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The complex error functions and various extensive results together with implications pertaining to certain special functions

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Abstract: The error functions play very important roles in science and technology. In this investigation, the error functions in the complex plane will be introduced, then comprehensive results together with several nonlinear implications in relation to the related complex functions will be indicated, and some possible special results of them will be next presented. Furthermore, various interesting or important suggestions will be also made for the scientific researchers who are interested in this topic.

Key words: Open unit disc, the complex error functions, equations and inequalities in the complex plane, complex series, special functions with complex variable, Hermite polynomials

1. Introduction, definition and preliminaries

In the literature, sometimes we deal with the well-known error functions whose independent variables can be both real and complex variables. These are known as special functions, which are defined by integral in the complex plane or represented by complex series, or certain forms defined or represented by those functions occur in many branches of science and technology. Particularly, for a long time now, more comprehensive properties or computational methods of these functions have been studied or used extensively in many scientific fields. For these, as a useful survey, one may check out [1, 18, 23, 30], and the earlier works in [9, 26, 27, 29] and also see the other works, as for example, in the references. Especially, in many applications, those functions must be evaluated a large number of times. Therefore, it can be important or interesting research for both methods and investigations. For certain studies, as an example, one may refer to the works given in [9, 11, 12, 15, 16, 19] for probability and statistics, in [15] for data analysis, in [10] for astronomy, in [5, 27, 31] for physics, in [32, 34, 37, 38] for a fundamental role in asymptotic expansions, in [6] for exponential asymptotics and see also certain different results given in [2–5, 7, 8, 14, 18, 22, 24, 32–37] in the references.

Under favour of different methods, for certain possible theoretical *or* practical results, which will be determined by error functions in the certain domains of the complex plane, it can be revealed various results, which will be unusual (or useful) for certain branches of science and technology. So, the proof technique to be

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used in this investigation plays a significant role in the complex error functions in the sense of its applicable or theoretical results in the related fields. For those, there is a need to introduce or reminder a few basic definitions and the well-known identities specified by the error functions in the complex plane, which are given below.

First of all, *here and throughout this present paper*, the following notations should noted:

$$\mathbb{C} \quad , \quad \mathbb{R} \quad , \quad \mathbb{N} \quad \text{and} \quad \mathbb{U}$$

the set of complex numbers, the set of real numbers, the set of natural numbers and the open unit disc in the complex plane, i.e.

$$\mathbb{U} = \big\{ x : x \in \mathbb{C} \text{ and } |x| < 1 \big\}.$$

The *error* function with the complex variable (or parameter) x, denoted by the notation $\mathrm{erf}(x)$, is defined by

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\big(-\xi^2 \big)\, d\xi \qquad (1.1),$$

for an arbitrary integration path in the certain domains of the complex plane $\mathbb{C}$. In terms of simplicity, let us denote it as:

$$\Xi(x) := \mathrm{erf}\,(x) \quad (x \in \mathbb{C}).$$

In the light of the Taylor–Mclaurin series expansion of the function $f(\xi) = \exp(\xi),$, which is as follows in the well-known form:

$$\begin{aligned} \exp(\xi) &= \frac{1}{0!} + \frac{\xi}{1!} + \frac{\xi^2}{2!} + \cdots + \frac{\xi^k}{k!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \ , \end{aligned}$$

the following series:

$$\begin{aligned} \exp\big(-\xi^2 \big) &= 1 - \frac{\xi^2}{1!} + \frac{\xi^4}{2!} - \cdots + (-1)^k \frac{\xi^{2k}}{k!} + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{\xi^{2k}}{k!} \ , \end{aligned} \qquad (1.2)$$

can be easily obtained when putting $\xi := -\xi^2$ there.

It is easily found that the above series is uniform convergent on any open connected set of $\mathbb{R}$. So, by taking the integration of the function in the series form, given by (1.2), from the point 0 to the point x by parts, the second definition of the complex error function $\Xi(x)$, identified by the definition (1.1), is also stated by the series expansion in the form:

$$\begin{aligned} \Xi\,(x) &= \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots + \frac{(-1)^k x^{2k+1}}{k!(2k+1)} + \cdots \right) \\ &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(2k+1)} \ , \end{aligned} \qquad (1.3)$$

where $x \in \mathbb{C}$.

In view of the (real)complex error function, defined by (1.1), the *complementary error* function in the complex plane or the *complementary complex error* function, denoted by $\mathrm{erfc}(x)$, is also defined by

$$\mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp\big(-\xi^2\big)\, d\xi \ , \tag{1.4}$$

where $x \in \mathbb{C}$. In terms of simplicity, let us also denote it as

$$\Xi_c(x) := \mathrm{erfc}\,(x) \quad (x \in \mathbb{C}).$$

Since the well-known property between the (complex) error function and the complementary (complex) error function, which is

$$\Xi(x) = 1 - \Xi_c(x) \quad (x \in \mathbb{C}),$$

the complementary complex error function (*or* the series expansion of the complementary (complex) error function) can also be expressed as

$$\begin{aligned}\Xi_c\,(x) &= 1 - \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots + \frac{(-1)^k x^{2k+1}}{k!(2k+1)} + \cdots \right) \\ &= 1 - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(2k+1)} \ , \end{aligned} \tag{1.5}$$

where $k \in \mathbb{C}$.

As various implications and comments of the related error functions with the complex variable x, some of those specified by $\Xi\,(x)$ and/or $\Xi_c\,(x)$ may represent certain important (differential) inequalities *or* equations and also have certain properties relating to their real and imaginary parts, especially, between 0 and 1 for the values of x in the first quadrant of the complex plane $\mathbb{C}$. Those possible properties may well have been one of the motivations for considering the possible functions determining by the error functions, as the basic form of the error function for complex arguments.

So, for certain novel and nonlinear applications of complex error functions and also their implications, namely, for our main results, there is a need to indicate the well-known assertion, which is given in [28] and was earlier proven by the help of the famous lemma in [20]. In addition, extra examples or researches, for some of its applications as well as some of its relevant assertions, one may refer to the works given by [16, 17, 26, 27].

Lemma 1.1 *Let*

$$p(x) = 1 + a_k x^k + a_{k+1} x^{k+1} + a_{k+2} x^{k+2} + \cdots \ \big(a_k \neq 0\big) \tag{1.6}$$

be an analytic function in $\mathbb{U}$ *and also suppose that there exists a point* x_0 *belonging to* $\mathbb{U}$ *such that*

$$\Re e\Big(p(x)\Big) > 0 \quad when \ \ |x| \leq |x_0| \tag{1.7}$$

and

$$p(x_0) \neq 0 \quad \ and \quad \ \Re e\Big(p(x_0)\Big) = 0\,. \tag{1.8}$$

Then

$$2z_0p'\big(x_0\big) \leq -\ k\Big(1+\big|p(k_0)\big|^2\Big) \quad \big(k\in\mathbb{N}\big)\,. \tag{1.9}$$

2. Main Results and Implications

Due to too many complex processes in the basic results of this chapter, it is useful to introduce some special sets. So, for convenience, we also identify a number of the following sets consisting of natural numbers, which are described by

$$\mathbf{A}=\{1,5,9,13,\cdots\} \quad , \quad \mathbf{B}=\{2,6,10,14,\cdots\}$$

$$\mathbf{C}=\{3,7,11,15,\cdots\} \quad \text{and} \quad \mathbf{D}=\{4,8,12,16,\cdots\}.$$

Moreover, let $\mathbf{F}:=\mathbf{A}\cup\mathbf{C}$ and also let $\mathbf{O}$ and $\mathbf{E}$ denote the sets of odd and even positive integers, respectively.

We now Start by fixing and then by proving the results consisting of various (differential) inequalities and/or equations specified by the complex error functions, defined as (1.1)-(1.5) and certain relations *or* special results relating to those are presented by the following theorems.

Theorem 2.1 *Let* $\tau\in\mathbb{N},\ \kappa\in\mathbb{N}$ *and* $x\in\mathbb{U}$ *and also let* $\Phi(x)$ *be a complex function that is in the form:*

$$\Phi(x):=\left[\frac{d^2}{dz^2}\Big(\Xi(x)\Big)\right]^{\tau}+\left[\frac{d}{dx}\Big(\Xi(x)\Big)\right]^{\kappa}, \tag{2.1}$$

where $\Xi(x)$ *is defined by (1.1) (or (1.3)) and also the values of the complex powers above are taken their principal values. For the mentioned function* $\Phi(x)$, *if any one of the cases of the proposition given by the forms*

$$\Re e\Big(\Phi(x)\Big)\begin{cases} > -\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if}\quad \kappa\in\boldsymbol{F},\tau\in\boldsymbol{O}\\ < \quad \left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if}\quad \kappa\in\boldsymbol{F},\tau\in\boldsymbol{E}\\ > -\left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}-\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if}\quad \kappa\in\boldsymbol{B},\tau\in\boldsymbol{O}\\ < -\left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if}\quad \kappa\in\boldsymbol{B},\tau\in\boldsymbol{E}\\ > \quad \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}-\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if}\quad \kappa\in\boldsymbol{D},\tau\in\boldsymbol{O}\\ < \quad \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if}\quad \kappa\in\boldsymbol{D},\tau\in\boldsymbol{E}\end{cases}, \tag{2.2}$$

is provided, then

$$\Re e\left(\frac{d}{dx}\Big(\Xi(x)\Big)\right)>0 \tag{2.3}$$

is provided for all $\alpha\in\mathbb{R}-\{0\}$ *and* $x\in\mathbb{U}$.

Proof By considering the series expansion of the function $\Xi(x)$, given (1.3), define a function $p(x)$ as

$$p(x) = \frac{\sqrt{\pi}}{2} \frac{d}{dx}\Big(\Xi(x)\Big) \tag{2.4}$$

to prove that $\Re e\big(p(x)\big) > 0$ *for all* $x \in \mathbb{U}$.

After a simple observation, clearly, the function $p(x)$ above has the series form given by the expansion as in (1.5) and also is an analytic function in the disc $\mathbb{U}$, which satisfies the conditions of Lemma 1.1, *i.e.,* $p(0) = 1$ and $n \geq 2$.

Differentiating the both sides of (2.4) with respect to the complex variable x, we then obtain

$$\frac{d}{dx}\left(\frac{d}{dx}\Big(\Xi(x)\Big)\right) = \frac{d^2}{dx^2}\Big(\Xi(x)\Big),$$

or, equivalently,

$$x\frac{d^2}{dx^2}\Big(\Xi(x)\Big) = \frac{2}{\sqrt{\pi}} x \frac{d}{dx}\Big(p(x)\Big) \quad \big(x \in \mathbb{U}\big). \tag{2.5}$$

By combining of (2.4) and (2.5), it is also arrived at the following equality:

$$\Phi(x) = \Big(\frac{2}{\sqrt{\pi}}\Big)^{\kappa} \ \big[p(x)\big]^{\kappa} \ + \Big(\frac{2}{\sqrt{\pi}}\Big)^{\tau} \ \Big[(x\frac{d}{dx}\Big(p(x)\Big)\Big]^{\tau}, \tag{2.6}$$

where $\kappa \in \mathbb{N}$ and $x \in \mathbb{U}$ and the function $\Phi(x)$ is defined as in (2.1).

For the desired proof, initially, we assume that there exists a x_0 in the punctured unit disc $\mathbb{U} - \{0\}$ satisfying the hypothesis of Lemma 1.1 in (1.8), which is

$$\Re e\Big(p(x_0)\Big) = 0 \quad \big(p(x_0) \neq 0\big)\,. \tag{2.7}$$

Then, at the same time, in defiance of the mentioned lemma, the followings

$$p(x)\big|_{x=x_0} = p(x_0) = i\alpha\,, \tag{2.8}$$

$$x\frac{d}{dx}\Big(p(x)\Big)\Big|_{x=x_0} = xp'(x_0) = \beta \tag{2.9}$$

and

$$\beta \leq -\frac{n}{2}\Big(1 + \big|p(x_0)\big|^2\Big) = -\frac{n}{2}\Big(1 + \alpha^2\Big) \tag{2.10}$$

can easily be constituted, *where*

$$n \geq 2 \ \ ; \ \ n \in \mathbb{N} \ \ ; \ \ \beta \in \mathbb{R} \quad \text{and} \quad \alpha \in \mathbb{R} - \{0\}. \tag{2.11}$$

Thereby, under the conditions restricted as in (2.11) and also in the light of information presented by (2.4)-(2.10), the following-extensive statement

$$\begin{aligned} \Phi(x)\big|_{x=x_0} &= \left(\tfrac{2}{\sqrt{\pi}}\right)^{\kappa}\left[p(x)\big|_{x:=x_0}\right]^{\kappa}+\left(\tfrac{2}{\sqrt{\pi}}\right)^{\tau}\left[x\tfrac{d}{dx}\Big(p(x)\Big)\Big|_{x:=x_0}\right]^{\tau} \\ &= \left(\tfrac{2}{\sqrt{\pi}}\right)^{\kappa}(i\alpha)^{\kappa}+\left(\tfrac{2}{\sqrt{\pi}}\right)^{\tau}\beta^{\tau} \\ &= \left(\tfrac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}i^{\kappa}+\left(\tfrac{2}{\sqrt{\pi}}\right)^{\tau}\beta^{\tau} \\ &= \begin{cases} i\left(\tfrac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\tfrac{2}{\sqrt{\pi}}\right)^{\tau}\beta^{\tau} & \text{if} \quad \kappa\in\mathbf{A} \\ -\left(\tfrac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\tfrac{2}{\sqrt{\pi}}\right)^{\tau}\beta^{\tau} & \text{if} \quad \kappa\in\mathbf{B} \\ -i\left(\tfrac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\tfrac{2}{\sqrt{\pi}}\right)^{\tau}\beta^{\tau} & \text{if} \quad \kappa\in\mathbf{C} \\ \left(\tfrac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\tfrac{2}{\sqrt{\pi}}\right)^{\tau}\beta^{\tau} & \text{if} \quad \kappa\in\mathbf{D} \end{cases} \end{aligned} \tag{2.12}$$

is then obtained. By considering the assertions given by (1.7) and also taking real part of the both sides of the equality given by (2.12), we get that the following inequalities:

$$\Re e\Big(\Phi(x)\Big)\begin{cases} \leq\ -\left(\tfrac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \text{if} \quad \kappa\in\mathbf{F}\,,\tau\in\mathbf{O} \\ \geq\ \left(\tfrac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \text{if} \quad \kappa\in\mathbf{F}\,,\tau\in\mathbf{E} \\ \leq\ -\left(\tfrac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}-\left(\tfrac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \text{if} \quad \kappa\in\mathbf{B}\,,\tau\in\mathbf{O} \\ \geq\ -\left(\tfrac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\tfrac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \text{if} \quad \kappa\in\mathbf{B}\,,\tau\in\mathbf{E} \\ \leq\ \left(\tfrac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}-\left(\tfrac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \text{if} \quad \kappa\in\mathbf{D}\,,\tau\in\mathbf{O} \\ \geq\ \left(\tfrac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\tfrac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \text{if} \quad \kappa\in\mathbf{D}\,,\tau\in\mathbf{E} \end{cases}. \tag{2.13}$$

But, the cases of the inequality given by (2.13), respectively, contradictions with the cases of the hypothesis given in (2.1). This says that there is no a point $x_0\in\mathbb{U}$ satisfying the mentioned condition in (2.7). Thus, it has to be in the form:

$$\Re e\Big(p(x)\Big)>0 \quad \big(\forall x\in\mathbb{U}\big)\,.$$

Hereby, the definition constituted by (2.4) immediately yields the provision of Theorem 2.1 given in (2.3). Therefore, this completes the proof of Theorem 2.1. □

Through the instrument of Theorem 2.1 together with its proof *or* in consideration of the well-known identity:

$$\Re e\big(\mathrm{w}\big)=\Im m\big(i\mathrm{w}\big)$$

for all $\mathrm{w}\in\mathbb{C}$, the following-extensive proposition can be easily represented.

Proposition 2.2 *Let $\tau\in\mathbb{N}$, $\kappa\in\mathbb{N}$ and $x\in\mathbb{U}$ and also let the function $\Phi(x)$ be in the form given by (2.1). If any one of the cases of the following inequality:*

$$\Im m\Big(i\,\Phi(x)\Big)\begin{cases} > -\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if} \quad \kappa\in\boldsymbol{F}, \tau\in\boldsymbol{O} \\ < \quad \left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if} \quad \kappa\in\boldsymbol{F}, \tau\in\boldsymbol{E} \\ > -\left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}-\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if} \quad \kappa\in\boldsymbol{B}, \tau\in\boldsymbol{O} \\ < -\left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if} \quad \kappa\in\boldsymbol{B}, \tau\in\boldsymbol{E} \\ > \quad \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}-\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if} \quad \kappa\in\boldsymbol{D}, \tau\in\boldsymbol{O} \\ < \quad \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{\kappa}+\left(\frac{2[1+\alpha^2]}{\sqrt{\pi}}\right)^{\tau} & \textit{if} \quad \kappa\in\boldsymbol{D}, \tau\in\boldsymbol{E} \end{cases} \tag{2.14}$$

is satisfied, then the inequality:

$$\Re e\left(\frac{d}{dx}\Big(\Xi(x)\Big)\right) > 0$$

holds for all $\alpha\in\mathbb{R}-\{0\}$ *and for any* $x\in\mathbb{U}$.

As we know, nearly all of the well-known properties relating to complex variables *or* complex functions are always available and also valid for complex error functions defined as in (1.1) and (1.3). At the same time, there are also many important special functions defined by those *or* related to the complex error functions given by (1.1)-(1.3). One can see the works in [1, 7, 15, 21, 23, 27–31, 34]. In terms of the scope of this research, we would like to remind only some of the issues raised. These are also given below.

$$\Xi\left(-x\right) = -\Xi\left(x\right), \tag{2.15}$$

$$\Xi\left(\overline{x}\right) = \overline{\Xi\left(x\right)}, \tag{2.16}$$

$$\Xi_c\left(\overline{x}\right) = \overline{\Xi_c\left(x\right)}, \tag{2.17}$$

$$\Xi\left(x\right) = 1-\Xi_c\left(x\right), \tag{2.18}$$

$$\Xi_c\left(-x\right) = 2-\Xi_c\left(x\right), \tag{2.19}$$

$$\sqrt{\pi}\,\Xi\left(x\right) = 2\,x\,{}_1F_1\big(1/2;3/2;-x^2\big), \tag{2.20}$$

and

$$\sqrt{\pi}\,\Xi\left(x\right) = \Gamma\big(1/2,x^2\big) \quad \big(\Re e(x)>0\big) \tag{2.21}$$

and also

$$\sqrt{\pi}\,\Xi\left(x\right) = 2\,x^2\,e^{-x^2}\,{}_1F_1\big(1;3/2;x^2\big), \tag{2.22}$$

where the special functions:

$${}_1F_1\big(a;b;x\big) \quad \text{and} \quad \Gamma\big(a,x\big)$$

are the *confluent hypergeometric function of the first kind* and *incomplete gamma function*, respectively. Specifically, for more applications relating to these special-complex functions above, one can also look over the works in [7, 11, 16, 17].

Especially, the derivatives of the complex error function given by the form in (1.1):

$$\frac{d}{dx}\Big(\Xi\,(x)\Big) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}d\xi$$

$$= \frac{2}{\sqrt{\pi}}e^{-x^2}, \tag{2.23}$$

$$\frac{d^2}{dx^2}\Big(\Xi\,(x)\Big) = \frac{d}{dx}\Big(\frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}d\xi\Big)$$

$$= -\frac{2}{\sqrt{\pi}}(2x)e^{-x^2}, \tag{2.24}$$

$$\frac{d^3}{dx^3}\Big(\Xi\,(x)\Big) = \frac{d}{dx}\Big(-\frac{2}{\sqrt{\pi}}(2x)e^{-x^2}\Big)$$

$$= -\frac{2}{\sqrt{\pi}}(4x^2-2)e^{-x^2}, \tag{2.25}$$

and, in general, for all n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$,

$$\frac{d^n}{dx^n}\Big(\Xi\,(x)\Big) = (-1)^n\frac{2}{\sqrt{\pi}}\mathrm{H}_n(x)e^{-x^2} \tag{2.26}$$

is also obtained, *where* $\mathrm{H}_n(x)$ is well-known polynomials called by Hermite polynomials in the literature.

In the literature, as a result of some simple researches and in the light of the information given in (2.23)–(2.26), it can be easily seen that the solution of a large number of certain differential equations is the (complex) error function given in (1.1) (or (1.3)) *and* the (complex) complementary error function (1.4) (or (1.5)). For example, the function $\omega(x) := \Xi(x)$, defined by (1.1), is a solution of the following initial-value problem:

$$\left.\begin{array}{rcl} \omega\,''(x)\,+\,2\,x\,\omega(x) & = & 0 \\ \omega\,'(0) & = & \frac{2}{\sqrt{\pi}} \\ \omega\,(0) & = & 0 \end{array}\right\} \tag{2.27}$$

In consideration of the initial-value problem above, we can now present the following extensive theorem, which is important for a wide range of relations between certain differential equations and the error functions in the complex plane.

Theorem 2.3 *Let a function* $\omega := \omega(x)$ *be one of the solutions of the initial-value problem given by (2.27) and also let the inequality:*

$$\Re e\Big(\Phi(x)\Big) < \frac{2}{\sqrt{\pi}} \quad \big(x \in \mathbb{U}\big) \tag{2.28}$$

be satisfied. If the function ω *is a solution of the following third-order nonlinear and nonhomogenous differential equation:*

$$x\,\frac{d^3\omega}{dx^3}\,+\,2\,\frac{d\omega}{dx}\,=\,\Phi(x) \quad \big(x \in \mathbb{U}\big)\,, \tag{2.29}$$

then

$$\Re e\Big(\omega(x)\Big) > 0 \quad \big(x \in \mathbb{U}\big). \tag{2.30}$$

Proof In the light of the propositions stated in the hypothesis of Theorem 2.3 and since that function $\omega := \omega(x)$ is a solution of the equation given in (2.27), it must be in the form of the error function specified by (1.3). In the light of these explanations, it is enough to follow the following steps.

Firstly, we define a function $p(x)$ as in the (implicit) form:

$$\frac{2}{\sqrt{\pi}}p(x) = \frac{d}{dx}\Big(\omega(x)\Big) = \frac{d\omega}{dx} \tag{2.31}$$

to show that $\Re e\big(p(x)\big) > 0$ for all $x \in \mathbb{U}$. It is obvious that this function, *i.e.* $p(x)$ has the form in (1.6) and is an analytic function in the open domain $\mathbb{U}$ of $\subset \mathbb{C}$. Additionally, it satisfies the conditions of Lemma 1.1, which are $p(0) = 1$ and $n = 2$.

The statement (2.30) also gives us

$$\begin{aligned} \frac{d}{dx}\Big(\frac{2}{\sqrt{\pi}}p(x)\Big) \;&=\; \frac{2}{\sqrt{\pi}}\frac{d}{dx}\Big(p(x)\Big) \\ &= \frac{d}{dx}\Big(\frac{d\omega}{dx}\Big) = \frac{d^2\omega}{dx^2}\,. \end{aligned} \tag{2.32}$$

Moreover, in view of the equation in (2.28) and by the help of (2.31) and (2.32), the following relationship:

$$\frac{d^3}{dx^3}\Big(\omega(x)\Big) \;+\; 2\,\frac{d}{dx}\Big(\omega(x)\Big) \;=\; -\,2\,x\,\frac{d^2}{dx^2}\Big(\omega(x)\Big)$$

or, equivalently,

$$\begin{aligned} \frac{d^3\omega}{dx^3} \;+\; 2\,\frac{d\omega}{dx} \;&=\; -\,2\,x\,\frac{d^2\omega}{dx^2} \\ &= -\frac{4}{\sqrt{\pi}}\,x\,\frac{d}{dx}\Big(p(x)\Big) \\ &\equiv\; \Phi(x) \;\; (say) \end{aligned} \tag{2.33}$$

is easily obtained.

We now assume that there exists a point x_0 in $\mathbb{U}$, and, in consideration of Lemma 1.1, the condition in (2.2) is also satisfied. Then, with the help of the conditions given by (1.7) and (1.8) of the lemma, by making use of

$$p(x)\Big|_{x=x_0} = p\big(x_0\big) = i\alpha \quad \big(\alpha \in \mathbb{R} - \{0\}\big)$$

and

$$x\frac{d}{dx}\big[p(x)\big]\Big|_{x=x_0} = \beta \quad \big(\beta \in \mathbb{R}\big)\,,$$

and, of course, by taking in consideration of the assumptions (2.10), it follows from (2.33) that

$$-\Re e\Big(\Phi(x_0)\Big) = \frac{4}{\sqrt{\pi}}\,\Re e\left(x\frac{d}{dx}\left[p(x)\right]\Big|_{x=x_0}\right)$$

$$= \frac{4}{\sqrt{\pi}}\,\beta \tag{2.34}$$

and also by using (1.9), the equality (2.34) immediately follows that

$$\begin{aligned}-\Re e\Big(\Phi(x_0)\Big) &= \frac{4}{\sqrt{\pi}}\beta \\ &\leq -\frac{4}{\sqrt{\pi}}\,\frac{1+\alpha^2}{2} \\ &= -\frac{2(1+\alpha^2)}{\sqrt{\pi}} \\ &\leq -\frac{2}{\sqrt{\pi}}\ ,\end{aligned}$$

, which is a contradiction with the inequality given in (2.28). This shows that there is no point $x_0 \in \mathbb{U}$ satisfying the mentioned condition (2.7). Thus, the function $p(x)$, defined by (2.31), immediately yields that the inequality given by (2.30). Therefore, the proof of Theorem 2.3 is completed. □

The explanation, highlighted for Proposition 2.2, will be a sufficient explanation for the following proposition. So, we think that there is no need to present the details of its proof.

Proposition 2.4 *Let a function $\omega := \omega(x)$ be one of the solutions of the initial-value problem given by (2.27) and let the inequality:*

$$\Im m\Big(i\,\Phi(x)\Big) < \frac{2}{\sqrt{\pi}} \quad \Big(x \in \mathbb{U}\Big).$$

be also satisfied. If the function $\omega(x)$ is a solution of the differential equation given by (2.29), then

$$\Re e\Big(\omega(x)\Big) > 0 \quad \Big(x \in \mathbb{U}\Big).$$

As applications, implications and scientific interpretations of our main results above, when one focuses on those results, it is possible to reveal a great number of different *or* new results (*or* propositions). All right, these results can be obtained in different ways. They can basically obtain those in the three ways, which are presented in the following ways:

(i) All of the mentioned results (*or* propositions) can be redetermined (*or* reconstituted) by selections of the suitable values of the related parameters.

(ii) The mentioned results can be redetermined (*or* reinvented) by taking into consideration the basic identities signified as in (2.1)–(2.19).

(iii) The mentioned results can be determined (*or* reconstituted) by making use of the special-different functions indicated in (2.19)–(2.122).

(iv) Numerous new results like the main results, can be obtained from the available relationships between certain differential equations (*or* inequalities) and their solutions will be determined (*or* calculated) by the derivatives in (2.23) –(2.26).

(v) The ideas mentioned in the previous steps (*or* ways) can also be extended to new (*or* different) results (*or* propositions) determined by the complementary error function given by (1.3) (*or* (1.4)).

(vi) Especially, the mentioned results can be reproduced by taking into account various special results of the Hermite polynomials specified in (2.26), as comprehensive new results.

(vii) Most specifically, with the help of various package programs, two and three dimensional graphs of each one of various special results of possible results proposed above can also be created and then analyzed as their certain applications of them.

From the point of view of the results of this research, of course, it is not possible to list all the possible outcomes above. But, we want to present only one of them, as an example, and also leave the others to the researchers who have been working on the related topics.

Proposition 2.5 *If the inequality*

$$\Re e\Big(\big[\,\Xi''(x)\big]^2+\big[\,\Xi'(x)\big]^2\Big)<\frac{2+\alpha^4}{\pi}$$

is satisfied, then the inequality:

$$\Re e\Big(\Xi'(x)\Big)>0$$

is also satisfied, where the function $\Xi(x)$ *is defined by (1.1) (or (1.3)),* $\alpha\in\mathbb{R}-\{0\}$ *and* $x\in\mathbb{U}$.

Proof By taking the values of the related parameters τ and κ as $\tau:=2$ and $\kappa:=2$ in Theorem 2.1, the proof of the proposition above can be easily achieved. Its proof is here omitted. □

3. Conclusion

We conclude our investigation by remarking that here, we introduced the error functions in the complex plane and then comprehensive results together with several nonlinear implications in relation to the related complex functions are indicated and some possible special results of them are presented. Some interesting suggestions have been made for the scientific researchers who are interested in this topic.

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