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Limited frequency band diffusive representation for nabla fractional order transfer functions

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Abstract: Though infinite-dimensional characteristic is the natural property of nabla fractional order systems and it is the foundation of stability analysis, controller synthesis and numerical realization, there are few research focusing on this topic. Under this background, this paper concerns the diffusive representation of nabla fractional order systems. Firstly, several variants are developed for the elementary equality in frequency domain, i.e. $\frac{1}{s^{\alpha}} = \int_0^{+\infty} \frac{\mu_\alpha(\omega)}{s+\omega} \mathrm{d}\omega$. Afterwards, the limited frequency band diffusive representation and the unit impulse response are derived for a series of nabla fractional order transfer functions. Finally, an attempt to find the diffusive representation for general nabla fractional order transfer functions is made. Some conclusions are presented.

Key words: Frequency distributed model, diffusive representation, nabla fractional calculus, nabla Laplace transform, limited frequency band

1. Introduction

Fractional calculus, as an emerging and active branch of mathematical analysis, is actually as old as the classical calculus that we know today [6–8, 19]. It has shown its strong capability when the problem being considered involves long memory dynamics (or nonlocality) given that fractional calculus has the nonlocality [42]. A considerable amount of results have been reported in various related fields due to continuous efforts, which makes a positive and profound impact and many potential applications have been proposed [36, 37].

At present, a consensus has been reached on the infinite-dimensional property of fractional order systems. The commonly used description state space description for fractional order systems is questionable [34]. To be precise, it should be called as pseudo state space model. The real state space model is actually with infinite dimension and usually named as diffusive representation [20, 21], diffusive realisation [11] or frequency distributed model [43]. In this direction, a great deal of work has been done. [2, 40, 41, 43] proposed the indirect Lyapunov method to study the stability of fractional order systems. [17] developed the inverse Lypunov thoerem for indirect Lyapunov method and proposed a systematic scheme to construct the desired Lyapunov functions. By using the the infinite state approach, two monographs are published on the analysis, modeling and stability of fractional order differential systems [38, 39].

Because of the infinite-dimensional nature, it is generally difficult to realize the fractional order system in

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physical platform or simulation platform. Realization in simulation platform is commonly referred to numerical realization which is common but crucial. Giving up the pursuit of exact solutions, various of attempts have been taken to obtain the approximate solutions. Heleschewitz developed a powerful method to approximate fractional integrals and thoroughly examined its performance. However, this method cannot give a clear relationship between the parameters and the precision. From a different point of view, Oustaloup presented a recursive algorithm to approximate fractional differentiator s^α in [23]. It is a seminal and promising work, which has aroused a series of proceeding work [5, 15, 18, 24, 31, 46, 49–51, 53] and played a critical role in practice. The main idea of this work is the discretization of diffusive representation [22]. In order to adapt to different definitions, the initial value configuration was discussed in [16, 44]. Along this way, the numerical simulation model and analog circuit realization of $\frac{1}{\tau s^\alpha+1}$ were presented [4]. By combining the diffusive representation and the Laplace transform, the state space description was deduced for $\frac{1}{(\tau s+1)^\alpha}$ in [33]. With the help of the Taylor expansion at zero, the infinite dimensional state space realization for linear filter $\left(\frac{s+\omega_l}{s+\omega_h}\right)^\alpha$ was developed [26]. Notably, Sabatier investigated the diffusive representation of a class of fractional order transfer functions [27, 29, 30, 32] and derived the corresponding unit impulsive response. More details can be founded in [47] and the references therein.

Despite the plentiful achievements, there are few papers concerning the infinite-dimensional characteristic of nabla fractional order system. [47] examined the infinite-dimensional characteristic of nabla fractional order system for the first time and developed the corresponding indirect Lyapunov method. Afterwards, the converse theorem for indirected Lyapunov method was established in [45]. By using the discretization approach, the numerical approximation method was proposed in [52]. However, this method in nabla discrete time case does not draw enough attention. i) The relationship between time domain and frequency domain of nabla fractional order systems is not clear. ii) The equivalent expression in frequency domain is extremely rare. iii) Limited frequency band diffusive representation for nabla fractional order transfer functions is absent now. iv) How to calculate the diffusive representation for general fractional order transfer functions has not been discussed yet. All of these motivates this work.

The rest of this paper is structured as follows. Section 2 presents some preliminaries on nabla fractional calculus and nabla Laplace transform. Section 3 is the main body of the paper where a series of diffusive representations of fractional calculus are provided and the theoretical results are proved strictly. Section 4 provides numerical evidences about the practicality of the developed methods. The paper ends in Section 5 with some concluding remarks and comments on the future work.

2. Preliminary

In this section, some basic definitions and properties of nabla fractional calculus are introduced here.

For $f:\mathbb{N}_{a+1-n}\to\mathbb{R}$, its n-th integer order backward difference is defined by [9]

$$\nabla^n f(k) := \textstyle\sum_{i=0}^{n}(-1)^i\binom{n}{i}f(k-i), \tag{2.1}$$

where $n\in\mathbb{Z}_+$, $k\in\mathbb{N}_{a+1}:=\{a+1,a+2,a+3,\cdots\}$, $a\in\mathbb{R}$, $\binom{p}{q}:=\frac{\Gamma(p+1)}{\Gamma(q+1)\Gamma(p-q+1)}$ is the generalized binomial coefficient and $\Gamma(z):=\int_0^{+\infty}t^{z-1}\mathrm{e}^{-t}\mathrm{d}t$ is the Gamma function.

For $f:\mathbb{N}_{a+1}\to\mathbb{R}$, its α-th nabla fractional sum/difference is defined by

$${}_a^{\mathrm{G}}\nabla_k^{-\alpha}f(k) := \textstyle\sum_{i=0}^{k-a-1}(-1)^i\binom{-\alpha}{i}f(k-i), \tag{2.2}$$

where $\alpha \in \mathbb{R}$, $k \in \mathbb{N}_{a+1}$ and $a \in \mathbb{R}$.

With the previous definitions, the α-th Riemann–Liouville and Caputo fractional difference for $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$, $\alpha \in (n-1,n)$, $n \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$ and $a \in \mathbb{R}$ can be respectively defined as

$$ {}_a^{\mathrm{R}}\nabla_k^{\alpha} f(k) := \nabla^n {}_a^{\mathrm{G}}\nabla_k^{\alpha-n} f(k), \tag{2.3}$$

$$ {}_a^{\mathrm{C}}\nabla_k^{\alpha} f(k) := {}_a^{\mathrm{G}}\nabla_k^{\alpha-n} \nabla^n f(k). \tag{2.4}$$

To highlight the effect, ${}_a^{\mathrm{G}}\nabla_k^{-\alpha}$ is called fractional sum operator in time domain and ${}_a^{\mathrm{G}}\nabla_k^{\alpha}$, ${}_a^{\mathrm{R}}\nabla_k^{\alpha}$, ${}_a^{\mathrm{C}}\nabla_k^{\alpha}$ are called fractional difference operators in time domain when $\alpha \in (n-1,n)$, $n \in \mathbb{Z}_+$.

For $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $a \in \mathbb{R}$, the nabla Laplace transform of $f(\cdot)$ is defined as

$$ \mathscr{N}_a\{f(k)\} := \sum_{k=1}^{+\infty} (1-s)^{k-1} f(k+a), \tag{2.5}$$

for those values of s such that the infinite series converges.

Assume $F(s) := \mathscr{N}_a\{f(k)\}$. Then the final value of $f(k)$ exists as k tends to infinity, i.e. $\lim_{k\to+\infty} f(k) = \sigma$, where σ is a finite constant, if and only if the multiplicity of pole $s = 0$ for $F(s)$ is no more than 1 or all the principal poles of $F(s)$ satisfy $|s-1| > 1$. Furthermore, if $F(s)$ converges for $|s-1| < r$ with some $r > 0$, then for any $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$, $a \in \mathbb{R}$, $\alpha \in (n-1,n)$ and $n \in \mathbb{Z}_+$, one has [48]

$$ \mathscr{N}_a\{{}_a^{\mathrm{G}}\nabla_k^{-\alpha} f(k)\} = s^{-\alpha} F(s), \tag{2.6}$$

$$ \mathscr{N}_a\{{}_a^{\mathrm{G}}\nabla_k^{\alpha} f(k)\} = s^{\alpha} F(s), \tag{2.7}$$

$$ \mathscr{N}_a\{{}_a^{\mathrm{R}}\nabla_k^{\alpha} f(k)\} = s^{\alpha} F(s) - \sum_{i=0}^{n-1} s^i [{}_a^{\mathrm{R}}\nabla_k^{\alpha-i-1} f(k)]_{k=a}, \tag{2.8}$$

$$ \mathscr{N}_a\{{}_a^{\mathrm{C}}\nabla_k^{\alpha} f(k)\} = s^{\alpha} F(s) - \sum_{i=0}^{n-1} s^{\alpha-i-1} [\nabla^i f(k)]_{k=a}, \tag{2.9}$$

where $|s-1| < r$. In this work, s is the complex variable from nabla Laplace transform instead of Laplace transform. Since $\frac{\mathscr{N}_a\{{}_a^{\mathrm{G}}\nabla_k^{-\alpha} f(k)\}}{\mathscr{N}_a\{f(k)\}} = \frac{1}{s^\alpha}$, $\frac{1}{s^\alpha}$ is called fractional sum operator in frequency domain. Under zero initial conditions, one has $\frac{\mathscr{N}_a\{{}_a^{\mathrm{G}}\nabla_k^{\alpha} f(k)\}}{\mathscr{N}_a\{f(k)\}} = \frac{\mathscr{N}_a\{{}_a^{\mathrm{R}}\nabla_k^{\alpha} f(k)\}}{\mathscr{N}_a\{f(k)\}} = \frac{\mathscr{N}_a\{{}_a^{\mathrm{C}}\nabla_k^{\alpha} f(k)\}}{\mathscr{N}_a\{f(k)\}} = s^\alpha$. Thus, s^α is called fractional difference operator in frequency domain. Notably, since (2.3) is defined on $k \in \mathbb{N}_{a+1}$, $[{}_a^{\mathrm{R}}\nabla_k^{\alpha-i-1} f(k)]_{k=a} \equiv 0$ cannot be directly obtained. The similar problem has been discussed in [3, Chapter 3] and [1].

3. Main results

In this section, a series of frequency distributed models are derived for fractional order transfer functions.

Before giving the main result, a useful lemma is given here.

Lemma 3.1 *For any* $\alpha \in (0,1)$, $s \in \mathbb{C}\backslash\mathbb{R}_-$, *one has*

$$ \frac{1}{s^\alpha} = \int_0^{+\infty} \frac{\mu_\alpha(\omega)}{s+\omega} \mathrm{d}\omega, \tag{3.1}$$

where $\mu_\alpha(\omega) := \frac{\sin(\alpha\pi)}{\omega^\alpha \pi}$ *is the weighting function.*

Lemma 3.1 gives the description of fractional integral operator or fractional sum operator. It is implied in (3.6) of [20] and Lemma 2.7 of [44]. Besides, it is proved in Theorem 1 of [50] and Proposition 4.1 of [13]. According to $s^\alpha = s\frac{1}{s^{1-\alpha}}$ and $s^\alpha = \frac{1}{\frac{1}{s^\alpha}}$, the following corollary can be developed for fractional differential operator or fractional difference operator.

Corollary 3.2 *The following equalities hold*

$$s^\alpha = \int_0^{+\infty} \frac{s\mu_{1-\alpha}(\omega)}{s+\omega}\mathrm{d}\omega = \int_0^{+\infty} \frac{\mu_{1-\alpha}(\omega)}{1+\omega s^{-1}}\mathrm{d}\omega, \tag{3.2}$$

$$s^\alpha = \int_0^{+\infty} \frac{\mu_\alpha(\omega)}{s^{-1}+\omega}\mathrm{d}\omega = \int_0^{+\infty} \frac{s\mu_\alpha(\omega)}{1+s\omega}\mathrm{d}\omega, \tag{3.3}$$

where $\alpha \in (0,1)$, $s \in \mathbb{C}\backslash\mathbb{R}_-$, $\mu_\alpha(\varsigma) = \frac{\sin(\alpha\pi)}{\varsigma^\alpha \pi}$.

For a real positive constant ι, let $\omega := \varrho^\iota$. (3.1) can be expressed as $\frac{1}{s^\alpha} = \int_0^{+\infty} \frac{\mu_\alpha(\varrho^\iota) l \varrho^{\iota-1}}{s+\varrho^\iota}\mathrm{d}\varrho$, which is implied in Lemma 2 of [35]. Likewise, five variants of (3.1) can be derived as follows.

Theorem 3.3 *The following equalities hold*

$$\frac{1}{s^\alpha} = \int_{-\infty}^{+\infty} \frac{\omega\mu_\alpha(\omega^2)}{s+\omega^2}\mathrm{d}\omega, \alpha \in (0,1), s \in \mathbb{C}\backslash\mathbb{R}_-, \tag{3.4}$$

$$\sin\left(\frac{\alpha\pi}{2}\right)\frac{1}{s^{1+\alpha}} = \int_0^{+\infty} \frac{\mu_\alpha(\omega)}{s^2+\omega^2}\mathrm{d}\omega,\ \alpha \in (-1,1), s \in \mathbb{C}\backslash\mathbb{I}_p, \tag{3.5}$$

$$\cos\left(\frac{\alpha\pi}{2}\right)\frac{1}{s^{\alpha}} = \int_0^{+\infty} \frac{\omega\mu_\alpha(\omega)}{s^2+\omega^2}\mathrm{d}\omega,\ \alpha \in (0,2), s \in \mathbb{C}\backslash\mathbb{I}_p, \tag{3.6}$$

$$\sin\left(\frac{\alpha\pi}{2}\right)\frac{1}{s^{1-\alpha}} = \int_0^{+\infty} \frac{\omega\mu_{1-\alpha}(\omega)}{s^2+\omega^2}\mathrm{d}\omega,\ \alpha \in (-1,1), s \in \mathbb{C}\backslash\mathbb{I}_p, \tag{3.7}$$

$$\cos\left(\frac{\alpha\pi}{2}\right)\frac{1}{s^{2-\alpha}} = \int_0^{+\infty} \frac{\mu_{1-\alpha}(\omega)}{s^2+\omega^2}\mathrm{d}\omega,\ \alpha \in (0,2), s \in \mathbb{C}\backslash\mathbb{I}_p, \tag{3.8}$$

where $\mu_\gamma(\varsigma) := \frac{\sin(\gamma\pi)}{\varsigma^\gamma\pi}$, $\gamma \in (0,1)$, $\mathbb{I}_p := \{z = \mathrm{i}x, x \in \mathbb{R}, x \neq 0\}$, $\mathrm{i}^2 = -1$.

Proof Five equalities will be proved successively.

Part I▶ For (3.1), letting $\omega := \tau^2$, one has $\mathrm{d}\omega = 2\tau\mathrm{d}\tau$ and

$$\frac{1}{s^\alpha} = \int_0^{+\infty} \frac{2\tau\mu_\alpha(\tau^2)}{s+\tau^2}\mathrm{d}\tau. \tag{3.9}$$

Setting $\tau := -x$, it follows $\mathrm{d}\tau = -\mathrm{d}x$ and

$$\int_0^{+\infty} \frac{\tau\mu_\alpha(\tau^2)}{s+\tau^2}\mathrm{d}\tau = \int_0^{-\infty} \frac{-x\mu_\alpha(x^2)}{s+x^2}\mathrm{d}x = \int_{-\infty}^0 \frac{x\mu_\alpha(x^2)}{s+x^2}\mathrm{d}x. \tag{3.10}$$

By combining (3.9) and (3.10), the desired result in (3.4) establishes.

Part II▶ From the property of trigonometric functions and the definition of weighting function, one has

$$\begin{aligned}\mu_\alpha(\omega) &= \tfrac{\sin(\alpha\pi)}{\omega^\alpha\pi}\\ &= 2\sin(\tfrac{\alpha\pi}{2})\tfrac{\cos(\frac{\alpha\pi}{2})}{\omega^\alpha\pi}\\ &= 2\sin(\tfrac{\alpha\pi}{2})\tfrac{\sin(\frac{\pi}{2}+\frac{\alpha\pi}{2})}{\omega^\alpha\pi}\\ &= 2\omega\sin(\tfrac{\alpha\pi}{2})\mu_{\frac{1+\alpha}{2}}(\omega^2).\end{aligned} \tag{3.11}$$

By using (3.9) and (3.11), one obtains

$$\int_0^{+\infty}\tfrac{\mu_\alpha(\omega)}{s^2+\omega^2}\mathrm{d}\omega = \sin\left(\tfrac{\alpha\pi}{2}\right)\int_0^{+\infty}\tfrac{2\omega\mu_{\frac{1+\alpha}{2}}(\omega^2)}{s^2+\omega^2}\mathrm{d}\omega = \sin\left(\tfrac{\alpha\pi}{2}\right)\tfrac{1}{s^{1+\alpha}}. \tag{3.12}$$

Due to the applicable condition of (3.9), one has $\frac{1+\alpha}{2}\in(0,1)$, $s^2\in\mathbb{C}\backslash\mathbb{R}_-$. Thus, $\alpha\in(-1,1)$ and $s\in\mathbb{C}\backslash\mathbb{I}_p$ should be satisfied for (3.5).

Part III▶ Similar to (3.11), one has

$$\begin{aligned}\mu_\alpha(\omega) &= \tfrac{\sin(\alpha\pi)}{\omega^\alpha\pi}\\ &= 2\cos(\tfrac{\alpha\pi}{2})\tfrac{\sin(\frac{\alpha\pi}{2})}{\omega^\alpha\pi}\\ &= 2\cos(\tfrac{\alpha\pi}{2})\mu_{\frac{\alpha}{2}}(\omega^2),\end{aligned} \tag{3.13}$$

which implies

$$\int_0^{+\infty}\tfrac{\omega\mu_\alpha(\omega)}{s^2+\omega^2}\mathrm{d}\omega = \cos\left(\tfrac{\alpha\pi}{2}\right)\int_0^{+\infty}\tfrac{2\omega\mu_{\frac{\alpha}{2}}(\omega^2)}{s^2+\omega^2}\mathrm{d}\omega = \cos\left(\tfrac{\alpha\pi}{2}\right)\tfrac{1}{s^\alpha}. \tag{3.14}$$

Due to the applicable condition of (3.9), one has $\frac{\alpha}{2}\in(0,1)$, $s^2\in\mathbb{C}\backslash\mathbb{R}_-$. Therefore $\alpha\in(0,2)$, $s\in\mathbb{C}\backslash\mathbb{I}_p$ can be obtained for (3.6).

Part IV▶ Similar to (3.11), one has

$$\begin{aligned}\mu_{1-\alpha}(\omega) &= \tfrac{\sin(\pi-\alpha\pi)}{\omega^{1-\alpha}\pi}\\ &= 2\sin(\tfrac{\alpha\pi}{2})\tfrac{\cos(\frac{\alpha\pi}{2})}{\omega^{1-\alpha}\pi}\\ &= 2\sin(\tfrac{\alpha\pi}{2})\tfrac{\sin(\frac{\pi}{2}-\frac{\alpha\pi}{2})}{\omega^{1-\alpha}\pi}\\ &= 2\sin(\tfrac{\alpha\pi}{2})\mu_{\frac{1-\alpha}{2}}(\omega^2),\end{aligned} \tag{3.15}$$

$$\int_0^{+\infty}\tfrac{\omega\mu_{1-\alpha}(\omega)}{s^2+\omega^2}\mathrm{d}\omega = \sin\left(\tfrac{\alpha\pi}{2}\right)\int_0^{+\infty}\tfrac{2\omega\mu_{\frac{1-\alpha}{2}}(\omega^2)}{s^2+\omega^2}\mathrm{d}\omega = \sin\left(\tfrac{\alpha\pi}{2}\right)\tfrac{1}{s^{1-\alpha}}. \tag{3.16}$$

Since $\frac{1-\alpha}{2}\in(0,1)$, $s^2\in\mathbb{C}\backslash\mathbb{R}_-$ for (3.9), the applicable condition of (3.7) is $\alpha\in(-1,1)$, $s\in\mathbb{C}\backslash\mathbb{I}_p$.

Part V▶ Similar to (3.11), one has

$$\begin{aligned}\mu_{1-\alpha}(\omega) &= \tfrac{\sin(\pi-\alpha\pi)}{\omega^{1-\alpha}\pi}\\ &= \tfrac{\sin(\alpha\pi)}{\omega^{1-\alpha}\pi}\\ &= 2\sin(\tfrac{\alpha\pi}{2})\tfrac{\sin(\frac{\alpha\pi}{2})}{\omega^{1-\alpha}\pi}\\ &= 2\omega\cos(\tfrac{\alpha\pi}{2})\mu_{1-\frac{\alpha}{2}}(\omega^2),\end{aligned} \tag{3.17}$$

$$\int_0^{+\infty}\tfrac{\mu_{1-\alpha}(\omega)}{s^2+\omega^2}\mathrm{d}\omega = \cos\left(\tfrac{\alpha\pi}{2}\right)\int_0^{+\infty}\tfrac{2\omega\mu_{1-\frac{\alpha}{2}}(\omega^2)}{s^2+\omega^2}\mathrm{d}\omega = \cos\left(\tfrac{\alpha\pi}{2}\right)\tfrac{1}{s^{2-\alpha}}. \tag{3.18}$$

For (3.9), $1-\frac{\alpha}{2}\in(0,1)$ and $s^2\in\mathbb{C}\backslash\mathbb{R}_-$. Then the applicable condition of (3.8) is $\alpha\in(0,2)$, $s\in\mathbb{C}\backslash\mathbb{I}_p$.

All of these complete the proof of this theorem. □

Remark 3.4 *Actually, both (5) of [11] and (3.7) of [20] imply (3.4). In this work, we present this formula explicitly. (3.5) and (3.6) are implied in (59) and (65) of [40], (42) and (48) of [41]. In this work, a more accurate applicable condition is provided to replace the strict one* $\alpha\in(0,1)$, $s>0$. *Similarly, (3.7) and (3.8) emerged in Proposition 4.2 of [13], Proposition 4.10 of [12] and Proposition 10 of [14]. This work greatly reduces the conservativeness of the applicable condition. Besides, (3.5) and (3.7) can be equivalently transformed, if* α *is replaced by* $-\alpha$. *(3.6) and (3.8) can be converted into each other, if* α *is replaced by* $2-\alpha$.

In [47], the infinite dimensional state space realization for nabla fractional order systems is discussed and the following two lemmas are presented.

Lemma 3.5 *The nabla fractional sum operator* $G(s)=\frac{1}{s^\alpha}$, $\alpha\in(0,1)$, $s\in\mathbb{C}\backslash\mathbb{R}_-$, *has the state space realization as*

$$\left\{\begin{array}{r}\nabla z(\omega,k)=-\omega z(\omega,k)+v(k),\\ x(k)=\int_0^{+\infty}\mu_\alpha(\omega)z(\omega,k)\,\mathrm{d}\omega,\end{array}\right. \tag{3.19}$$

and its unit impulse response is

$$\mathscr{N}_a^{-1}\{G(s)\}=\frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\frac{1}{\omega^\alpha}\frac{1}{(1+\omega)^{k-a}}\mathrm{d}\omega, \tag{3.20}$$

where $\mu_\alpha(\omega)=\frac{\sin(\alpha\pi)}{\omega^\alpha\pi}$, $k\in\mathbb{N}_{a+1}$, $a\in\mathbb{R}$, $z(\omega,k)\in\mathbb{R}$, $z(\omega,a)=0$, $\omega\in[0,+\infty)$.

Note that the unit impulse response in Lemma 3.5 means the system output $x(k)$ generated by $G(s)$ with the system input $v(k)=\delta_d(k-a-1)$, where $\delta_d(\cdot)$ is discrete time unit impulsive function, i.e. $\delta_d(0)=1$, $\delta_d(z)=0$, $z\neq0$. Since $\mathscr{N}_a\{\delta_d(k-a-1)\}=1$, the unit impulse response satisfies $x(k)=\mathscr{N}_a^{-1}\{G(s)\mathscr{N}_a\{v(k)\}\}=\mathscr{N}_a^{-1}\{G(s)\}$.

The unit impulse response can be calculated via fractional sum as

$$\begin{array}{rl}x(k)&={}_a^{\mathrm{G}}\nabla_k^{-\alpha}\delta_d(k-a-1)\\ &=\sum_{i=0}^{k-a-1}(-1)^i\binom{-\alpha}{i}\delta_d(k-a-1-i)\\ &=(-1)^{k-a-1}\binom{-\alpha}{k-a-1}\\ &=(-1)^{k-a-1}\frac{\Gamma(-\alpha+1)}{\Gamma(k-a)\Gamma(-\alpha-k+a+2)}\\ &=\frac{\Gamma(\alpha+k-a-1)}{\Gamma(k-a)\Gamma(\alpha)}\\ &=\frac{(k-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)},\end{array} \tag{3.21}$$

where $p^{\overline{q}}:=\frac{\Gamma(p+q)}{\Gamma(p)}$ is the so-called rising function.

Letting $\tau:=\frac{\omega}{1+\omega}$, one has $\omega=\frac{\tau}{1-\tau}$. Using the definition and property of Beta function yields

$$\begin{array}{rl}\int_0^{+\infty}\frac{1}{\omega^\alpha}\frac{1}{(1+\omega)^{k-a}}\mathrm{d}\omega&=\int_0^1\tau^{-\alpha}(1-\tau)^{k-a+\alpha-2}\mathrm{d}\tau\\ &=\mathcal{B}(-\alpha+1,k-a+\alpha-1)\\ &=\frac{\Gamma(-\alpha+1)\Gamma(k-a+\alpha-1)}{\Gamma(k-a)}.\end{array} \tag{3.22}$$

Due to the reflection formula of Gamma function, one has $\frac{\sin(\alpha\pi)}{\pi}=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$. On this basis, the unit impulse response in (3.20) becomes

$$\mathscr{N}_a^{-1}\left\{\tfrac{1}{s^\alpha}\right\}=\tfrac{\Gamma(k-a+\alpha-1)}{\Gamma(\alpha)\Gamma(k-a)}=\tfrac{(k-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)},\tag{3.23}$$

which coincides with the fact $\mathscr{N}_a\left\{\frac{(k-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}=\frac{1}{s^\alpha}$ in [48].

Lemma 3.6 *The nabla transfer function* $G(s)=\frac{1}{(\tau s+1)^\alpha}$, $\alpha\in(0,1)$, $\tau\in(-1,0)\cup(0,+\infty)$, $\tau s+1\in\mathbb{C}\backslash\mathbb{R}_-$, *has the state space realization as*

$$\begin{cases}\tau\nabla z(\omega,k)=-(\omega+1)z(\omega,k)+v(k),\\ x(k)=\int_0^{+\infty}\mu_\alpha(\omega)z(\omega,k)\,\mathrm{d}\omega,\end{cases}\tag{3.24}$$

$$\begin{cases}\nabla z(\omega,k)=-\omega z(\omega,k)+(1+\lambda^{-1})^{a+1-k}v(k),\\ x(k)=\int_0^{+\infty}\mu_\alpha(\omega)\,\kappa(1+\tau^{-1})^{a+1-k}z(\omega,k)\,\mathrm{d}\omega,\end{cases}\tag{3.25}$$

and its unit impulse response is

$$\mathscr{N}_a^{-1}\{G(s)\}=\tfrac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\tfrac{1}{\omega^\alpha}\tfrac{\tau^{k-a-1}}{(\tau+\omega+1)^{k-a}}\mathrm{d}\omega,\tag{3.26}$$

where $\mu_\alpha(\omega)=\frac{\sin(\alpha\pi)}{\omega^\alpha\pi}$, $k\in\mathbb{N}_{a+1}$, $a\in\mathbb{R}$, $\kappa=(\tau+1)^{-\alpha}$, $\lambda=-1-\tau$, $z(\omega,k)\in\mathbb{R}$, $z(\omega,a)=0$, $\omega\in[0,+\infty)$.

Lemma 3.6 is the generalization of Lemma 3.5. $\frac{1}{(\tau s+1)^\alpha}$ is also called the Davidson–Cole transfer function [33, Section 2.2]. Letting $\varsigma:=\frac{\omega}{1+\omega+\tau}$, one has $\omega=\frac{\varsigma}{1-\varsigma}(1+\tau)$. Along this way, one has $\mathscr{N}_a^{-1}\left\{\frac{1}{(\tau s+1)^\alpha}\right\}=\frac{\tau^{k-a-1}}{(1+\tau)^{k-a+\alpha-1}}\frac{(k-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}$. Note that (3.25) is linear time invariant (LTI) and (3.26) is linear time varying (LTV). Inspired from Lemma 3.6, the following corollary can be obtained.

Corollary 3.7 *The tempered fractional sum operator* $G(s)=\left(\frac{s-\lambda}{1-\lambda}\right)^{-\alpha}$, $\alpha\in(0,1)$, $\lambda\in\mathbb{R}$, $\lambda\neq1$, $\frac{s-\lambda}{1-\lambda}\in\mathbb{C}\backslash\mathbb{R}_-$, *has the state space realization as*

$$\begin{cases}\nabla z(\omega,k)=-(\omega-\omega\lambda-\lambda)z(\omega,k)+(1-\lambda)v(k),\\ x(k)=\int_0^{+\infty}\mu_\alpha(\omega)z(\omega,k)\,\mathrm{d}\omega,\end{cases}\tag{3.27}$$

$$\begin{cases}\nabla z(\omega,k)=-\omega z(\omega,k)+(1-\lambda)^{k-a}v(k),\\ x(k)=\int_0^{+\infty}\mu_\alpha(\omega)\,(1-\lambda)^{a-k}z(\omega,k)\,\mathrm{d}\omega,\end{cases}\tag{3.28}$$

and its unit impulse response is

$$\mathscr{N}_a^{-1}\{G(s)\}=\tfrac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\tfrac{1}{\omega^\alpha}\tfrac{(1-\lambda)^{a+1-k}}{(1+\omega)^{k-a}}\mathrm{d}\omega=(1-\lambda)^{a+1-k}\tfrac{(k-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)},\tag{3.29}$$

where $\mu_\alpha(\omega)=\frac{\sin(\alpha\pi)}{\omega^\alpha\pi}$, $k\in\mathbb{N}_{a+1}$, $a\in\mathbb{R}$, $z(\omega,k)\in\mathbb{R}$, $z(\omega,a)=0$, $\omega\in[0,+\infty)$.

On this basis, a series of valuable conclusions can be obtained further. In order to fully demonstrate the process of change, the relevant proof adopts forward reasoning instead of reverse reasoning.

Theorem 3.8 *The nabla transfer function* $G(s) = \left(\frac{s}{\omega_l}+1\right)^{-\alpha}$, $\alpha \in (0,1)$, $\omega_l > 0$, *has the state space realization as*

$$\begin{cases} \nabla z(\omega,k) = -\omega z(\omega,k) + v(k), \\ x(k) = \frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{+\infty}\frac{\omega_l^\alpha}{(\omega-\omega_l)^\alpha} z(\omega,k)\,\mathrm{d}\omega, \end{cases} \tag{3.30}$$

$$\begin{cases} \nabla z(\omega,k) = -\omega z(\omega,k) + (1+\omega_l)^{k-a} v(k), \\ x(k) = \frac{\omega_l^\alpha}{(1+\omega_l)^\alpha}\int_{0}^{+\infty}\mu_\alpha(\omega)(1+\omega_l)^{a-k} z(\omega,k)\,\mathrm{d}\omega, \end{cases} \tag{3.31}$$

and its unit impulse response is

$$\mathscr{N}_a^{-1}\{G(s)\} = \frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{+\infty}\frac{\omega_l^\alpha}{(\omega-\omega_l)^\alpha}\frac{1}{(1+\omega)^{k-a}}\mathrm{d}\omega = \frac{\omega_l^\alpha}{(1+\omega_l)^{k-a+\alpha-1}}\frac{(k-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}, \tag{3.32}$$

where $\mu_\alpha(\omega) = \frac{\sin(\alpha\pi)}{\omega^\alpha\pi}$, $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$, $z(\omega,k) \in \mathbb{R}$, $z(\omega,a) = 0$, $\omega \in [0,+\infty)$.

Proof Since $\alpha \in (0,1)$, $s+\omega_l \in \mathbb{C}\backslash\mathbb{R}_-$, (3.1) bring in

$$\begin{aligned} G(s) &= \frac{\omega_l^\alpha}{(s+\omega_l)^\alpha} \\ &= \frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\frac{\omega_l^\alpha}{w^\alpha}\frac{1}{s+\omega_l+w}\mathrm{d}w \\ &= \frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{+\infty}\frac{\omega_l^\alpha}{(\omega-\omega_l)^\alpha}\frac{1}{s+\omega}\mathrm{d}\omega. \end{aligned} \tag{3.33}$$

By using the nabla Laplace transform, one obtains that system (3.30) is the state space realization of $G(s)$. Due to the fact that $\mathscr{N}_a^{-1}\{\frac{1}{s+\omega}\} = \frac{1}{(1+\omega)^{k-a}}$, (3.30) follows from (3.33) immediately. By using the similar derivation method, the detail expression of $\mathscr{N}_a^{-1}\{G(s)\}$ can be obtained. □

Similar to (3.33), one has $\left(\frac{s}{\omega_h}+1\right)^{\alpha-1} = \frac{\sin(\alpha\pi)}{\pi}\int_{\omega_h}^{+\infty}\frac{(\omega-\omega_h)^{\alpha-1}}{\omega_h^{\alpha-1}}\frac{1}{s+\omega}\mathrm{d}\omega$, where $\alpha \in (0,1)$, $s+\omega_h \in \mathbb{C}\backslash\mathbb{R}_-$, and the following corollary can be obtained.

Corollary 3.9 *The nabla transfer function* $G(s) = \left(\frac{s}{\omega_h}+1\right)^{\alpha-1}$, $\alpha \in (0,1)$, $\omega_h > 0$, *has the state space realization as*

$$\begin{cases} \nabla z(\omega,k) = -\omega z(\omega,k) + v(k), \\ x(k) = \frac{\sin(\alpha\pi)}{\pi}\int_{\omega_h}^{+\infty}\frac{(\omega-\omega_h)^{\alpha-1}}{\omega_h^{\alpha-1}} z(\omega,k)\,\mathrm{d}\omega, \end{cases} \tag{3.34}$$

$$\begin{cases} \nabla z(\omega,k) = -\omega z(\omega,k) + (1+\omega_h)^{k-a} v(k), \\ x(k) = \frac{\omega_h^\alpha}{(1+\omega_h)^\alpha}\int_{0}^{+\infty}\mu_{1-\alpha}(\omega)(1+\omega_h)^{a-k} z(\omega,k)\,\mathrm{d}\omega, \end{cases} \tag{3.35}$$

and its unit impulse response is

$$\mathscr{N}_a^{-1}\{G(s)\} = \frac{\sin(\alpha\pi)}{\pi}\int_{\omega_h}^{+\infty}\frac{(\omega-\omega_h)^{\alpha-1}}{\omega_h^{\alpha-1}}\frac{1}{(1+\omega)^{k-a}}\mathrm{d}\omega = \frac{\omega_h^\alpha}{(1+\omega_h)^{k-a-\alpha}}\frac{(k-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}, \tag{3.36}$$

where $\mu_\gamma(\varsigma) := \frac{\sin(\gamma\pi)}{\varsigma^\gamma\pi}$, $\gamma \in (0,1)$, $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$, $z(\omega,k) \in \mathbb{R}$, $z(\omega,a) = 0$, $\omega \in [0,+\infty)$.

Based on Theorem 3.8 and Corollary 3.9, the infinite dimensional description theory on limited frequency band $\omega \in [\omega_l, \omega_h]$ can be developed.

Theorem 3.10 *The nabla transfer function* $G(s)=\left(\frac{s}{\omega_h}+1\right)^{\alpha-1}\left(\frac{s}{\omega_l}+1\right)^{-\alpha}$, $\alpha\in(0,1)$, $\omega_h>\omega_l>0$, *has the state space realization as*

$$\left\{\begin{array}{rl}\nabla z(\omega,k)=&-\omega z(\omega,k)+v(k),\\ x(k)=&\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^{\alpha-1}(\omega-\omega_l)^\alpha}z(\omega,k)\,\mathrm{d}\omega,\end{array}\right. \tag{3.37}$$

and its unit impulse response is

$$\mathscr{N}_a^{-1}\left\{G(s)\right\}=\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^{\alpha-1}(\omega-\omega_l)^\alpha}\frac{1}{(1+\omega)^{k-a}}\mathrm{d}\omega, \tag{3.38}$$

where $k\in\mathbb{N}_{a+1}$, $a\in\mathbb{R}$, $z(\omega,k)\in\mathbb{R}$, $z(\omega,a)=0$, $\omega\in[0,+\infty)$.

Proof When $\alpha\in(0,1)$, $\omega_h>\omega_l>0$, $s\in\mathbb{C}\backslash\mathbb{R}_-$, (3.1) leads to

$$\begin{aligned}G(s)&=\frac{1}{(s+\omega_h)}\frac{\omega_l^\alpha}{\omega_h^{\alpha-1}}\frac{(s+\omega_h)^\alpha}{(s+\omega_l)^\alpha}\\&=\frac{1}{(s+\omega_h)}\frac{\omega_l^\alpha}{\omega_h^{\alpha-1}}\frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\frac{1}{\tau^\alpha}\frac{1}{\frac{s+\omega_l}{s+\omega_h}+\tau}\mathrm{d}\tau\\&=\frac{\omega_l^\alpha}{\omega_h^{\alpha-1}}\frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\frac{1}{\tau^\alpha}\frac{1}{s+\omega_l+\tau s+\tau\omega_h}\mathrm{d}\tau\\&=\frac{\omega_l^\alpha}{\omega_h^{\alpha-1}}\frac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty}\frac{1}{\tau^\alpha(1+\tau)}\frac{1}{s+\frac{\omega_l+\tau\omega_h}{1+w}}\mathrm{d}\tau.\end{aligned} \tag{3.39}$$

Setting $\omega:=\frac{\omega_l+\tau\omega_h}{1+\tau}$, one has $\tau=\frac{\omega-\omega_l}{\omega_h-\omega}$, $\mathrm{d}\tau=\frac{\omega_h-\omega_l}{(\omega_h-\omega)^2}\mathrm{d}\omega$ and then

$$\begin{aligned}G(s)&=\frac{\omega_l^\alpha}{\omega_h^{\alpha-1}}\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{(\omega_h-\omega)^\alpha}{(\omega-\omega_l)^\alpha}\frac{\omega_h-\omega}{\omega_h-\omega_l}\frac{1}{s+\omega}\frac{\omega_h-\omega_l}{(\omega_h-\omega)^2}\mathrm{d}\omega\\&=\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^{\alpha-1}(\omega-\omega_l)^\alpha}\frac{1}{s+\omega}\mathrm{d}\omega.\end{aligned} \tag{3.40}$$

By applying the nabla Laplace transform, it can be concluded that (3.37) is the state space realization of $G(s)$. Considering $\mathscr{N}_a^{-1}\left\{\frac{1}{s+\omega}\right\}=\frac{1}{(1+\omega)^{k-a}}$, (3.40) leads to the desired result (3.38). □

Theorem 3.11 *The nabla transfer function* $G(s)=\left(\frac{s}{\omega_h}+1\right)^{\alpha}\left(\frac{s}{\omega_l}+1\right)^{-\alpha}$, $\alpha\in(0,1)$, $\omega_h>\omega_l>0$, *has the state space realization as*

$$\left\{\begin{array}{rl}\nabla z(\omega,k)=&-\omega z(\omega,k)+v(k),\\ x(k)=&\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^{\alpha}}{\omega_h^{\alpha}(\omega-\omega_l)^\alpha}z(\omega,k)\,\mathrm{d}\omega+\frac{\omega_l^\alpha}{\omega_h^\alpha}v(k),\end{array}\right. \tag{3.41}$$

and its unit impulse response is

$$\mathscr{N}_a^{-1}\left\{G(s)\right\}=\frac{\omega_l^\alpha}{\omega_h^\alpha}\delta_d(k-a-1)+\frac{\sin(\alpha\pi)}{\pi}\frac{\omega_l^\alpha}{\omega_h^\alpha}\int_{\omega_l}^{\omega_h}\frac{(\omega_h-\omega)^{\alpha}}{(\omega-\omega_l)^\alpha}\frac{1}{(1+\omega)^{k-a}}\mathrm{d}\omega, \tag{3.42}$$

where $k\in\mathbb{N}_{a+1}$, $a\in\mathbb{R}$, $z(\omega,k)\in\mathbb{R}$, $z(\omega,a)=0$, $\omega\in[0,+\infty)$.

Proof From (3.40), it follows

$$\begin{aligned}G(s)&=\left(\frac{s}{\omega_h}+1\right)\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^{\alpha-1}(\omega-\omega_l)^\alpha}\frac{1}{s+\omega}\mathrm{d}\omega\\&=\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^{\alpha}(\omega-\omega_l)^\alpha}\frac{s+\omega_h}{s+\omega}\mathrm{d}\omega\\&=\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^{\alpha}(\omega-\omega_l)^\alpha}\mathrm{d}\omega+\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^{\alpha}}{\omega_h^{\alpha}(\omega-\omega_l)^\alpha}\frac{1}{s+\omega}\mathrm{d}\omega.\end{aligned} \tag{3.43}$$

By using the reflection formula of gamma function and the property of beta function, one has

$$\begin{aligned}\int_0^{+\infty} \tfrac{1}{w^\alpha}\tfrac{1}{1+w}\mathrm{d}w &= \int_1^0 t(1-t^{-1})^{-\alpha}(-t^{-2})\mathrm{d}t\\ &= \int_0^1 t^{\alpha-1}(1-t)^{-\alpha}\mathrm{d}t\\ &= \mathcal{B}\left(\alpha,1-\alpha\right)\\ &= \tfrac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)}\\ &= \tfrac{\pi}{\sin(\alpha\pi)}.\end{aligned} \tag{3.44}$$

Setting $\omega = \frac{\omega_l + w\omega_h}{1+w}$, one has $\mathrm{d}\omega = \frac{\omega_h-\omega_l}{(1+w)^2}\mathrm{d}w$ and then

$$\begin{aligned}\tfrac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h} \tfrac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^\alpha(\omega-\omega_l)^\alpha}\mathrm{d}\omega &= \tfrac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} \tfrac{\omega_l^\alpha}{\omega_h^\alpha}\tfrac{1}{w^\alpha}\tfrac{1}{\omega_h-\frac{\omega_l+w\omega_h}{1+w}}\tfrac{\omega_h-\omega_l}{(1+w)^2}\mathrm{d}w\\ &= \tfrac{\sin(\alpha\pi)}{\pi}\int_0^{+\infty} \tfrac{\omega_l^\alpha}{\omega_h^\alpha}\tfrac{1}{w^\alpha}\tfrac{1}{1+w}\mathrm{d}w\\ &= \tfrac{\omega_l^\alpha}{\omega_h^\alpha}.\end{aligned} \tag{3.45}$$

By substituting (3.45) into (3.43), one obtains

$$G\left(s\right) = \tfrac{\omega_l^\alpha}{\omega_h^\alpha} + \tfrac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h} \tfrac{\omega_l^\alpha(\omega_h-\omega)^{\alpha}}{\omega_h^\alpha(\omega-\omega_l)^\alpha}\tfrac{1}{s+\omega}\mathrm{d}\omega. \tag{3.46}$$

By applying the nabla Laplace transform, it can be concluded that (3.41) is the state space realization of $G\left(s\right)$. Due to $\mathscr{N}_a^{-1}\left\{1\right\} = \delta_d\left(k-a-1\right)$ and $\mathscr{N}_a^{-1}\left\{\frac{1}{s+\omega}\right\} = \frac{1}{(1+\omega)^{k-a}}$, (3.46) leads to the desired result (3.42). □

Theorem 3.12 *The nabla transfer function* $G\left(s\right) = \frac{1}{s}\left(\frac{s}{\omega_h}+1\right)^\alpha\left(\frac{s}{\omega_l}+1\right)^{-\alpha}$, $\alpha \in (0,1)$, $\omega_h > \omega_l > 0$, *has the state space realization as*

$$\left\{\begin{aligned}\nabla z\left(\omega,k\right) &= -\omega z\left(\omega,k\right) + v\left(k\right),\\ x\left(k\right) &= -\tfrac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h} \tfrac{\omega_l^\alpha(\omega_h-\omega)^{\alpha}}{\omega_h^\alpha(\omega-\omega_l)^\alpha\omega} z\left(\omega,k\right)\mathrm{d}\omega + \xi\tfrac{\omega_l^\alpha}{\omega_h^\alpha}v\left(k\right),\end{aligned}\right. \tag{3.47}$$

and its unit impulse response is

$$\mathscr{N}_a^{-1}\left\{G\left(s\right)\right\} = u_d\left(k-a-1\right) - \tfrac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h} \tfrac{\omega_l^\alpha(\omega_h-\omega)^{\alpha}}{\omega_h^\alpha(\omega-\omega_l)^\alpha\omega}\tfrac{1}{(1+\omega)^{k-a}}\mathrm{d}\omega, \tag{3.48}$$

where $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$, $z\left(\omega,k\right) \in \mathbb{R}$, $z\left(\omega,a\right) = 0$, $\omega \in [0,+\infty)$, $u_d\left(\cdot\right)$ *is discrete time unit step function.*

Proof From (3.40), it follows

$$\begin{aligned}G\left(s\right) &= \tfrac{1}{s}\left(\tfrac{s}{\omega_h}+1\right)\tfrac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h} \tfrac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^{\alpha-1}(\omega-\omega_l)^\alpha}\tfrac{1}{s+\omega}\mathrm{d}\omega\\ &= \tfrac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h} \tfrac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^\alpha(\omega-\omega_l)^\alpha}\tfrac{s+\omega_h}{s(s+\omega)}\mathrm{d}\omega.\end{aligned} \tag{3.49}$$

Since $\alpha \in (0,1)$, $\omega_h > \omega_l > 0$, it can be concluded that $\mathscr{N}_a^{-1}\left\{G\left(s\right)\right\}$ exists as $k \to +\infty$. Using the final value theorem of the nabla Laplace transform yields

$$\begin{aligned}\lim_{k\to+\infty}\mathscr{N}_a^{-1}\left\{G\left(s\right)\right\} &= \lim_{s\to 0} sG\left(s\right)\\ &= \lim_{s\to 0}\tfrac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h} \tfrac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^\alpha(\omega-\omega_l)^\alpha}\tfrac{s+\omega_h}{s+\omega}\mathrm{d}\omega\\ &= \tfrac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h} \tfrac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^\alpha(\omega-\omega_l)^\alpha}\tfrac{\omega_h}{\omega}\mathrm{d}\omega\\ &= \lim_{s\to 0}\left(\tfrac{s}{\omega_h}+1\right)^\alpha\left(\tfrac{s}{\omega_l}+1\right)^{-\alpha}\\ &= 1.\end{aligned} \tag{3.50}$$

In this connection, one has

$$\begin{array}{rl} G\left(s\right) = & \frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^{\alpha-1}}{\omega_h^\alpha(\omega-\omega_l)^\alpha}\left(\frac{\omega_h}{\omega}\frac{1}{s}-\frac{\omega_h-\omega}{\omega}\frac{1}{s+\omega}\right)\mathrm{d}\omega \\ = & \frac{1}{s}-\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{\omega_l^\alpha(\omega_h-\omega)^\alpha}{\omega_h^\alpha(\omega-\omega_l)^\alpha\omega}\frac{1}{s+\omega}\mathrm{d}\omega. \end{array} \tag{3.51}$$

With the similar method, it can be concluded that (3.47) is the state space realization of $G\left(s\right)$ and (3.48) is the unit impulse response of $G\left(s\right)$. □

Remark 3.13 *Theorems 3.8–3.12 provide limited frequency band diffusive representation for several nabla transfer functions. This work extends the result in [32, Section IV], [30, Section III], [29, Section 3], [27, Table A1.1] and [28, Section 3] from the continuous time case to the discrete time case. Due to (3.45), one has* $\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{(\omega_h-\omega)^{\alpha-1}}{(\omega-\omega_l)^\alpha}\mathrm{d}\omega = 1$. *According to (3.50), one has* $\frac{\sin(\alpha\pi)}{\pi}\int_{\omega_l}^{\omega_h}\frac{(\omega_h-\omega)^{\alpha-1}}{(\omega-\omega_l)^\alpha}\frac{\omega_h}{\omega}\mathrm{d}\omega = \frac{\omega_h^\alpha}{\omega_l^\alpha}$. *The two formulas might be potentially useful.*

Theorem 3.14 *If the unit impulse response of transfer function $G(s)$ can be expressed as*

$$\mathscr{N}_a^{-1}\{G(s)\} = \int_0^{+\infty}\lambda\left(\omega\right)\left(1+\omega\right)^{a-k}\mathrm{d}\omega, \tag{3.52}$$

then the state space realization of $G(s)$ can be written as

$$\left\{\begin{array}{rl} \nabla z\left(\omega,k\right) = & -\omega z\left(\omega,k\right)+v\left(k\right), \\ x\left(k\right) = & \int_0^{+\infty}\lambda\left(\omega\right)z\left(\omega,k\right)\mathrm{d}\omega, \end{array}\right. \tag{3.53}$$

where $k\in\mathbb{N}_{a+1}$, $a\in\mathbb{R}$, $z\left(\omega,k\right)\in\mathbb{R}$, $z\left(\omega,a\right)=0$, $\omega\in[0,+\infty)$.

Proof Under the given conditions, one has $x\left(k\right) = \mathscr{N}_a^{-1}\{G(s)\}*v\left(k\right)$. Defining $Z\left(\omega,s\right) := \mathscr{N}_a\left\{z\left(\omega,t\right)\right\}$, $V\left(s\right) := \mathscr{N}_a\left\{v\left(t\right)\right\}$, $X\left(s\right) := \mathscr{N}_a\left\{x\left(t\right)\right\}$, it follows that $X\left(s\right) = G\left(s\right)V\left(s\right)$. Thanks to nabla Laplace transform, (3.53) leads to

$$\begin{array}{rl} X\left(s\right) = & \int_0^{+\infty}\lambda_\alpha\left(\omega\right)Z\left(\omega,s\right)\mathrm{d}\omega \\ = & \int_0^{+\infty}\frac{\lambda_\alpha(\omega)}{s+\omega}\mathrm{d}\omega V\left(s\right). \end{array} \tag{3.54}$$

From the definition of nabla Laplace transform, (3.52) leads to

$$\begin{array}{rl} G\left(s\right) = & \sum_{k=1}^{+\infty}\int_0^{+\infty}\lambda\left(\omega\right)\left(1+\omega\right)^{-k}\mathrm{d}\omega(1-s)^{k-1} \\ = & \int_0^{+\infty}\lambda\left(\omega\right)\sum_{k=1}^{+\infty}\left(1-s\right)^{k-1}\left(1+\omega\right)^{-k}\mathrm{d}\omega \\ = & \int_0^{+\infty}\frac{\lambda(\omega)}{s+\omega}\mathrm{d}\omega, \end{array} \tag{3.55}$$

where $|1-s|<|1+\omega|$. Combining (3.54) and (3.55), the equivalence of $G(s)$ and (3.53) can be obtained. □

Theorem 3.14 provides a more general idea to derive the diffusive representation of a class of nabla transfer functions. For the given transfer function $G\left(s\right)$, by using the definition of inverse nabla Laplace transform yields

$$\begin{array}{rl} \mathscr{N}_a^{-1}\left\{G\left(s\right)\right\} = & \frac{1}{2\pi\mathrm{i}}\oint_c G\left(s\right)\left(1-s\right)^{a-k}\mathrm{d}s \\ = & \int_0^{+\infty}\lambda\left(\omega\right)\left(1+\omega\right)^{a-k}\mathrm{d}\omega. \end{array} \tag{3.56}$$

The pioneering works in [10, 25] discuss how to get the desired weighting function $\lambda(\omega)$ in continuous time case. However, there are many difficulties and challenges in the discrete time case. One of our future research topics is to construct suitable $\lambda(\omega)$ satisfying (3.56). From this, (3.52) and (3.53) establish naturally.

In general, it is difficult or impossible to use the developed diffusive representation for the calculation of the system response. If we discretize the continuous time frequency band $[0,+\infty)$ or $[\omega_l,\omega_h]$ with $\omega_0,\omega_1,\cdots,\omega_n$ under special parameter configuration, then the resulting model can approximate the considered infinite-dimensional diffusive representation to arbitrary accuracy.

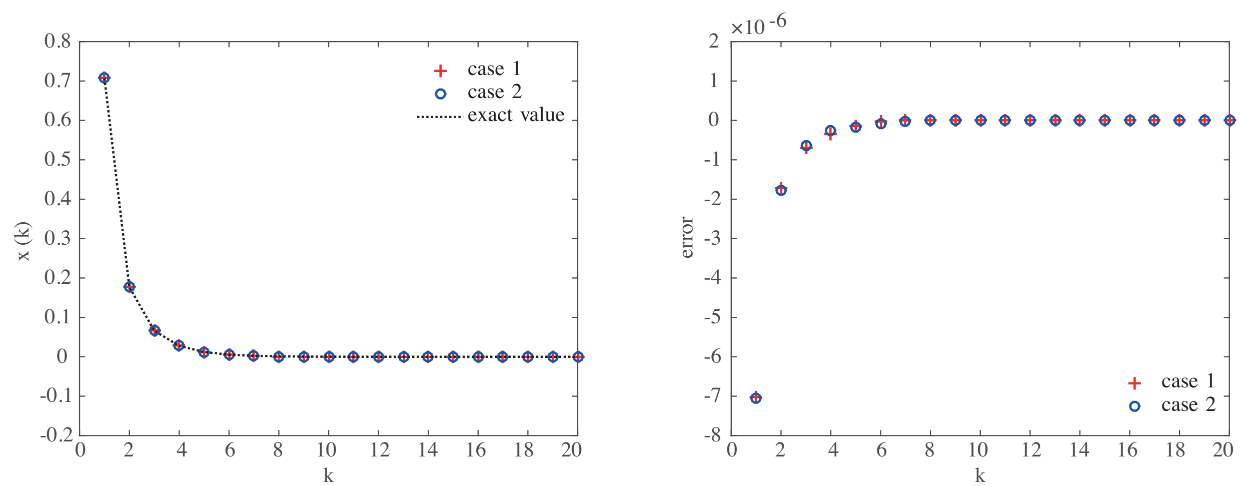
4. Simulation study

In this section, two detailed numerical examples are provided to verify the effectiveness of the presented results. The adopted numerical calculation algorithm is modified from the approximation method in [52].

Example 4.1 *Consider the time domain response of* $G(s)=\left(\frac{s}{\omega_l}+1\right)^{-\alpha}$ *with the following two approximation models.*

$$\begin{cases} \text{case 1: the approximation model from (3.30);} \\ \text{case 2: the approximation model from (3.31).} \end{cases}$$

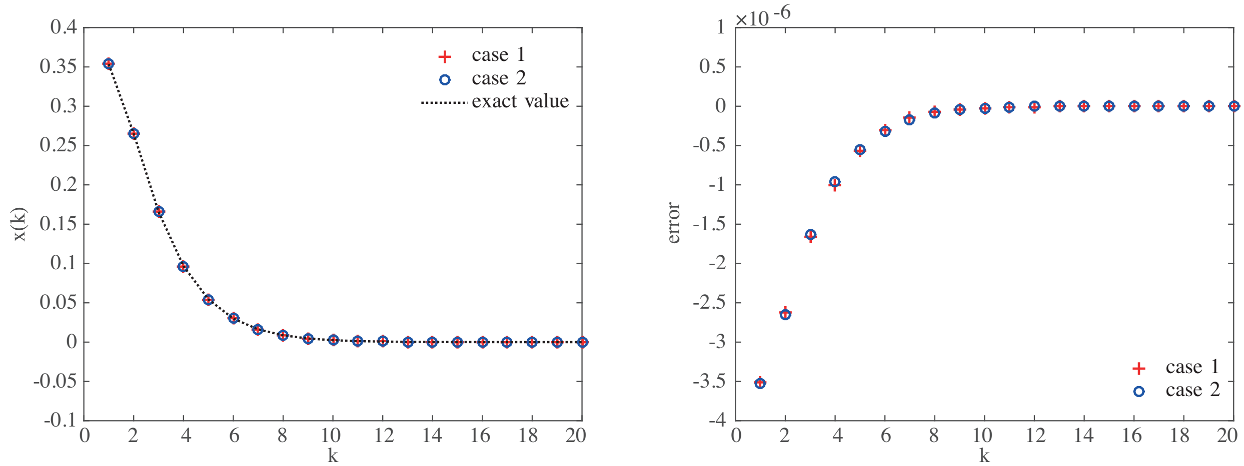
Firstly, setting the input $v(k)=\delta_d(k-a-1)$, *the output satisfies* $x(k)=\frac{\omega_l^\alpha}{(1+\omega_l)^{k-a+\alpha-1}}\frac{(k-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}$. *The simulation parameters are configured as* $\alpha=0.5$, $\omega_l=1$, $n=20$, $a=0$, *and then the obtained results are shown as Figures 1a and 1b. It can be checked that both the LTI model and the LTV model could generate high precision approximation results.*



(a) Time evolution of $x(k)$. (b) Approximation error of $x(k)$.

Figure 1. The approximation of system response with $v(k)=\delta_d(k-a-1)$.

Secondly, setting the input $v(k)=(1+\omega_l)^{a-k}$, *the output satisfies* $x(k)=\frac{\omega_l^\alpha}{(1+\omega_l)^{k-a+\alpha}}\frac{(k-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)}$. *With the same parameters configuration, the simulation results are displayed in Figures 2a and 2b. Also, the precision with* 10^{-6} *level can be achieved.*



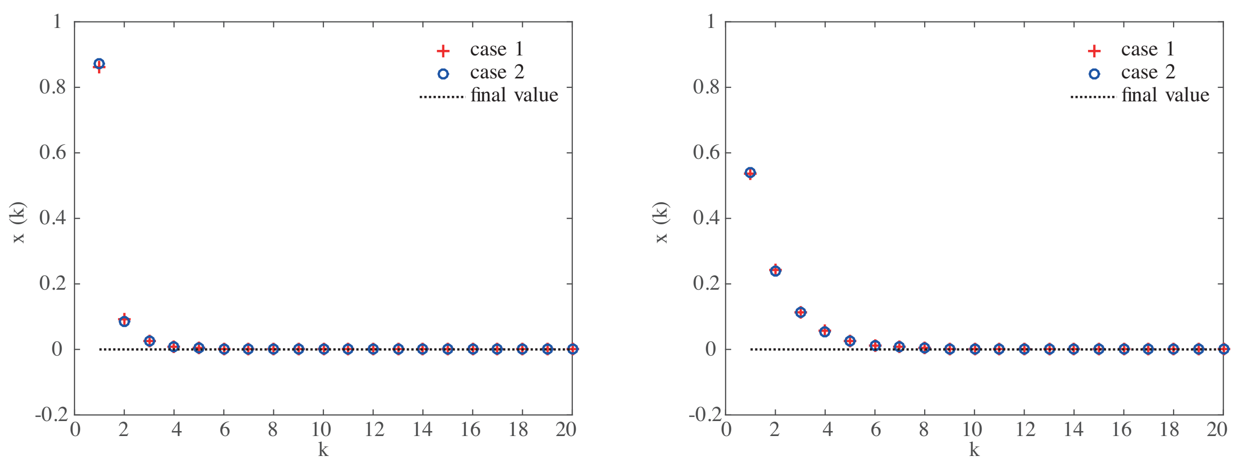
(a) Time evolution of x(k).   (b) Approximation error of x(k).

Figure 2. The approximation of system response with $v(k) = (1+\omega_l)^{a-k}$.

Example 4.2 *Consider the time domain response of $G(s)$ with the following two approximation models.*

$$\begin{cases} \textit{case 1: the approximation model from Theorem 3.10;} \\ \textit{case 2: the approximation model from Theorem 3.11.} \end{cases}$$

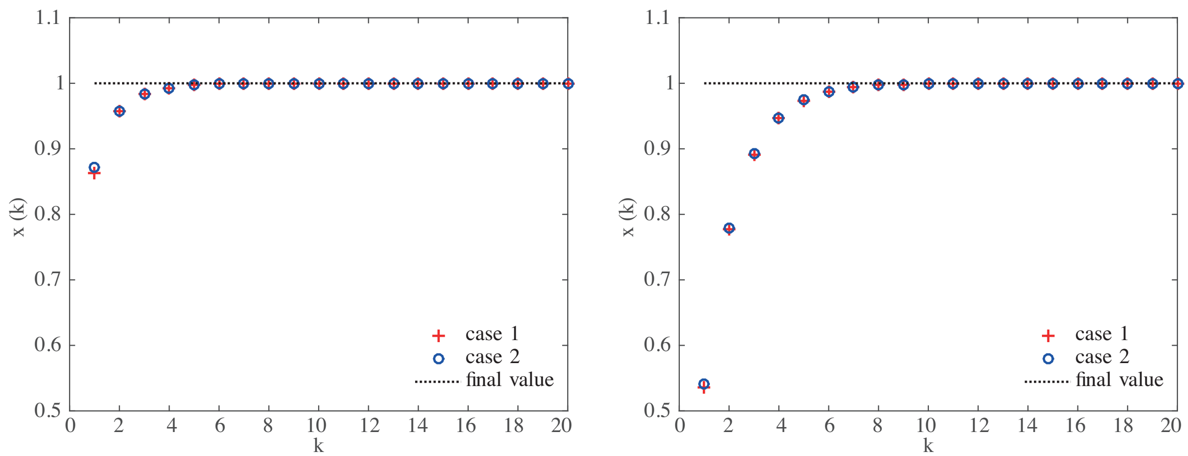
Firstly, setting the input $v(k) = \delta_d(k-a-1)$, the final value of the output satisfies $\lim_{k\to+\infty} x(k) = 0$. The simulation parameters are configured as $\alpha = 0.2$, $\omega_l = 1$, $\omega_h = 100$, $n = 20$, $a = 0$, and then the obtained results are shown as Figures 3a and 3b, which clearly demonstrate the effectiveness.



(a) The case of α = 0.2.   (b) The case of α = 0.9.

Figure 3. The approximation of system response with $v(k) = \delta_d(k-a-1)$.

Secondly, setting the input $v(k) = u_d(k-a-1)$, the final value of the output satisfies $\lim_{k\to+\infty} x(k) = 1$. The results in Figures 4a and 4b also evaluate the effectiveness of the developed diffusive representation.



(a) The case of α = 0.2.

(b) The case of α = 0.9.

Figure 4. The approximation of system response with $v(k) = u_d(k - a - 1)$.

5. Conclusion

In this paper, diffusive representation of a class of nabla transfer functions has been investigated originally. Based on the basic equality on fractional sum operator, several useful equalities are further developed. From this, the equivalent expression for fractional difference operator can be built in two ways. Some theorems are deduced with rigorous mathematical proof, from which equivalent diffusive representation, unit impulse response and the numerical implementation scheme for nabla transfer functions can be obtained conveniently. It is believed that this work reveals the relationship between time domain and frequency domain of nabla fractional order systems and enriches the knowledge of nabla fractional order system theory. The following topics may be interesting and promising in future work.

(a) Derive the equivalent diffusive representation for general nabla transfer function $G(s)$.

(b) Develop the stability analysis and controller design method with the help of the diffusive representation of nabla fractional order systems.

(c) Design efficient and high precision numerical calculation algorithm based on the diffusive representation of nabla fractional order systems.

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