

A semisymmetric metric connection on almost contact B -metric manifolds

Şenay BULUT* 

Department of Mathematics, Faculty of Science, Eskişehir Technical University, Eskişehir, Turkey

Received: 25.02.2020

Accepted/Published Online: 11.09.2021

Final Version: ..2021

Abstract: The object of the present paper is to study a semisymmetric metric connection on an almost contact B -metric manifold. We deduce a relation between the Levi-Civita connection and the semisymmetric metric connection on the considered manifold. We determined the class of the torsion tensor corresponding to the semisymmetric connection. We study Ricci-like solitons on almost contact B -metric manifolds with the semisymmetric connection. Finally, we give some examples to considered manifolds with the semisymmetric connection.

Key words: Semisymmetric metric connection, Almost contact B -metric manifold, Ricci-like soliton

1. Introduction

In this paper we undertake a study of semisymmetric metric connection on an almost contact B -metric manifold. Firstly, the concept of semisymmetric connection in a Riemannian manifold was introduced by Yano in [17]. Later, in [14] a semisymmetric metric connection in an almost contact manifold was defined and the properties of curvature tensors was studied in [11]. Some properties of semisymmetric connections have been examined on manifolds equipped with special structures [2, 4, 13, 17]. Kenmotsu manifolds with this connection were investigated in [1, 12]. In [16] the existence of a new connection on a Riemannian manifold is proved. In particular case, this connection corresponds to a semisymmetric metric connection.

The decomposition of the space of torsion tensors on almost contact B -metric manifolds is studied in [6]. The class of these manifolds equipped with the semisymmetric connection is described. If the manifold is Sasaki-like, then we attain some identities and theorems.

The concept of Ricci solitons was firstly described in Riemannian geometry. But, recently, Ricci solitons and η -Ricci solitons have been studied ([7, 12, 15]). Motivated by the study in [7] we investigate Ricci-like solitons on almost contact B -metric manifolds endowed with the semisymmetric metric connection.

The organization of the present paper is as follows: In Section 2 we give some necessary facts about Sasaki-like and almost contact B -metric manifolds. In Section 3 we determine the class of torsion tensor with respect to the connection and achieve a relation among curvature quantities of the semisymmetric metric connection and Levi-Civita connection. We prove that Sasaki-like manifold admitting a Ricci-like soliton also admits a Ricci-like soliton with respect to the considered connection. Finally, we find some geometric characteristics of examples of 3-dimensional B -metric manifolds equipped with the considered connection.

*Correspondence: math@tubitak.gov.tr

2010 *AMS Mathematics Subject Classification*: 53C05; 53C15; 53D10.

2. Almost contact B -metric manifolds

Let M be a $(2n + 1)$ -dimensional C^∞ -manifold and suppose that there exists in M a vector-valued linear function φ , a vector field ξ and a 1-form η such that

$$\eta(\xi) = 1, \quad \varphi^2 x = -x + \eta(x)\xi, \tag{2.1}$$

for any vector field x . Then (M, φ, ξ, η) is called an almost contact manifold.

By virtue of the relation (2.1), the following relations hold

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.2}$$

In addition, if there exists a pseudo-Riemannian metric g of signature $(n + 1, n)$ satisfying

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y), \tag{2.3}$$

then $(M, \varphi, \xi, \eta, g)$ is said to be an almost contact B -metric manifold.

By using the contact 1-form η on M , we determine the $2n$ -dimensional contact distribution $\mathcal{H} = \text{Ker } \eta$, known as horizontal distribution. The sections of \mathcal{H} are called the horizontal vector fields. Throughout this article we shall use X, Y, Z to denote elements of the smooth horizontal vector fields on M . x, y, z stand for arbitrary smooth vector fields on M .

The following equations are some consequences of (2.1), (2.2), (2.3)

$$g(x, \varphi y) = g(\varphi x, y), \quad g(\xi, \xi) = 1, \quad \eta(\nabla_x^g \xi) = 0 \tag{2.4}$$

where ∇^g is the Levi-Civita connection of g . The associated metric \tilde{g} of g is given by

$$\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y).$$

A classification of almost contact B -metric manifolds is made in [6] by using the tensor field F of type $(0, 3)$ determined by

$$F(x, y, z) = g((\nabla_x^g \varphi)y, z).$$

Moreover, F satisfies the following identities:

$$\begin{aligned} F(x, y, z) &= F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi) \\ F(x, \varphi y, \xi) &= (\nabla_x^g \eta)y = g(\nabla_x^g \xi, y) \end{aligned} \tag{2.5}$$

See for more details [3, 8, 9].

2.1. Sasaki-like almost contact B -metric manifolds

Sasaki-like almost contact B -metric manifolds are determined by the conditions

$$\begin{aligned} F(X, Y, Z) &= F(\xi, Y, Z) = F(\xi, \xi, Z) = 0, \\ F(X, Y, \xi) &= -g(X, Y). \end{aligned} \tag{2.6}$$

The covariant derivative $\nabla^g \varphi$ satisfies the equality

$$(\nabla_x^g \varphi)y = -g(x, y)\xi - \eta(y)x + 2\eta(x)\eta(y)\xi = g(\varphi x, \varphi y)\xi + \eta(y)\varphi^2 x. \tag{2.7}$$

Hence, the following relations given in [5] hold:

$$\begin{aligned}
 \nabla_x^g \xi &= -\varphi x, & R(x, y)\xi &= \eta(y)x - \eta(x)y, \\
 \nabla_\xi^g X &= -\varphi X - [X, \xi], & R(\xi, y)\xi &= \varphi^2 y, \\
 (\nabla_x^g \eta)y &= -g(x, \varphi y), & Ric(x, \xi) &= 2n, \\
 \nabla_\xi^g \xi &= 0, & Ric(\xi, \xi) &= 2n, \\
 div \xi &= 0.
 \end{aligned}
 \tag{2.8}$$

where R and Ric stand for the curvature and the Ricci tensor, respectively.

3. A semisymmetric metric connection on almost contact B -metric manifolds

In this section we deal with a semisymmetric metric connection on an almost contact B -metric manifold. Let $(M, \varphi, \xi, \eta, g)$ be an almost contact B -metric manifold with the Levi-Civita connection ∇^g and we define a semisymmetric connection $\tilde{\nabla}$ on M by

$$\tilde{\nabla}_x y = \nabla_x^g y + g(x, y)\xi - \eta(y)x.
 \tag{3.1}$$

By using above equation the torsion tensor T of M with respect to the connection $\tilde{\nabla}$ is given by

$$T(x, y) = \tilde{\nabla}_x y - \tilde{\nabla}_y x - [x, y] = \eta(x)y - \eta(y)x.
 \tag{3.2}$$

Further, the semisymmetric connection $\tilde{\nabla}$ satisfying the condition

$$(\tilde{\nabla}_x g)(y, z) = 0
 \tag{3.3}$$

for all $x, y, z \in \chi(M)$ is said to be a semisymmetric metric connection, otherwise it is called a semisymmetric nonmetric connection.

Now we shall show the existence of the semisymmetric metric connection $\tilde{\nabla}$ on an almost contact B -metric manifold $(M, \varphi, \xi, \eta, g)$.

Theorem 3.1 *There exists a unique linear connection $\tilde{\nabla}$ satisfying (3.2) and (3.3) on an almost contact B -metric manifold $(M, \varphi, \xi, \eta, g)$.*

Proof Suppose that $\tilde{\nabla}$ is a linear connection on M given by

$$\tilde{\nabla}_x y = \nabla_x^g y + H(x, y),
 \tag{3.4}$$

where H denotes a tensor field of type $(1, 2)$. Now, we shall specify the tensor field H such that $\tilde{\nabla}$ satisfies the conditions (3.2) and (3.3). By the definition of torsion tensor, the equation (3.4) leads to

$$T(x, y) = H(x, y) - H(y, x)
 \tag{3.5}$$

for all $x, y \in \chi(M)$. We have

$$g(T(x, y), z) = g(H(x, y), z) - g(H(y, x), z)
 \tag{3.6}$$

The equation (3.3) leads to

$$(\tilde{\nabla}_x g)(y, z) = 0 = xg(y, z) - g(\tilde{\nabla}_x y, z) - g(\tilde{\nabla}_x z, y) = g(H(x, y), z) + g(H(x, z), y). \quad (3.7)$$

Then, we get

$$g(H(x, y), z) = -g(H(x, z), y). \quad (3.8)$$

From (3.6) and (3.8), we have

$$\begin{aligned} & g(T(x, y), z) + g(T(z, x), y) + g(T(z, y), x) \\ &= g(H(x, y), z) - g(H(y, x), z) + g(H(z, x), y) \\ &\quad - g(H(x, z), y) + g(H(z, y), x) - g(H(y, z), x) \\ &= 2g(H(x, y), z). \end{aligned} \quad (3.9)$$

By using Equations (3.2) and (3.9) we obtain

$$\begin{aligned} 2g(H(x, y), z) &= g(T(x, y), z) + g(T(z, x), y) + g(T(z, y), x) \\ &= g(\eta(x)y - \eta(y)x, z) + g(\eta(z)x - \eta(x)z, y) + g(\eta(z)y - \eta(y)z, x) \\ &= g(\eta(x)y - \eta(y)x, z) + 2\eta(z)g(x, y) - \eta(x)g(z, y) - \eta(y)g(z, x) \\ &= g(\eta(x)y - \eta(y)x, z) + 2g(z, \xi)g(x, y) - \eta(x)g(z, y) - \eta(y)g(z, x) \\ &= 2g(g(x, y)\xi - \eta(y)x, z) \end{aligned} \quad (3.10)$$

In consequence of (3.10), we get

$$H(x, y) = g(x, y)\xi - \eta(y)x. \quad (3.11)$$

In view of (3.10), we can get Equation (3.1). \square

Let us consider the torsion tensor T of a semisymmetric metric connection $\tilde{\nabla}$ on M . Then, the corresponding tensor of type $(0, 3)$ is determined by

$$T(x, y, z) = g(T(x, y), z) = \eta(x)g(y, z) - \eta(y)g(x, z). \quad (3.12)$$

Especially, we have

$$T(\varphi x, \varphi y, z) = 0, \quad \text{and} \quad T(x, y, \xi) = 0. \quad (3.13)$$

Proposition 3.2 *The torsion tensor T of a semisymmetric metric connection on M belongs to the class \mathcal{T}_9 .*

Proof Consider the vector space of all tensors of type $(0, 3)$ over $T_p(M)$ in the following way:

$$\mathcal{T} = \{T(x, y, z) \in \mathbb{R} \mid T(x, y, z) = -T(y, x, z), x, y, z \in T_p M\}. \quad (3.14)$$

By using the operator $p_1 : \mathcal{T} \rightarrow \mathcal{T}$ by

$$p_1(T)(x, y, z) = -T(\varphi^2 x, \varphi^2 y, \varphi^2 z) = -g(T(\varphi^2 x, \varphi^2 y), \varphi^2 z),$$

the orthogonal decomposition of \mathcal{T} by the image and the kernel of p_1 is obtained as follows:

$$W_1 = im(p_1) = \{T \in \mathcal{T} \mid p_1(T) = T\} \text{ and } W_1^\perp = ker(p_1) = \{T \in \mathcal{T} \mid p_1(T) = 0\}.$$

Since $p_1(T) = 0$ by (3.13), we have $T \in \ker(p_1) = W_1^\perp$. Let us define the operator $p_2 : W_1^\perp \rightarrow W_1^\perp$ by

$$p_2(T)(x, y, z) = \eta(z)T(\varphi^2x, \varphi^2y, \xi) = \eta(z)g(T(\varphi^2x, \varphi^2y), \xi).$$

Since the operator has the property $p_2 \circ p_2 = p_2$, the decomposition of W_1^\perp is given by

$$W_2 = \text{im}(p_2) = \{T \in W_1^\perp \mid p_2(T) = T\} \text{ and } W_2^\perp = \ker(p_2) = \{T \in W_1^\perp \mid p_2(T) = 0\}.$$

Again from (3.13) we obtain $p_2(T) = 0$, namely, $T \in \ker(p_2) = W_2^\perp$. Let us define the operator $p_3 : W_2^\perp \rightarrow W_2^\perp$ by

$$p_3(T)(x, y, z) = \eta(x)T(\xi, \varphi^2y, \varphi^2z) + \eta(y)T(\varphi^2x, \xi, \varphi^2z).$$

By making use of (2.3), (3.2) and (3.12), the above equality turns into

$$\begin{aligned} p_3(T)(x, y, z) &= \eta(x)T(\xi, \varphi^2y, \varphi^2z) + \eta(y)T(\varphi^2x, \xi, \varphi^2z) \\ &= \eta(x)g(T(\xi, \varphi^2y), \varphi^2z) + \eta(y)g(T(\varphi^2x, \xi), \varphi^2z) \\ &= \eta(x)g(\varphi^2y, \varphi^2z) - \eta(y)g(\varphi^2x, \varphi^2z) \\ &= -\eta(x)g(\varphi y, \varphi z) + \eta(y)g(\varphi x, \varphi z) \\ &= -\eta(x)(-g(y, z) + \eta(y)\eta(z)) + \eta(y)(-g(x, z) + \eta(x)\eta(z)) \\ &= \eta(x)g(y, z) - \eta(y)g(x, z) \\ &= g(T(x, y), z) = T(x, y, z). \end{aligned} \tag{3.15}$$

Hence, $p_3(T) = T$, i.e. $T \in \text{im}(p_3) = W_3$. The operators

$$\begin{aligned} L_{3,0}(T)(x, y, z) &= \eta(x)T(\xi, \varphi y, \varphi z) - \eta(y)T(\xi, \varphi x, \varphi z) \\ L_{3,1}(T)(x, y, z) &= \eta(x)T(\xi, \varphi^2z, \varphi^2y) - \eta(y)T(\xi, \varphi^2z, \varphi^2x) \end{aligned} \tag{3.16}$$

are involutive isometries on W_3 . By virtue of (3.12) we obtain $L_{3,0}(T) = -T$, that is, $T \in W_3^-$ and $L_{3,1}(T) = T$, that is, $T \in W_{3,1}$ where

$$W_3^- = \{T \in W_3 \mid L_{3,0}(T) = -T\} \text{ and } W_{3,1} = \{T \in W_3^- \mid L_{3,1}(T) = T\}.$$

$t(x) = g^{ij}T(x, e_i, e_j)$ and $t^*(x) = g^{ij}T(x, e_i, \varphi e_j)$ are torsion forms of T with respect to the basis $\{\xi, e_1, \dots, e_{2n}\}$. By the direct calculation we get $t \neq 0$, $t^* = 0$. Therefore, $T \in W_{3,1,1} = \mathcal{T}_9$, where

$$W_{3,1,1} = \{T \in W_{3,1} \mid t \neq 0, t^* = 0\}.$$

□

Lemma 3.3 *The relations between the curvature quantities of the semisymmetric metric connection $\widetilde{\nabla}$ and Levi-Civita connection ∇^g are given by the following formulas:*

1. *Curvature transformation:*

$$\begin{aligned} \widetilde{R}(x, y)z &= R(x, y)z + g(y, z)\nabla_x^g \xi - g(x, z)\nabla_y^g \xi + [g(\nabla_y^g \xi, z) + g(\varphi y, \varphi z)]x \\ &\quad - [g(\nabla_x^g \xi, z) + g(\varphi x, \varphi z)]y + g(T(x, y), z)\xi. \end{aligned} \tag{3.17}$$

2. *Ricci curvature:*

$$\widetilde{Ric}(x, y) = Ric(x, y) + [\text{div}(\xi) + (1 - 2n)]g(x, y) + (2n - 1)[\eta(x)\eta(y) + g(\nabla_x^g \xi, y)]. \tag{3.18}$$

3. *Scalar curvature:*

$$\tilde{s} = s^g + 4\text{ndiv}(\xi) + 2n(1 - 2n). \quad (3.19)$$

Proof

1. The curvature tensor \tilde{R} of type (1, 3) is defined by

$$\tilde{R}(x, y)z = \tilde{\nabla}_x \tilde{\nabla}_y z - \tilde{\nabla}_y \tilde{\nabla}_x z - \tilde{\nabla}_{[x, y]} z. \quad (3.20)$$

where $x, y, z \in \chi(M)$. Using (2.3), (3.1), (3.2), (3.3) in (3.20) we obtain the relation (3.17) by a routine computation.

2. The Ricci curvature tensor \widetilde{Ric} with respect to $\tilde{\nabla}$ is defined by

$$\widetilde{Ric}(x, y) = \sum_{i=0}^{2n} \varepsilon_i g(\tilde{R}(e_i, x)y, e_i). \quad (3.21)$$

with regard to the basis $\{\xi, e_1, \dots, e_{2n}\}$. The divergence of the vector field ξ is given by

$$\text{div}\xi = \sum_{i=0}^{2n} \varepsilon_i g(\nabla_{e_i}^g \xi, e_i). \quad (3.22)$$

By making use of (2.3), (3.1), (3.2), (3.3), (3.17) in (3.21) we get (3.18).

3. The scalar curvature tensor \tilde{s} with respect to $\tilde{\nabla}$ is defined by

$$\tilde{s} = \sum_{i=0}^{2n} \widetilde{Ric}(e_i, e_i).$$

By the way of the relations (2.3), (3.1), (3.2), (3.3), (3.18), (3.22) it can be easily computed the identity (3.19). □

Note that we have the following relations with respect to the basis $\{\xi, e_1, \dots, e_{2n}\}$ for $(M, \varphi, \xi, \eta, g)$ as a consequence of the Lemma (3.3):

$$(\tilde{\nabla}_x \eta)y = (\nabla_x^g \eta)y + g(\varphi x, \varphi y) \quad (3.23)$$

$$(\tilde{\nabla}_x \varphi)y = (\nabla_x^g \varphi)y + g(x, \varphi y)\xi + \eta(y)\varphi x \quad (3.24)$$

$$\tilde{\nabla}_\xi y = \nabla_\xi^g y \quad (3.25)$$

$$\tilde{\nabla}_x \xi = \nabla_x^g \xi + \varphi^2 x \quad (3.26)$$

$$\tilde{R}(x, y)\xi = R(x, y)\xi + \eta(y)\nabla_x^g \xi - \eta(x)\nabla_y^g \xi + \eta(\nabla_y^g \xi)x \quad (3.27)$$

$$\widetilde{Ric}(\xi, \xi) = Ric(\xi, \xi) + \text{div}(\xi) + \eta(\nabla_\xi^g \xi) \quad (3.28)$$

Theorem 3.4 *In a Sasaki-like almost contact B–metric manifold with a semisymmetric metric connection we have the following identities:*

1. $(\widetilde{\nabla}_x \eta)y = g(\varphi x - x, \varphi y)$
2. $(\widetilde{\nabla}_x \varphi)y = g(x + \varphi x, \varphi y)\xi + \eta(y)\varphi(x + \varphi x)$
3. $\widetilde{\nabla}_x \xi = -\varphi x + \varphi^2 x$
4. $\widetilde{\nabla}_\xi x = \nabla_\xi^g x = -\varphi x + [\xi, x]$
5. $\widetilde{Ric}(\xi, \xi) = 2n$
6. $\widetilde{s} = s + 2n(1 - 2n)$
7. $\widetilde{Ric}(x, y) = Ric(x, y) + (2n - 1)(-g(\varphi x, y) + \eta(x)\eta(y)) + (1 - 2n)g(x, y)$

Proof

1. Using the relations (2.8) and (3.23) it follows that

$$\begin{aligned} (\widetilde{\nabla}_x \eta)y &= (\nabla_x^g \eta)y + g(\varphi x, \varphi y) \\ &= -g(x, \varphi y) + g(\varphi x, \varphi y) \\ &= g(\varphi x - x, \varphi y) \end{aligned}$$

2. The equations (2.7) and (3.24) imply that

$$\begin{aligned} (\widetilde{\nabla}_x \varphi)y &= (\nabla_x^g \varphi)y + g(x, \varphi y)\xi + \eta(y)\varphi x \\ &= g(\varphi x, \varphi y)\xi + \eta(y)\varphi^2 x + g(x, \varphi y)\xi + \eta(y)\varphi x \\ &= g(\varphi x + x, \varphi y)\xi + \eta(y)\varphi(\varphi x + x) \end{aligned}$$

3. It comes directly from Equation (2.8).
4. It can be easily verified by using the identities (2.8) and (3.1).
5. For any Sasaki-like almost contact B–metric manifold we know that $\text{div } \xi = 0$ and $(\nabla_\xi^g \xi) = 0$. Hence, from (3.28) and (2.8) the given identity is verified.
6. Since $\text{div } \xi = 0$, Equation (3.19) yields the desired identity.
7. It is obtained by using (3.18) and (2.8).

□

Corollary 3.5

The manifold $(M, \varphi, \xi, \eta, g)$ is said to be Einstein-like if its Ricci tensor Ric satisfies the following condition:

$$Ric = \lambda g + \mu \widetilde{g} + \nu \eta \otimes \eta \tag{3.29}$$

where λ, μ, ν are constants.

Theorem 3.6 *If $(M, \varphi, \xi, \eta, g)$ is a Einstein-like Sasaki-like almost contact B -metric manifold, then it is a Einstein-like almost contact B -metric manifold with respect to the semisymmetric metric connection $\tilde{\nabla}$.*

Proof Since $(M, \varphi, \xi, \eta, g)$ is a Einstein-like, its Ricci tensor satisfies (3.29). Hence, there exist the constants (λ, μ, ν) such that

$$Ric(x, y) = \lambda g(x, y) + \mu \tilde{g}(x, y) + \nu \eta(x)\eta(y). \tag{3.30}$$

Using the Theorem (3.4)(7) and (3.30) we derive that

$$\begin{aligned} \widetilde{Ric}(x, y) &= Ric(x, y) + (2n - 1)(-g(\varphi x, y) + \eta(x)\eta(y)) + (1 - 2n)g(x, y) \\ &= \lambda g(x, y) + \mu \tilde{g}(x, y) + \nu \eta(x)\eta(y) + (2n - 1)(-g(\varphi x, y) + \eta(x)\eta(y)) + (1 - 2n)g(x, y) \\ &= (\lambda + 1 - 2n)g(x, y) + (\mu - 2n + 1)\tilde{g}(x, y) + (\nu + 4n - 2)\eta(x)\eta(y). \end{aligned}$$

Therefore, M is a Einstein-like with respect to $\tilde{\nabla}$ with the constants $(\lambda + 1 - 2n, \mu - 2n + 1, \nu + 4n - 2)$. \square

The manifold M is said to be an almost contact B -metric manifold with a torse-forming Reeb vector field ξ provided that $\nabla_x^g \xi = fx + \alpha(x)\xi$, where f is a smooth function and α is a one-form on M . If ξ is a torse-forming vector field on M , then the 1-form α must be $-f\eta$ because of $\eta(\nabla_x^g \xi) = 0$. Hence, we derive the following equivalent identities:

$$\nabla_x^g \xi = -f\varphi^2 x \text{ and } (\nabla_x^g)y = -fg(\varphi x, \varphi y). \tag{3.31}$$

Taking into account the relations (3.31) we state the following:

Proposition 3.7 *If the Reeb vector field ξ of $(M, \varphi, \xi, \eta, g)$ is a torse-forming with respect to the Levi-Civita connection ∇^g , then it is a torse-forming with respect to the semisymmetric connection $\tilde{\nabla}$.*

$(M, \varphi, \xi, \eta, g)$ is a Ricci-like soliton if its Ricci tensor Ric satisfies the following condition with the constants (λ, μ, ν) :

$$\frac{1}{2}\mathcal{L}_\xi g + Ric + \lambda g + \mu \tilde{g} + \nu \eta \otimes \eta = 0, \tag{3.32}$$

where $\mathcal{L}_\xi g$ is a Lie derivative defined by

$$(\mathcal{L}_\xi g)(x, y) = g(\nabla_x^g \xi, y) + g(x, \nabla_y^g \xi).$$

A generalization of the Ricci soliton and the η -Ricci soliton was introduced in [7].

Theorem 3.8 *If a Sasaki-like almost contact B -metric manifold M admits a Ricci-like soliton with constants (λ, μ, ν) , then M admits a Ricci-like soliton with the constants $(\lambda + 2n, \mu + 2n - 1, \nu - 4n + 1)$ with respect to the semisymmetric connection $\tilde{\nabla}$.*

Proof The manifold M satisfies Equation (3.32). Taking into account Theorem (3.4) and the equalities of $\nabla_x^g \xi$ and $\tilde{\nabla}_x \xi$ by (2.8), (3.26) the main assertion

$$\frac{1}{2}\tilde{\mathcal{L}}_\xi g + \widetilde{Ric} + (\lambda + 2n)g + (\mu + 2n - 1)\tilde{g} + (\nu - 4n + 1)\eta \otimes \eta = 0 \tag{3.33}$$

is valid. \square

4. Examples

4.1. Example

Let us consider a 3–dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 | z \neq 0\}$ where (x, y, z) are standard coordinates in \mathbb{R}^3 . A linearly independent global frame field on M is $\{\xi = e_0, e_1, e_2\}$ defined by

$$e_0 = \frac{\partial}{\partial z}, \quad e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right). \quad (4.1)$$

Let g be a semi-Riemannian metric defined by $g(e_0, e_0) = g(e_1, e_1) = -g(e_2, e_2) = 1$. Define the 1–form η by $\eta(x) = g(x, e_0)$ for any vector field $x \in \chi(M)$. Let φ be the $(1, 1)$ –tensor field defined by $\varphi(e_0) = 0$, $\varphi(e_1) = e_2$, $\varphi(e_2) = -e_1$. The linearity property of φ and g yields

$$\begin{aligned} \eta(e_0) &= 1, \\ \varphi^2 x &= -x + \eta(x)e_0, \\ g(\varphi x, \varphi y) &= -g(x, y) + \eta(x)\eta(y), \end{aligned} \quad (4.2)$$

for any vector fields x, y . Hence, (φ, ξ, η, g) is an almost contact B –metric structure on M . But, the considered manifold is not Sasaki-like.

Now we have

$$[e_1, e_2] = 0, \quad [e_0, e_1] = e_1, \quad [e_0, e_2] = e_2. \quad (4.3)$$

By using Koszul’s formula we obtain the following nonzero components of the Levi–Civita connection ∇^g .

$$\nabla_{e_1}^g e_1 = e_0, \quad \nabla_{e_1}^g e_0 = -e_1, \quad \nabla_{e_2}^g e_2 = -e_0, \quad \nabla_{e_2}^g e_0 = -e_2. \quad (4.4)$$

The nonzero components of the semisymmetric connection $\tilde{\nabla}$ are calculated by

$$\tilde{\nabla}_{e_1} e_1 = 2e_0, \quad \tilde{\nabla}_{e_1} e_0 = -2e_1, \quad \tilde{\nabla}_{e_2} e_2 = -2e_0, \quad \tilde{\nabla}_{e_2} e_0 = -2e_2. \quad (4.5)$$

With the help of above results the nonzero components of curvature tensors R and \tilde{R} can be easily obtained by

$$\begin{aligned} R(e_1, e_2)e_1 &= e_2, & R(e_2, e_0)e_2 &= -e_3, & R(e_1, e_0)e_1 &= e_0, \\ R(e_1, e_2)e_2 &= e_1, & R(e_2, e_0)e_0 &= -e_2, & R(e_1, e_0)e_0 &= -e_1, \end{aligned} \quad (4.6)$$

$$\begin{aligned} \tilde{R}(e_1, e_2)e_1 &= 4e_2, & \tilde{R}(e_2, e_0)e_2 &= -2e_3, & \tilde{R}(e_1, e_0)e_1 &= 2e_0, \\ \tilde{R}(e_1, e_2)e_2 &= 4e_1, & \tilde{R}(e_2, e_0)e_0 &= -2e_2, & \tilde{R}(e_1, e_0)e_0 &= -2e_1. \end{aligned} \quad (4.7)$$

All components of Ricci curvatures Ric and $\tilde{\text{Ric}}$ with respect to ∇ and $\tilde{\nabla}$, respectively are zero. Therefore, the scalar curvatures Scal and $\tilde{\text{Scal}}$ are zero.

4.2. Example

Let us consider the real connected 3–dimensional Lie group L with a global basis $\{\xi = e_0, e_1, e_2\}$ of the left invariant vector fields on L such that the commutators of its associated Lie algebra are defined as follows:

$$[\xi, e_1] = e_2, \quad [\xi, e_2] = -e_1. \quad (4.8)$$

Hence, L is endowed with an almost contact B -metric structure by

$$\begin{aligned} g(e_0, e_0) = g(e_1, e_1) = -g(e_2, e_2) = 1, \quad g(e_i, e_j) = 0 \text{ for } i \neq j, \\ \varphi(e_1) = e_2, \quad \varphi(e_2) = -e_1. \end{aligned} \tag{4.9}$$

This example is given in [10]. The nonzero components of the Levi-Civita connection ∇ and the semisymmetric connection $\tilde{\nabla}$ are respectively calculated by

$$\nabla_{e_1} e_0 = -e_2, \quad \nabla_{e_2} e_0 = e_1, \quad \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = -e_0, \tag{4.10}$$

$$\begin{aligned} \tilde{\nabla}_{e_1} e_0 = -e_1 - e_2, \quad \tilde{\nabla}_{e_2} e_0 = e_1 - e_2, \quad \tilde{\nabla}_{e_1} e_2 = \tilde{\nabla}_{e_2} e_1 = -e_0, \\ \tilde{\nabla}_{e_1} e_1 = e_0, \quad \tilde{\nabla}_{e_2} e_2 = -e_0. \end{aligned} \tag{4.11}$$

It can be easily proved that the constructed manifold L is a Sasaki-like. Taking into account (4.8), (4.9), (4.10) and (4.11), we compute the nonzero components of curvature tensors R and \tilde{R} with regard to ∇ and $\tilde{\nabla}$ as follows:

$$\begin{aligned} R_{010} = -e_1, \quad R_{020} = -e_2, \quad R_{011} = e_0, \quad R_{121} = e_2, \\ R_{022} = -e_0, \quad R_{122} = e_1, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \tilde{R}_{010} = -e_1 + e_2, \quad \tilde{R}_{020} = -e_1 - e_2, \quad \tilde{R}_{011} = e_0, \quad \tilde{R}_{121} = 2e_2, \\ \tilde{R}_{021} = e_0, \quad \tilde{R}_{012} = e_0, \quad \tilde{R}_{022} = -e_0, \quad \tilde{R}_{122} = 2e_1. \end{aligned} \tag{4.13}$$

The nonzero components of Ricci tensors Ric and \widetilde{Ric} are determined by the following equalities:

$$\begin{aligned} Ric_{00} = 2, \\ \widetilde{Ric}_{00} = 2, \quad \widetilde{Ric}_{11} = -1, \quad \widetilde{Ric}_{22} = 1, \\ \widetilde{Ric}_{12} = 1 = \widetilde{Ric}_{21} \end{aligned} \tag{4.14}$$

Scalar curvatures $Scal$ and \widetilde{Scal} are given by $Scal = 2$ and $\widetilde{Scal} = 0$. Since we have the condition $Ric = 2\eta \otimes \eta$, L is Einstein-like with constants $(0, 0, 2)$. The Ricci tensor \widetilde{Ric} corresponding to the semisymmetric metric connection $\tilde{\nabla}$ satisfies the relation

$$\widetilde{Ric} = -g - \tilde{g} + 4\eta \otimes \eta.$$

Hence, L is an Einstein-like with regard to the semisymmetric metric connection $\tilde{\nabla}$ with constants $(-1, -1, 4)$. Moreover, L is a Ricci-like soliton with potential vector field ξ since Ric has the following relation

$$\frac{1}{2}(\mathcal{L}_\xi g)(x, y) + Ric(x, y) + \tilde{g} - 3\eta(x)\eta(y) = 0$$

, where $\mathcal{L}_\xi g$ is the Lie derivative of g along ξ . It can be easily proved that ξ is not a torse-forming vector field.

Let us consider the Lie derivative of g along ξ corresponding to the connection $\tilde{\nabla}$ defined by

$$\tilde{\mathcal{L}}_\xi g(x, y) = g(\tilde{\nabla}_x \xi, y) + g(x, \tilde{\nabla}_y \xi).$$

The following equality is valid:

$$\frac{1}{2}\tilde{\mathcal{L}}_\xi g + \widetilde{Ric} + 2g + 2\tilde{g} - 6\eta \otimes \eta = 0.$$

Therefore, L admits a Ricci-like soliton with the potential ξ with respect to the connection $\tilde{\nabla}$ by constants $(2, 2, -6)$. It can be easily checked that Theorem (3.8) is confirmed.

Acknowledgment

The author would like to thank anonymous referees for their valuable comments which helped to improve the manuscript.

References

- [1] Chaubey SK, Yadav SK. Study of Kenmotsu manifolds with semi-symmetric connection. *Universal Journal of Mathematics and Applications* 2018; 1 (2): 89-97.
- [2] De UC, Pathok G. On a semi-symmetric metric connection in a Kenmotsu manifold. *Bulletin of the Calcutta Mathematical Society* 2002; 94: 319-324.
- [3] Ganchev G, Mihova V, Gribachev K. Almost contact manifolds with B -metric. *Mathematica Balkanica, New Series* 1993; 7: 261-276.
- [4] Imai T. Notes on semi-symmetric metric connections. *Tensor (N.S.)* 1972; 24: 293-296.
- [5] Ivanov S, Manev H, Manev M. Sasaki-like almost contact complex Riemannian manifolds. *Journal of Geometry and Physics* 2016; 107: 136-148.
- [6] Manev M, Ivanova M. A classification of the torsion tensors on an almost contact manifolds with B -metric. *Central European Journal of Mathematics* 2014; 12 (10): 1416-1432.
- [7] Manev M. Ricci-like solitons on almost contact B -metric manifolds. *Journal of Geometry and Physics* 2020; 154: 103734.
- [8] Manev M, Ivanova M. Canonical-type connection on almost contact manifolds with B -metric. *Annals of Global Analysis and Geometry* 2013; 43: 397-408.
- [9] Manev M. Contactly conformal transformations of general type of almost contact manifold with B -metric. *Applications. Mathematica Balkanica, New Series* 1997; 11: 347-357.
- [10] Manev H, Mekerov D. Lie groups as 3-dimensional almost contact B -metric manifolds. *Journal of Geometry* 2015; 106: 229-242.
- [11] Mishra RS, Pandey SN. Semi-symmetric metric connections in an almost contact manifold. *Indian Journal of Pure and Applied Mathematics* 1978; 9 (6): 570-580.
- [12] Nagaraja HG, Premalatha CR. Ricci solitons in Kenmotsu manifolds. *Journal of Mathematical Analysis* 2012; 3 (2): 18-24.
- [13] Pravanović M, Pušić N. On manifolds admitting some semi-symmetric metric connection. *Indian Journal of Mathematics* 1995; 37: 37-67.
- [14] Sharfuddin A, Husain SI. Semi-symmetric metric connections in almost contact manifolds. *Tensor (N.S.)* 1976; 30: 133-139.
- [15] Sharma R. Certain results on K -contact and (κ, μ) -contact manifolds. *Journal of Geometry* 2008; 89 (1-2): 138-147.
- [16] Tripathi MM. A new connection in a Riemannian manifold. *International Electronic Journal of Geometry* 2008; 1: 15-24.
- [17] Yano K. On semi-symmetric metric connection. *Revue Roumaine de Mathématique Pures et Appliquées* 1970; 15: 1579-1586.