3x_2^1x_3^3x_4^2x_5^2x_6^1$	15. $x_1^3x_2^1x_3^1x_4^2x_5^2x_6^3$	16. $x_1^1x_2^3x_3^3x_4^2x_5^2x_6^1$
17. $x_1^1x_2^3x_3^1x_4^2x_5^2x_6^3$	18. $x_1^1x_2^1x_3^3x_4^2x_5^2x_6^3$	19. $x_1^3x_2^1x_3^2x_4^3x_5^1x_6^2$	20. $x_1^3x_2^1x_3^2x_4^1x_5^3x_6^2$
21. $x_1^1x_2^3x_3^2x_4^3x_5^1x_6^2$	22. $x_1^1x_2^3x_3^2x_4^1x_5^3x_6^2$	23. $x_1^1x_2^1x_3^2x_4^3x_5^3x_6^2$	24. $x_1^3x_2^1x_3^2x_4^3x_5^2x_6^1$
25. $x_1^3x_2^1x_3^2x_4^1x_5^2x_6^3$	26. $x_1^1x_2^3x_3^2x_4^3x_5^2x_6^1$	27. $x_1^1x_2^3x_3^2x_4^1x_5^2x_6^3$	28. $x_1^1x_2^1x_3^2x_4^3x_5^2x_6^3$
29. $x_1^3x_2^1x_3^2x_4^2x_5^3x_6^1$	30. $x_1^3x_2^1x_3^2x_4^2x_5^1x_6^3$	31. $x_1^1x_2^3x_3^2x_4^2x_5^3x_6^1$	32. $x_1^1x_2^3x_3^2x_4^2x_5^1x_6^3$
33. $x_1^1x_2^1x_3^2x_4^2x_5^3x_6^3$	34. $x_1^1x_2^2x_3^3x_4^3x_5^1x_6^2$	35. $x_1^1x_2^2x_3^3x_4^1x_5^3x_6^2$	36. $x_1^1x_2^2x_3^1x_4^3x_5^3x_6^2$
37. $x_1^1x_2^2x_3^3x_4^3x_5^2x_6^1$	38. $x_1^1x_2^2x_3^3x_4^1x_5^2x_6^3$	39. $x_1^1x_2^2x_3^1x_4^3x_5^2x_6^3$	40. $x_1^1x_2^2x_3^3x_4^2x_5^3x_6^1$
41. $x_1^1x_2^2x_3^3x_4^2x_5^1x_6^3$	42. $x_1^1x_2^2x_3^1x_4^2x_5^3x_6^3$	43. $x_1^1x_2^2x_3^2x_4^3x_5^3x_6^1$	44. $x_1^1x_2^2x_3^2x_4^3x_5^1x_6^3$
45. $x_1^1x_2^2x_3^2x_4^1x_5^3x_6^3$			

We next prove that the vectors $[c_i], 1 \leqslant i \leqslant 45$, are linearly independent in $\mathbb{Z}_2{\otimes}_{\mathcal{A}}\mathcal{P}_6$. Denote

$$\mathcal{N}_n = \{(j;J) : J = (j_1, j_2, \ldots, j_t), 1 \leqslant j < j_1 < \ldots < j_t \leqslant n, 0 \leqslant t < n\},$$

where by convention $J = \emptyset$ if $t = 0$. Writing $t = \ell(J)$ for the length of J.

For any $(j;J) \in \mathcal{N}_6$, we define $\varphi_{(j;J)} : \mathcal{P}_6 \to \mathcal{P}_5$ by substituting:

$$\varphi_{(j;J)}(x_i) = \begin{cases} x_i, & \text{if } 1 \leqslant i \leqslant j-1, \\ \sum_{s\in J} x_{s-1}, & \text{if } i = j, \\ x_{i-1}, & \text{if } j < i \leqslant 6. \end{cases}$$

It is straightforward to determine whether these homomorphisms are $\mathcal{A}$-modules homomorphisms. We utilize them to establish that a given set of monomials is the set of admissible monomials in $\mathcal{P}_6$ by demonstrating that they are linearly independent in $\mathbb{Z}_2{\otimes}_{\mathcal{A}}\mathcal{P}_6$.

Suppose that there is a linear relation

$$\mathcal{U} = \sum_{1\leqslant i\leqslant 45} \gamma_i c_i \equiv 0, \tag{3.1}$$

with $\gamma_i \in \mathbb{Z}_2,\ i \in I = \{1, 2, \ldots, 45\}$.

We recall the result in [8] that $(\mathbb{Z}_2{\otimes}_{\mathcal{A}}\mathcal{P}_5^+)_{12}$ is a $\mathbb{Z}_2$-vector space of dimension 85 with a basis consisting of all the classes represented by the following admissible monomials:

$$\begin{array}{llll}
s_1 = x_1^1x_2^1x_3^1x_4^2x_5^7 & s_2 = x_1^1x_2^1x_3^1x_4^3x_5^6 & s_3 = x_1^1x_2^1x_3^1x_4^6x_5^3 & s_4 = x_1^1x_2^1x_3^1x_4^7x_5^2 \\
s_5 = x_1^1x_2^1x_3^2x_4^1x_5^7 & s_6 = x_1^1x_2^1x_3^2x_4^3x_5^5 & s_7 = x_1^1x_2^1x_3^2x_4^5x_5^3 & s_8 = x_1^1x_2^1x_3^2x_4^7x_5^1 \\
s_9 = x_1^1x_2^1x_3^3x_4^1x_5^6 & s_{10} = x_1^1x_2^1x_3^3x_4^2x_5^5 & s_{11} = x_1^1x_2^1x_3^3x_4^5x_5^2 & s_{12} = x_1^1x_2^1x_3^3x_4^6x_5^1 \\
s_{13} = x_1^1x_2^1x_3^6x_4^1x_5^3 & s_{14} = x_1^1x_2^1x_3^6x_4^3x_5^1 & s_{15} = x_1^1x_2^1x_3^7x_4^1x_5^2 & s_{16} = x_1^1x_2^1x_3^7x_4^2x_5^1 \\
s_{17} = x_1^1x_2^2x_3^1x_4^1x_5^7 & s_{18} = x_1^1x_2^2x_3^1x_4^3x_5^5 & s_{19} = x_1^1x_2^2x_3^1x_4^5x_5^3 & s_{20} = x_1^1x_2^2x_3^1x_4^7x_5^1 \\
s_{21} = x_1^1x_2^2x_3^3x_4^1x_5^5 & s_{22} = x_1^1x_2^2x_3^3x_4^5x_5^1 & s_{23} = x_1^1x_2^2x_3^5x_4^1x_5^3 & s_{24} = x_1^1x_2^2x_3^5x_4^3x_5^1 \\
s_{25} = x_1^1x_2^2x_3^7x_4^1x_5^1 & s_{26} = x_1^1x_2^3x_3^1x_4^1x_5^6 & s_{27} = x_1^1x_2^3x_3^1x_4^2x_5^5 & s_{28} = x_1^1x_2^3x_3^1x_4^5x_5^2 \\
s_{29} = x_1^1x_2^3x_3^1x_4^6x_5^1 & s_{30} = x_1^1x_2^3x_3^2x_4^1x_5^5 & s_{31} = x_1^1x_2^3x_3^2x_4^5x_5^1 & s_{32} = x_1^1x_2^3x_3^4x_4^1x_5^3 \\
s_{33} = x_1^1x_2^3x_3^4x_4^3x_5^1 & s_{34} = x_1^1x_2^3x_3^5x_4^1x_5^2 & s_{35} = x_1^1x_2^3x_3^5x_4^2x_5^1 & s_{36} = x_1^1x_2^3x_3^6x_4^1x_5^1 \\
s_{37} = x_1^1x_2^6x_3^1x_4^1x_5^3 & s_{38} = x_1^1x_2^6x_3^1x_4^3x_5^1 & s_{39} = x_1^1x_2^6x_3^3x_4^1x_5^1 & s_{40} = x_1^1x_2^7x_3^1x_4^1x_5^2 \\
s_{41} = x_1^1x_2^7x_3^1x_4^2x_5^1 & s_{42} = x_1^1x_2^7x_3^2x_4^1x_5^1 & s_{43} = x_1^3x_2^5x_3^2x_4^1x_5^1 & s_{44} = x_1^3x_2^1x_3^1x_4^1x_5^6 \\
s_{45} = x_1^3x_2^1x_3^1x_4^2x_5^5 & s_{46} = x_1^3x_2^1x_3^1x_4^5x_5^2 & s_{47} = x_1^3x_2^1x_3^1x_4^6x_5^1 & s_{48} = x_1^3x_2^1x_3^2x_4^1x_5^5 \\
s_{49} = x_1^3x_2^1x_3^2x_4^5x_5^1 & s_{50} = x_1^3x_2^1x_3^5x_4^1x_5^2 & s_{51} = x_1^3x_2^1x_3^5x_4^2x_5^1 & s_{52} = x_1^3x_2^1x_3^6x_4^1x_5^1 \\
s_{53} = x_1^3x_2^5x_3^1x_4^1x_5^2 & s_{54} = x_1^3x_2^5x_3^1x_4^2x_5^1 & s_{55} = x_1^7x_2^1x_3^1x_4^1x_5^2 & s_{56} = x_1^7x_2^1x_3^1x_4^2x_5^1 \\
s_{57} = x_1^7x_2^1x_3^2x_4^1x_5^1 & s_{58} = x_1^1x_2^1x_3^3x_4^3x_5^4 & s_{59} = x_1^1x_2^1x_3^3x_4^4x_5^3 & s_{60} = x_1^1x_2^3x_3^1x_4^3x_5^4 \\
s_{61} = x_1^1x_2^3x_3^1x_4^4x_5^3 & s_{62} = x_1^1x_2^3x_3^3x_4^1x_5^4 & s_{63} = x_1^1x_2^3x_3^3x_4^4x_5^1 & s_{64} = x_1^3x_2^1x_3^1x_4^3x_5^4 \\
s_{65} = x_1^3x_2^1x_3^1x_4^4x_5^3 & s_{66} = x_1^3x_2^1x_3^3x_4^1x_5^4 & s_{67} = x_1^3x_2^1x_3^3x_4^4x_5^1 & s_{68} = x_1^3x_2^1x_3^4x_4^1x_5^3 \\
s_{69} = x_1^3x_2^1x_3^4x_4^3x_5^1 & s_{70} = x_1^3x_2^3x_3^1x_4^1x_5^4 & s_{71} = x_1^3x_2^3x_3^1x_4^4x_5^1 & s_{72} = x_1^3x_2^3x_3^4x_4^1x_5^1 \\
s_{73} = x_1^3x_2^4x_3^1x_4^1x_5^3 & s_{74} = x_1^3x_2^4x_3^1x_4^3x_5^1 & s_{75} = x_1^3x_2^4x_3^3x_4^1x_5^1 & s_{76} = x_1^1x_2^3x_3^3x_4^3x_5^2 \\
s_{77} = x_1^1x_2^3x_3^3x_4^2x_5^3 & s_{78} = x_1^1x_2^3x_3^2x_4^3x_5^3 & s_{79} = x_1^1x_2^2x_3^3x_4^3x_5^3 & s_{80} = x_1^3x_2^1x_3^3x_4^3x_5^2 \\
s_{81} = x_1^3x_2^1x_3^3x_4^2x_5^3 & s_{82} = x_1^3x_2^1x_3^2x_4^3x_5^3 & s_{83} = x_1^3x_2^3x_3^1x_4^3x_5^2 & s_{84} = x_1^3x_2^3x_3^1x_4^2x_5^3 \\
s_{85} = x_1^3x_2^3x_3^3x_4^1x_5^2. & & &
\end{array}$$

Acting the homomorphism $\varphi_{(5;6)}$ on both sides of (3.1), and explicitly computing $\varphi_{(5;6)}(\mathcal{U})$ in terms of admissible monomials in $\mathcal{P}_5(\text{mod } (\mathcal{A}^+\mathcal{P}_5))$, we obtain

$$\begin{aligned}
\varphi_{(5;6)}(\mathcal{U}) \equiv{}& (\gamma_{23}+\gamma_{28})s_6 + (\gamma_{12}+\gamma_{18})s_{10} + (\gamma_{36}+\gamma_{39})s_{18} + (\gamma_{35}+\gamma_{38})s_{21} \\
&+ (\gamma_{11}+\gamma_{17})s_{27} + (\gamma_{22}+\gamma_{27})s_{30} + (\gamma_9+\gamma_{15})s_{45} + (\gamma_{20}+\gamma_{25})s_{48} \\
&+ \gamma_6 s_{58} + \gamma_5 s_{60} + \gamma_4 s_{62} + \gamma_3 s_{64} + \gamma_2 s_{66} + \gamma_1 s_{70} + (\gamma_{10}+\gamma_{16})s_{77} \\
&+ (\gamma_{21}+\gamma_{26})s_{78} + (\gamma_{34}+\gamma_{37})s_{79} + (\gamma_8+\gamma_{14})s_{81} + (\gamma_{19}+\gamma_{24})s_{82} \\
&+ (\gamma_7+\gamma_{13})s_{84} \equiv 0.
\end{aligned}$$

From the above equalities, we get $\gamma_i = 0$, for all $1 \leqslant i \leqslant 6$.

So, the relation (3.1) becomes

$$\mathcal{U} = \sum_{i \in I \setminus \mathbb{N}_6} \gamma_i c_i \equiv 0, \tag{3.2}$$

The homomorphisms $\varphi_{(4;5)}$ and $\varphi_{(3;6)}$ send the relation (3.2) to the following relations in $\mathcal{P}_5(\text{mod } (\mathcal{A}^+\mathcal{P}_5))$

$$\begin{aligned}
\varphi_{(4;5)}(\mathcal{U}) \equiv{}& (\gamma_{28}+\gamma_{33})s_7 + \gamma_{12}s_{11} + (\gamma_{39}+\gamma_{42})s_{19} + (\gamma_{37}+\gamma_{40})s_{22} + \gamma_{11}s_{28} \\
&+ (\gamma_{26}+\gamma_{31})s_{31} + \gamma_9 s_{46} + (\gamma_{24}+\gamma_{29})s_{49} + \gamma_{18}s_{59} + (\gamma_{27}+\gamma_{32})s_{78} \\
&+ \gamma_{17}s_{61} + \gamma_{16}s_{63} + \gamma_{15}s_{65} + \gamma_{14}s_{67} + \gamma_{13}s_{71} + \gamma_{10}s_{76} + \gamma_8 s_{80} \\
&+ (\gamma_{38}+\gamma_{41})s_{79} + (\gamma_{25}+\gamma_{30})s_{82} + \gamma_7 s_{83} \equiv 0,
\end{aligned}$$

$$\begin{aligned} \text{and } \varphi_{(3;6)}(\mathcal{U}) \equiv\ & \gamma_{33}s_6 + \gamma_{28}s_{10} + (\gamma_{35}+\gamma_{45})s_{18} + (\gamma_{34}+\gamma_{44})s_{21} + \gamma_{27}s_{27} + \gamma_{32}s_{30} \\ & + \gamma_{25}s_{45} + \gamma_{30}s_{48} + \gamma_{23}s_{58} + \gamma_{22}s_{60} + \gamma_{21}s_{62} + \gamma_{20}s_{64} + \gamma_{19}s_{66} \\ & + \gamma_{26}s_{77} + \gamma_{31}s_{78} + (\gamma_{36}+\gamma_{43})s_{79} + \gamma_{24}s_{81} + \gamma_{29}s_{82} \equiv 0. \end{aligned}$$

From the above equalities, we get $\gamma_i = 0,$ for all $7 \leqslant i \leqslant 33$.

Therefore, the relation (3.2) becomes

$$\mathcal{U} = \sum_{i \in I \setminus \mathbb{N}_{33}} \gamma_i c_i \equiv 0, \tag{3.3}$$

Similarly, the homomorphisms $\varphi_{(2;4)}$ and $\varphi_{(1;5)}$ to the relation (3.3), it shows that $\gamma_i = 0, 34 \leqslant i \leqslant 45$. From this, the vectors $[c_i], 1 \leqslant i \leqslant 45$, are linearly independent in $\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6$.

In summary, the set $\big\{[c_i] : 1 \leqslant i \leqslant 45\big\}$ is a basis of the $\mathbb{Z}_2$-vector space $Q\mathcal{P}_6^+(\widetilde{\omega_1})$. Consequently, $\dim Q\mathcal{P}_6^+(\widetilde{\omega_1}) = 45$.

Case 2. Consider the weight vector $\omega = \widetilde{\omega_2}$. By similar arguments, we also see that $\mathcal{P}_6^+(\widetilde{\omega_2}) = \mathrm{Span}\{\mathcal{C}_6^2\}$. Here,

$$\mathcal{C}_6^2 := \{x_ix_jx_kx_\ell.u^2 : 1 \leqslant i < j < k < \ell \leqslant 6, \omega(u) = (2,1)\} \cap \mathcal{P}_6^+, \text{and } |\mathcal{C}_6^2| = 150.$$

Remarkably, by Theorem 2.7, if $u \in \mathcal{K}_6(12)$ such that $\omega(u) = (4,2,1)$, then u belongs to $\mathcal{C}_6^2$. By direct calculations, using Theorem 2.7, we remove the inadmissible monomials in $\mathcal{C}_6^2$, and we get $\mathcal{P}_6^+(\widetilde{\omega_2})$ which is generated by 90 elements $d_i,\ 1 \leqslant i \leqslant 90$ as follows:

1. $x_1^1x_2^1x_3^1x_4^1x_5^2x_6^6$	2. $x_1^1x_2^1x_3^1x_4^1x_5^6x_6^2$	3. $x_1^3x_2^1x_3^1x_4^1x_5^2x_6^4$	4. $x_1^1x_2^3x_3^1x_4^1x_5^2x_6^4$
5. $x_1^1x_2^1x_3^3x_4^1x_5^2x_6^4$	6. $x_1^1x_2^1x_3^1x_4^3x_5^2x_6^4$	7. $x_1^3x_2^1x_3^1x_4^1x_5^4x_6^2$	8. $x_1^1x_2^3x_3^1x_4^1x_5^4x_6^2$
9. $x_1^1x_2^1x_3^3x_4^1x_5^4x_6^2$	10. $x_1^1x_2^1x_3^1x_4^3x_5^4x_6^2$	11. $x_1^1x_2^1x_3^1x_4^2x_5^1x_6^6$	12. $x_1^1x_2^1x_3^1x_4^6x_5^1x_6^2$
13. $x_1^3x_2^1x_3^1x_4^2x_5^1x_6^4$	14. $x_1^1x_2^3x_3^1x_4^2x_5^1x_6^4$	15. $x_1^1x_2^1x_3^3x_4^2x_5^1x_6^4$	16. $x_1^1x_2^1x_3^1x_4^2x_5^3x_6^4$
17. $x_1^3x_2^1x_3^1x_4^4x_5^1x_6^2$	18. $x_1^1x_2^3x_3^1x_4^4x_5^1x_6^2$	19. $x_1^1x_2^1x_3^3x_4^4x_5^1x_6^2$	20. $x_1^1x_2^1x_3^1x_4^2x_5^5x_6^2$
21. $x_1^1x_2^1x_3^1x_4^2x_5^6x_6^1$	22. $x_1^1x_2^1x_3^1x_4^6x_5^2x_6^1$	23. $x_1^3x_2^1x_3^1x_4^2x_5^4x_6^1$	24. $x_1^1x_2^3x_3^1x_4^2x_5^4x_6^1$
25. $x_1^1x_2^1x_3^3x_4^2x_5^4x_6^1$	26. $x_1^1x_2^1x_3^1x_4^2x_5^2x_6^5$	27. $x_1^3x_2^1x_3^1x_4^4x_5^2x_6^1$	28. $x_1^1x_2^3x_3^1x_4^4x_5^2x_6^1$
29. $x_1^1x_2^1x_3^3x_4^4x_5^2x_6^1$	30. $x_1^1x_2^1x_3^1x_4^2x_5^4x_6^3$	31. $x_1^1x_2^1x_3^2x_4^1x_5^1x_6^6$	32. $x_1^1x_2^1x_3^6x_4^1x_5^1x_6^2$
33. $x_1^3x_2^1x_3^2x_4^1x_5^1x_6^4$	34. $x_1^1x_2^3x_3^2x_4^1x_5^1x_6^4$	35. $x_1^1x_2^1x_3^2x_4^3x_5^1x_6^4$	36. $x_1^1x_2^1x_3^2x_4^1x_5^3x_6^4$
37. $x_1^3x_2^1x_3^4x_4^1x_5^1x_6^2$	38. $x_1^1x_2^3x_3^4x_4^1x_5^1x_6^2$	39. $x_1^1x_2^1x_3^2x_4^5x_5^1x_6^2$	40. $x_1^1x_2^1x_3^2x_4^1x_5^5x_6^2$
41. $x_1^1x_2^1x_3^2x_4^1x_5^6x_6^1$	42. $x_1^1x_2^1x_3^6x_4^1x_5^2x_6^1$	43. $x_1^3x_2^1x_3^2x_4^1x_5^4x_6^1$	44. $x_1^1x_2^3x_3^2x_4^1x_5^4x_6^1$
45. $x_1^1x_2^1x_3^2x_4^3x_5^4x_6^1$	46. $x_1^1x_2^1x_3^2x_4^1x_5^2x_6^5$	47. $x_1^3x_2^1x_3^4x_4^1x_5^2x_6^1$	48. $x_1^1x_2^3x_3^4x_4^1x_5^2x_6^1$
49. $x_1^1x_2^1x_3^2x_4^5x_5^2x_6^1$	50. $x_1^1x_2^1x_3^2x_4^1x_5^4x_6^3$	51. $x_1^1x_2^1x_3^2x_4^6x_5^1x_6^1$	52. $x_1^1x_2^1x_3^6x_4^2x_5^1x_6^1$
53. $x_1^3x_2^1x_3^2x_4^4x_5^1x_6^1$	54. $x_1^1x_2^3x_3^2x_4^4x_5^1x_6^1$	55. $x_1^1x_2^1x_3^2x_4^2x_5^5x_6^1$	56. $x_1^1x_2^1x_3^2x_4^2x_5^1x_6^5$
57. $x_1^3x_2^1x_3^4x_4^2x_5^1x_6^1$	58. $x_1^1x_2^3x_3^4x_4^2x_5^1x_6^1$	59. $x_1^1x_2^1x_3^2x_4^4x_5^3x_6^1$	60. $x_1^1x_2^1x_3^2x_4^4x_5^1x_6^3$
61. $x_1^1x_2^2x_3^1x_4^1x_5^1x_6^6$	62. $x_1^1x_2^6x_3^1x_4^1x_5^1x_6^2$	63. $x_1^1x_2^2x_3^3x_4^1x_5^1x_6^4$	64. $x_1^1x_2^2x_3^1x_4^3x_5^1x_6^4$
65. $x_1^1x_2^2x_3^1x_4^1x_5^3x_6^4$	66. $x_1^3x_2^4x_3^1x_4^1x_5^1x_6^2$	67. $x_1^1x_2^2x_3^5x_4^1x_5^1x_6^2$	68. $x_1^1x_2^2x_3^1x_4^5x_5^1x_6^2$
69. $x_1^1x_2^2x_3^1x_4^1x_5^5x_6^2$	70. $x_1^1x_2^2x_3^1x_4^1x_5^6x_6^1$	71. $x_1^1x_2^6x_3^1x_4^1x_5^2x_6^1$	72. $x_1^1x_2^2x_3^3x_4^1x_5^4x_6^1$
73. $x_1^1x_2^2x_3^1x_4^3x_5^4x_6^1$	74. $x_1^1x_2^2x_3^1x_4^1x_5^2x_6^5$	75. $x_1^3x_2^4x_3^1x_4^1x_5^2x_6^1$	76. $x_1^1x_2^2x_3^5x_4^1x_5^2x_6^1$
77. $x_1^1x_2^2x_3^1x_4^5x_5^2x_6^1$	78. $x_1^1x_2^2x_3^1x_4^1x_5^4x_6^3$	79. $x_1^1x_2^2x_3^1x_4^6x_5^1x_6^1$	80. $x_1^1x_2^6x_3^1x_4^2x_5^1x_6^1$
81. $x_1^1x_2^2x_3^3x_4^4x_5^1x_6^1$	82. $x_1^1x_2^2x_3^1x_4^2x_5^5x_6^1$	83. $x_1^1x_2^2x_3^1x_4^2x_5^1x_6^5$	84. $x_1^3x_2^4x_3^1x_4^2x_5^1x_6^1$
85. $x_1^1x_2^2x_3^5x_4^2x_5^1x_6^1$	86. $x_1^1x_2^2x_3^1x_4^4x_5^3x_6^1$	87. $x_1^1x_2^2x_3^1x_4^4x_5^1x_6^3$	88. $x_1^1x_2^2x_3^4x_4^3x_5^1x_6^1$
89. $x_1^1x_2^2x_3^4x_4^1x_5^3x_6^1$	90. $x_1^1x_2^2x_3^4x_4^1x_5^1x_6^3$		

Suppose that there is a linear relation

$$\mathcal{S} = \sum_{1\leqslant i\leqslant 90} \gamma_i d_i \equiv 0, \tag{3.4}$$

with $\gamma_i \in \mathbb{Z}_2,\ 1 \leqslant i \leqslant 90$.

By the same calculation as above, we explicitly compute $\varphi_{(j;J)}(\mathcal{S}),\ (j;J) \in \mathcal{N}_6$, in terms of $s_j, 1 \leqslant j \leqslant 85$. From the relation $\varphi_{(j;J)}(\mathcal{S}) \equiv 0,\ 1 \leqslant \ell(J) \leqslant 2,$ we also get $\gamma_i = 0$ for all $1 \leqslant i \leqslant 90$.

That means, the vectors $[d_i], 1 \leqslant i \leqslant 90$, are linearly independent in $\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6$. Hence, $\big\{[d_i],\ 1 \leqslant i \leqslant 90\big\}$ is a basis of $Q\mathcal{P}_6^+(\widetilde{\omega_2})$. Consequently, $|\mathcal{F}_6^+(\widetilde{\omega_2})| = 90$. The theorem is proved. $\square$

From the results of Proposition 3.2, Corollary 3.3, and Theorems 3.4, 3.5, we obtain the following.

Corollary 3.6 *The set* $\{b_i\}_{i=1}^{825} \cup \{c_j\}_{j=1}^{45} \cup \{d_\ell\}_{\ell=1}^{90} \cup \{e_k\}_{k=1}^{41}$ *is a minimal set of* $\mathcal{A}$*-generators for* $\mathcal{P}_6$ *in degree twelve. Consequently,* $\dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^1-1)+3.2^1} = 1001$*. Here,* $e_k \in \big\{\prod_{i=1}^6 x_i x^2 : x \in \mathcal{K}_6(3)\big\}$*, for all* $k = 1, \ldots, 41$.

Note that Mothebe-Kaelo-Ramatebele [4] used a different method to verify the dimension result in the above Corollary.

For $s = 2$, then $d_2 = 6(2^2-1)+3.2^2$. We also see that the Kameko's homomorphism $(\widetilde{Sq}^0_*)_{(6;6(2^2-1)+3.2^2)}$ is a $\mathbb{Z}_2$-epimorphism, it follows that

$$(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{30} \cong (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^0)_{30} \bigoplus \big(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;30)} \cap (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^+)_{30}\big) \bigoplus \mathrm{Im}(\widetilde{Sq}^0_*)_{(6;30)}$$

The following proposition is an immediate consequence of Corollary 3.6.

Proposition 3.7 *Im*$(\widetilde{Sq}^0_*)_{(6;30)}$ *is isomorphic to a subspace of* $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{30}$ *generated by all the classes represented by the admissible monomials of the form* $\prod_{i=1}^6 x_i v^2$*, for every* $v \in \mathcal{K}_6(12)$*. This implies Im*$(\widetilde{Sq}^0_*)_{(6;30)}$ *has dimension* 1001.

We can prove the following theorem using the same argument as the former.

Theorem 3.8 *The following statements are true:*

(i) *If* u *is an admissible monomial in* $(\mathcal{P}_6)_{6(2^2-1)+3.2^2}$ *such that* $[u]$ *belongs to* $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;30)}$*, then* $\omega(u)$ *is one of the following sequences:*

$$\omega_{[1]} = (2,2,2,2), \omega_{[2]} = (2,2,4,1), \omega_{[3]} = (2,4,3,1), \omega_{[4]} = (2,4,5),$$

$$\omega_{[5]} = (4,3,3,1),\ \omega_{[6]} = (4,5,2,1),\ \omega_{[7]} = (4,3,5),\ \omega_{[8]} = (4,5,4).$$

Moreover, we have an isomorphism of the $\mathbb{Z}_2$*-vector spaces:*

$$\big(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;30)} \cap (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^+)_{30}\big) \cong \bigoplus_{m=1}^{8} Q\mathcal{P}_6^+(\omega_{[m]}).$$

(ii) *The set* $\big\{[v_i] : v_i \in \bigcup_{t=1}^{6}\mathcal{T}_t(\mathcal{K}_5(6(2^2-1)+3.2^2)),\ 1\leqslant i\leqslant 4115\big\}$ *is a basis of* $\mathbb{Z}_2$*-vector space* $(\mathbb{Z}_2\otimes_{\mathcal{A}}\mathcal{P}_6^0)_{6(2^2-1)+3.2^2}$. *This implies* $(\mathbb{Z}_2\otimes_{\mathcal{A}}\mathcal{P}_6^0)_{6(2^2-1)+3.2^2}$ *has dimension* 4115.

Proof First, we prove Part (i) of the above theorem. We set

$$QP_6^{\omega} := \mathrm{Span}\big\{[x]\in\mathbb{Z}_2\otimes_{\mathcal{A}}\mathcal{P}_6 : \omega(x)=\omega,\ \text{and } x \text{ is admissible monomial}\big\}.$$

By the same argument as in the proof of Theorem 3.4, we also see that the map $Q\mathcal{P}_6(\omega)\longrightarrow Q\mathcal{P}_6^{\omega}$, $[u]_\omega\longrightarrow[u]$ is an isomorphism of $\mathbb{Z}_2$-vector spaces. Hence, one gets

$$(\mathbb{Z}_2\otimes_{\mathcal{A}}\mathcal{P}_6)_{6(2^2-1)+3.2^2} = \bigoplus_{\deg\omega=30} Q\mathcal{P}_6^{\omega} \cong \bigoplus_{\deg\omega=30} Q\mathcal{P}_6(\omega).$$

As a result of this, we may deduce that $(\mathbb{Z}_2\otimes_{\mathcal{A}}\mathcal{P}_6^{+})_{6(2^2-1)+3.2^2} = \bigoplus_{\deg\omega=30} Q\mathcal{P}_6^{+}(\omega)$.

Assume that u is an admissible monomial of degree $6(2^2-1)+3.2^2$ in $\mathcal{P}_6$ such that $[u]$ belongs to $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;30)}$. Observe that $z=x_1^{15}x_2^{15}$ is the minimal spike of degree $6(2^2-1)+3.2^2$ in $\mathcal{P}_6$ and $\omega(z)=(2,2,2,2)$. By Theorem 2.9, one gets $\omega_1(u)\geqslant 2$. Since the degree of (u) is even number, it shows that $\omega_1(u)=2$, $\omega_1(u)=4$, or $\omega_1(u)=6$.

Case 1. If $\omega_1(u)=2$, then $u=x_ix_jX^2$ with X a monomial of degree fourteen in $\mathcal{P}_6$, and $1\leqslant i<j\leqslant 6$. By Theorem 2.7, X is admissible monomial. Since $u\in\mathcal{K}_6(14)$, it shows that $\omega(X)=(2,2,2)$, or $\omega(X)=(2,4,1)$, or $\omega(X)=(4,5)$, or $\omega(X)=(4,3,1)$, or $\omega(X)=(6,2,1)$, or $\omega(X)=(6,4)$ (see Tin [22]).

Using the results in Sum [15], we see that if v is a monomial in $\mathcal{P}_6$ such that $\omega(v)=(2,6,2,1)$, or $\omega(v)=(2,6,4)$, then v is strictly inadmissible. Therefore, v is inadmissible. From this, $\omega(u)=(2,2,2,2)$, or $\omega(u)=(2,2,4,1)$, or $\omega(u)=(2,4,5)$, or $\omega(u)=(2,4,3,1)$.

Case 2. If $\omega_1(u)=4$, then $u=x_ix_jx_kx_\ell Y^2$, where Y is an admissible monomial of degree thirteen in $\mathcal{P}_6$, and $1\leqslant i<j<k<\ell\leqslant 6$. It is a simple matter to see that the monomial $w=x_1^7x_2^3x_3^3$ is a minimal spike of degree thirteen in $\mathcal{P}_6$, and $\omega(w)=(3,3,1)$. By Theorem 2.9, one gets $\omega_1(Y)\geqslant 3$. Since the degree of (Y) is odd number, it shows that either $\omega_1(Y)=3$, or $\omega_1(Y)=5$.

If $\omega_1(Y)=3$, then $Y=x_mx_nx_tZ^2$, where Z is an admissible monomial of degree five in $\mathcal{P}_6$, and $1\leqslant m<n<t\leqslant 6$. An easy computation shows that

$$\mathcal{K}_6(5)=\big\{x_ix_jx_k^3;\, x_ix_jx_kx_\ell x_t;\, x_ix_jx_kx_mx_n^2 : 1\leqslant i,j,k,\ell,t,m,n\leqslant 6; m<n\big\}.$$

Since $Z\in\mathcal{K}_6(5)$, it shows that either $\omega(Z)=(3,1)$, or $\omega(Z)=(5,0)$. Thus, either $\omega(Y)=(3,3,1)$, or $\omega(Y)=(3,5,0)$.

If $\omega_1(Y)=5$, then $Y=x_mx_nx_tx_rx_sT^2$, where T is an admissible monomial of degree four in $\mathcal{P}_6$, and $1\leqslant m<n<t<r<s\leqslant 6$. Since $T\in\mathcal{K}_6(4)$, it yields either $\omega(T)=(2,1)$, or $\omega(T)=(4,0)$. Hence, either $\omega(Y)=(5,2,1)$, or $\omega(Y)=(5,4,0)$.

So, $\omega(u)=(4,3,3,1)$, or $\omega(u)=(4,3,5)$, or $\omega(u)=(4,5,2,1)$, or $\omega(u)=(4,5,4)$.

Case 3. If $\omega_1(u)=6$ then $u=\prod_{i=1}^{6}x_iF^2$, with F a monomial of degree nine in $\mathcal{P}_6$. Since u is admissible monomial, using Theorem 2.7, F is also an admissible monomial. Hence, $[F]\neq 0$. Moreover, $[F]=(\widetilde{Sq}^0_*)_{(6;30)}([u])\neq 0$. This contradicts the fact that $[u]\in\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;30)}$.

In summary, we obtain $\omega(u) = \omega_{[m]}$, for all $m = 1, \ldots, 8$. Therefore, we have an isomorphism of the $\mathbb{Z}_2$-vector spaces:

$$\big(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;30)} \cap (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^+)_{30}\big) \cong \bigoplus_{m=1}^{8} Q\mathcal{P}_6^+(\omega_{[m]}).$$

Next, we prove Part (ii) of the theorem. Since $\mu(6(2^2-1)+3.2^2) = 2$, it follows that if $t < 2$ then the spaces $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_t^+)_{6(2^2-1)+3.2^2}$ are trivial. From the results in Peterson [7], Kameko [3], and Sum [16], one gets

$$\dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_t^+)_{6(2^2-1)+3.2^2} = \begin{cases} 1, & \text{if } t = 2, \\ 4, & \text{if } t = 3, \\ 48, & \text{if } t = 4. \end{cases}$$

At the same time, Tin showed in [20] that $\big|\mathcal{K}_5(6(2^2-1)+3.2^2)\big| = 840$. That means, the space $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_5)_{6(2^2-1)+3.2^2}$ has dimension 840. Note that the result dimension of the space $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_5)_{6(2^2-1)+3.2^2}$ has been verified by using a computer calculation program in SAGE (Software for Algebra and Geometry Experimentation) by V. H. Viet (we would like to thank for his support). Moreover, we have $\dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_5^0)_{6(2^2-1)+3.2^2} = 290$, and $\dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_5^+)_{6(2^2-1)+3.2^2} = 550$.

From the above results, one gets

$$\begin{aligned} \dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^0)_{6(2^2-1)+3.2^2} &= \sum_{2 \leqslant m \leqslant 5} \binom{6}{m} \dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_m^+)_{6(2^2-1)+3.2^2} \\ &= \binom{6}{2}.1 + \binom{6}{3}.4 + \binom{6}{4}.48 + \binom{6}{5}.550 = 4115. \end{aligned}$$

Suppose that $\mathcal{K}_5(6(2^2-1)+3.2^2) = \big\{u_i : \ 1 \leqslant i \leqslant 840\big\}$. An easy computation shows that

$$\big|\big\{\bigcup_{t=1}^{6} \mathcal{T}_t(u_i), \ 1 \leqslant i \leqslant 550\big\}\big| = 4115.$$

Furthermore, we obtain the set

$$\big\{[v_i] : v_i \in \bigcup_{t=1}^{6} \mathcal{T}_t(\mathcal{K}_5(6(2^2-1)+3.2^2)), \ 1 \leqslant i \leqslant 4115\big\}$$

is a basis of $\mathbb{Z}_2$-vector space $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6^0)_{6(2^2-1)+3.2^2}$. The theorem is proved. □

Consider the degrees $d_s = 6(2^s-1)+3.2^s$, for any $s \geqslant 3$. Let $GL(n;\mathbb{Z}_2)$ be the general linear group over the field $\mathbb{Z}_2$. This group acts naturally on $\mathcal{P}_n$ by matrix substitution. Since the two actions of $\mathcal{A}$ and $GL(n;\mathbb{Z}_2)$ upon $\mathcal{P}_n$ commute with each other, there is an action of $GL(n;\mathbb{Z}_2)$ on $\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$. We set

$$\zeta(n;d) = \max\{0, n - \alpha(d+n) - \xi(d+n)\},$$

where $\xi(n)$ the greatest integer m such that n is divisible by 2^m. We have the following theorem.

Theorem 3.9 (Tin-Sum [19]) *Let d be an arbitrary nonnegative integer. Then*

$$(\widetilde{Sq}^0_*)^{r-s} : (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_{n(2^r-1)+2^r d} \longrightarrow (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_{n(2^s-1)+2^s d}$$

is an isomorphism of $GL(n;\mathbb{Z}_2)$-modules for every $r \geqslant s$ if and only if $s \geqslant \zeta(n;d)$.

It is not difficult to see that for $n = 6$ and $d = 66$ then $\alpha(d+n) = \alpha(72) = 2$, and $\xi(d+n) = \xi(2^3.9) = 3$. Therefore $\zeta(n;d) = \zeta(6;72) = 1$. Using the above theorem, we get an isomorphism of $\mathbb{Z}_2$-vector spaces:

$$(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^r-1)+66.2^r} \cong (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^1-1)+66.2^1} \quad \text{for all } r \geqslant 1.$$

Therefore, we obtain

$$(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^s-1)+3.2^s} \cong (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^1-1)+66.2^1} \text{ for all } s > 3.$$

That means, $\dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^s-1)+3.2^s} = \dim(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^1-1)+66.2^1}$ for all $s > 3$.

Moreover, we get the set $\big\{[x] : x \in \Phi^{s-4}\big(\mathcal{K}_6(6(2^4-1)+3.2^4)\big)\big\}$ is a basis of the $\mathbb{Z}_2$-vector space $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^s-1)+3.2^s}$ for all $s > 3$. Here, $\Phi : \mathcal{P}_6 \to \mathcal{P}_6$ is the homomorphism determined by $\Phi(x) = \prod_{i=1}^{6} x_i x^2$, for all $x \in \mathcal{P}_6$.

Therefore, we need only to study the $\mathbb{Z}_2$-vector space $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)_{6(2^s-1)+3.2^s}$ for $s \leqslant 4$. At the same time, these findings will also be applied to the study of the sixth algebraic transfer and the modular representation of the general linear group $GL(n;\mathbb{Z}_2)$.

Remark 3.10

Let $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d^{GL(n;\mathbb{Z}_2)}$ be the subspace of $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d$ consisting of all the $GL(n;\mathbb{Z}_2)$-invariant classes of degree d, and $\mathbb{Z}_2 \otimes_{GL(n;\mathbb{Z}_2)} PH_d((\mathbb{R}\mathcal{P}^\infty)^n)$ be dual to $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d^{GL(n;\mathbb{Z}_2)}$. One of the major applications of hit problem is in surveying a homomorphism introduced by W. M. Singer. It is a useful tool in describing the cohomology groups of the Steenrod algebra, $\mathrm{Ext}_{\mathcal{A}}^{n,n+*}(\mathbb{Z}_2, \mathbb{Z}_2)$.

Singer [13] defined the algebraic transfer, which is a homomorphism

$$\psi_n : \mathbb{Z}_2 \otimes_{GL(n;\mathbb{Z}_2)} PH_*((\mathbb{R}\mathcal{P}^\infty)^n) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{n,n+*}(\mathbb{Z}_2, \mathbb{Z}_2).$$

Singer has indicated the importance of the algebraic transfer by showing that ψ_n is an isomorphism with $n = 1, 2$ and at some other degrees with $n = 3, 4$, but he also disproved this for ψ_5 at degree 9, and then gave the following conjecture.

Conjecture 3.11 *For any $n \geqslant 0$, the algebraic transfer ψ_n is a monomorphism.*

It could be seen from the work of Singer the meaning and importance of the hit problem. In [1], Boardman confirmed this again by using the modular representation theory of linear groups to show that ψ_3 is also an isomorphism.

For $n \geqslant 4$, the Singer algebraic transfer was studied by many authors (See Boardman [1], Bruner-Ha-Hung [2], Minami [5], Sum-Tin [17], Phuc [9] and others). However, Singer's conjecture is still open for $n \geqslant 4$.

We will use the results of the hit problem to study and verify the Singer conjecture for the algebraic transfer in the above degrees. More specifically, we use the admissible monomial basis of degree $6(2^s-1)+3.2^s$ in $\mathcal{P}_6$ to explicitly compute the vector space $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)^{GL(6;\mathbb{Z}_2)}_{6(2^s-1)+3.2^s}$, and combining the computation of the cohomology groups of the Steenrod algebra $\mathrm{Ext}^{6,6(2^s-1)+3.2^s+6}_{\mathcal{A}}(\mathbb{Z}_2,\mathbb{Z}_2)$, to obtain information about the behavior of the sixth Singer algebraic transfer in these degrees.

Remarkably, by using Theorem 3.9, we also get

$$(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)^{GL(6;\mathbb{Z}_2)}_{6(2^s-1)+3.2^s} \cong (\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_6)^{GL(6;\mathbb{Z}_2)}_{6(2^4-1)+3.2^4}, \quad \text{for all } s>4.$$

Since $(\mathbb{Z}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)^{GL(n;\mathbb{Z}_2)}_{*}$ is dual to $\mathbb{Z}_2 \otimes_{GL(n;\mathbb{Z}_2)} PH_*((\mathbb{R}\mathcal{P}^\infty)^n)$, we have

$$\mathbb{Z}_2 \otimes_{GL(6;\mathbb{Z}_2)} PH_{6(2^s-1)+3.2^s}((\mathbb{R}\mathcal{P}^\infty)^6) \cong (\mathbb{Z}_2 \otimes_{GL(6;\mathbb{Z}_2)} PH_{6(2^4-1)+3.2^4}((\mathbb{R}\mathcal{P}^\infty)^6)), \quad \text{for all } s>4.$$

Therefore, we need only to compute the dimension of vector spaces $\mathbb{Z}_2 \otimes_{GL(6;\mathbb{Z}_2)} PH_{6(2^s-1)+3.2^s}((\mathbb{R}\mathcal{P}^\infty)^6)$ for $s \leqslant 4$. This is an open problem. We will continue to study this problem in the near future.

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