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Scattering properties of impulsive difference Dirac equations

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Abstract: In this paper, we explore the Jost solutions and the scattering matrix of the impulsive difference Dirac systems (IDDS) on the whole axis and study their analytic and asymptotic properties. Furthermore, characteristic properties of the scattering matrix of the IDDS have been examined.

Key words: Difference equations, Dirac systems, Jost solutions, Scattering matrix.

1. Introduction

In mathematics and development of theoretical physics, Dirac systems have many handy applications, and it is known that the Dirac operator finds its natural place in the attempt to obtain a relativistic wave equation for the electron. As is known, the dynamic of developing processes is constantly depended on abrupt changes happened at such points. Usually, these short-term perturbations are treated as having acted suddenly or in the form of "*impulses*". The general theory of impulsive differential equations is given in [2, 8, 9, 19]. In the literature, different names may be encountered in place of these conditions such as jump conditions, interface conditions, point interaction conditions and transmission conditions. Furthermore, impulsive differential equations that are mentioned with impulse effects have been evolved in modelling problems in physics, biotechnology, industrial robotics, pharmacokinetics, population dynamics, and so forth, and, since the late 1990s, the authors have produced an extensive portfolio of results on differential and difference equations undergoing impulse effects [2, 15, 26].

Differential inclusions subject to the impulse conditions have played a crucial role in modeling phenomena, especially in scenarios including automatic control systems. Also, while these processes include hereditary phenomena such as biological and social macrosystems, some of the modeling is made via functional differential equations with impulsive effects. Furthermore, in the studies [3, 10–14, 16, 17, 25, 29], authors have examined the spectral and scattering properties of Sturm–Liouville and Dirac operators. Also, two component scattering systems and the inverse problem of the one-dimensional Sturm–Liouville equations have attracted a great amount of interest for years [4–7, 18, 28]. This attention is guided by a great deal effects of such problems happening in fields as diversified as geophysics, relativistic quantum mechanics, voice synthesisfilter design, transmission-line analysis and so on. In recent years, the spectral features of singular dissipative operators have been investigated in [21–24, 27]. Also, singular Dirac systems in Weyl's limit-circle case with finite transmission conditions have been investigated in [1].

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This study provides a concise and selfcontained introduction to global theory of Dirac systems and to the analysis of spectral asymptotics with impulsive conditions. To solve the problems with impulsive conditions, we need to establish further conditions at the point where impulsive effect exists. Throughout this work in which we examined the scattering theory of Dirac systems with impulsive conditions in entire axis, we first obtained significant results for Jost solutions and scattering matrix of canonic Dirac systems.

2. Installation of the problem

Let $\ell_2\left(\mathbb{Z}^*, \mathbb{C}^2\right)$ be the space of vector-valued sequences with domain $\mathbb{Z}^*$ and range $\mathbb{C}^2$. We consider the operator created in the space $\ell_2\left(\mathbb{Z}^*, \mathbb{C}^2\right)$ by

$$\begin{cases} a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}, \qquad n \in \mathbb{Z}^* := \mathbb{Z} \backslash \{-1, 0, 1\}, \end{cases} \tag{2.1}$$

and the impulsive condition

$$\begin{pmatrix} y_1^{(1)}(z) \\ y_2^{(2)}(z) \end{pmatrix} = \mathbf{B} \begin{pmatrix} y_{-1}^{(2)}(z) \\ y_{-2}^{(1)}(z) \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}, \tag{2.2}$$

where $\mathbf{B}$ is a real matrix such that $\det \mathbf{B} > 0$, $\{a_n\}_{n\in\mathbb{Z}}, \{b_n\}_{n\in\mathbb{Z}}, \{p_n\}_{n\in\mathbb{Z}}$ and $\{q_n\}_{n\in\mathbb{Z}}$ are real sequences such that $a_n \neq 0$ and $b_n \neq 0, (n \in \mathbb{Z})$, and these sequences satisfy

$$\sum_{n\in\mathbb{Z}} |n| \left(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|\right) < \infty, \tag{2.3}$$

and also λ is a spectral parameter.

If $a_n \equiv 1, b_n \equiv -1$ for all $n \in \mathbb{Z}$, then the system (2.1), denoted by L, reduce to

$$\begin{cases} \Delta y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)}, \\ -\Delta y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}, \qquad n \in \mathbb{Z}^*, \end{cases} \tag{2.4}$$

where Δ is a forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$.

The system (2.4) is the difference analog of the well-known canonical Dirac system [10]:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

, and so the system (2.4) is called a *canonical difference Dirac system with impulsive condition at the point* $n = 0$.

In the light of [6], we know that, under the condition (2.3) for $\lambda = 2\sin\frac{z}{2}$, the system (2.1) has two solutions

$$\varphi(z) = \{\varphi_n(z)\}_{n\in\mathbb{Z}^+} = \left\{ \begin{pmatrix} \varphi_n^{(1)}(z) \\ \varphi_n^{(2)}(z) \end{pmatrix} \right\}_{n\in\mathbb{Z}^+}, \quad z \in \overline{\mathbb{C}}_+,$$

and

$$\psi(z)=\{\psi_n(z)\}_{n\in\mathbb{Z}^-}=\left\{\begin{pmatrix}\psi_n^{(1)}(z)\\ \psi_n^{(2)}(z)\end{pmatrix}\right\}_{n\in\mathbb{Z}^-}, \quad z\in\overline{\mathbb{C}}_+,$$

satisfying the following asymptotic conditions in terms of n:

$$\varphi_n(z)=[I_2+o(1)]\begin{pmatrix}e^{i\frac{z}{2}}\\ -i\end{pmatrix}e^{inz}, \quad z\in\overline{\mathbb{C}}_+, \quad n\to+\infty, \tag{2.5}$$

and

$$\psi_n(z)=[I_2+o(1)]\begin{pmatrix}-i\\ e^{i\frac{z}{2}}\end{pmatrix}e^{-inz}, \quad z\in\overline{\mathbb{C}}_+, \quad n\to-\infty, \tag{2.6}$$

in which $\mathbb{Z}^{\mp}:=\{\mp1,\mp2,\mp3,\cdots\}$ and $\overline{\mathbb{C}}_+:=\{z: z\in\mathbb{C}, \quad \Im(z)\geq 0\}$ such that $\Im(z)$ is imaginary part of z and also, I_2 is the 2x2 identity matrix. Furthermore, it's well-known that these solutions have the following representations [3]:

$$\varphi_n(z)=\begin{pmatrix}\varphi_n^{(1)}(z)\\ \varphi_n^{(2)}(z)\end{pmatrix}=\alpha_n\left(I_2+\sum_{m=1}^{\infty}A_{nm}e^{imz}\right)\begin{pmatrix}e^{i\frac{z}{2}}\\ -i\end{pmatrix}e^{inz}, \quad n\in\mathbb{Z}^+, \tag{2.7}$$

and

$$\psi_n(z)=\begin{pmatrix}\psi_n^{(1)}(z)\\ \psi_n^{(2)}(z)\end{pmatrix}=\beta_n\left(I_2+\sum_{m=-1}^{-\infty}B_{nm}e^{-imz}\right)\begin{pmatrix}-i\\ e^{i\frac{z}{2}}\end{pmatrix}e^{-inz}, \quad n\in\mathbb{Z}^-, \tag{2.8}$$

where

$$\alpha_n=\begin{pmatrix}\alpha_n^{11} & \alpha_n^{12}\\ \alpha_n^{21} & \alpha_n^{22}\end{pmatrix}, \ \beta_n=\begin{pmatrix}\beta_n^{11} & \beta_n^{12}\\ \beta_n^{21} & \beta_n^{22}\end{pmatrix}, \ A_{nm}=\begin{pmatrix}A_{nm}^{11} & A_{nm}^{12}\\ A_{nm}^{21} & A_{nm}^{22}\end{pmatrix}, \ B_{nm}=\begin{pmatrix}B_{nm}^{11} & B_{nm}^{12}\\ B_{nm}^{21} & B_{nm}^{22}\end{pmatrix},$$

and also, A_{nm}^{ij}, B_{nm}^{ij} are expressed in terms of $\{p_n\}_{n\in\mathbb{Z}}$ and $\{q_n\}_{n\in\mathbb{Z}}$ and these vector sequences satisfy that

$$|A_{nm}^{ij}|\leq C\sum_{k=n+[m/2]}^{\infty}(|1-a_k|+|1+b_k|+|p_k|+|q_k|), \quad i,j=1,2,$$

$$|B_{nm}^{ij}|\leq C\sum_{k=n+[m/2]+1}^{-\infty}(|1-a_k|+|1+b_k|+|p_k|+|q_k|), \quad i,j=1,2,$$

where $[m/2]$ is the integer part of $m/2$ and $C>0$ is a constant. Furthermore, these solutions are analytic with respect to z in $\mathbb{C}_+:=\{z: z\in\mathbb{C}, \quad \Im(z)>0\}$ and continuous in $\overline{\mathbb{C}}_+$. Taking into account the equations (2.7) and (2.8), we can easily prove that the solutions $\varphi(z)=\{\varphi_n(z)\}_{n\in\mathbb{Z}^+}$ and $\psi(z)=\{\psi_n(z)\}_{n\in\mathbb{Z}^-}$ satisfy the following asymptotic equations for all z in $\overline{\mathbb{C}}_+$:

$$\varphi_n(z)=e^{inz}\left[-i\begin{pmatrix}\alpha_n^{12}\\ \alpha_n^{22}\end{pmatrix}+o(1)\right], \qquad n\in\mathbb{Z}^+, \quad |z|\to+\infty, \tag{2.9}$$

and

$$\psi_n(z) = e^{-inz}\left[-i\begin{pmatrix}\beta_n^{11}\\ \beta_n^{21}\end{pmatrix} + o(1)\right], \qquad n \in \mathbb{Z}^-, \quad |z| \to +\infty. \tag{2.10}$$

Definition 2.1 *The wronskian of two solutions* $\{Y_n(z)\}_{n\in\mathbb{Z}^*} = \left\{\begin{pmatrix}y_n^{(1)}(z)\\ y_n^{(2)}(z)\end{pmatrix}\right\}_{n\in\mathbb{Z}^*}$ *and* $\{U_n(z)\}_{n\in\mathbb{Z}^*} = \left\{\begin{pmatrix}u_n^{(1)}(z)\\ u_n^{(2)}(z)\end{pmatrix}\right\}_{n\in\mathbb{Z}^*}$ *of the system* (2.1) *is defined by*

$$W\left[Y_n(z), U_n(z)\right] := a_n\left[y_n^{(1)}(z)u_{n+1}^{(2)}(z) - y_{n+1}^{(2)}(z)u_n^{(1)}(z)\right]. \tag{2.11}$$

It's easy to see that $\overline{\varphi(z)}$ and $\overline{\psi(z)}$ are also the solutions of the system (2.1) with the following asymptotic equations in terms of n, respectively

$$\overline{\varphi_n(z)} = [I_2 + o(1)]\begin{pmatrix}e^{-i\frac{z}{2}}\\ i\end{pmatrix}e^{-inz}, \quad z \in \overline{\mathbb{C}}_+, \quad n \to +\infty, \tag{2.12}$$

and

$$\overline{\psi_n(z)} = [I_2 + o(1)]\begin{pmatrix}i\\ e^{-i\frac{z}{2}}\end{pmatrix}e^{inz}, \quad z \in \overline{\mathbb{C}}_+, \quad n \to -\infty. \tag{2.13}$$

Taking into the asymptotic equations (2.7), (2.8) and (2.12), (2.13), we can easily write

$$W\left[\varphi(z), \overline{\varphi(z)}\right] = W\left[\overline{\psi(z)}, \psi(z)\right] = 2i\cos\frac{z}{2}. \tag{2.14}$$

Two fundamental system of solutions of (2.1) are constituted separately by the sets $\left\{\{\varphi_n(z)\}_{n\in\mathbb{Z}^+}, \left\{\overline{\varphi_n(z)}\right\}_{n\in\mathbb{Z}^+}\right\}$ and $\left\{\{\psi_n(z)\}_{n\in\mathbb{Z}^-}, \left\{\overline{\psi_n(z)}\right\}_{n\in\mathbb{Z}^-}\right\}$ for $\lambda = 2\sin\frac{z}{2}$ and $z \in \mathbb{R}\backslash\{(2k+1)\pi : k \in \mathbb{Z}\}$.

We consider the following vector sequences:

$$\Phi_n(z) = \begin{cases} a(z)\psi_n(z) + b(z)\overline{\psi_n(z)}, & n \in \mathbb{Z}^- \\ \varphi_n(z), & n \in \mathbb{Z}^+, \end{cases} \tag{2.15}$$

and

$$\Psi_n(z) = \begin{cases} \psi_n(z), & n \in \mathbb{Z}^- \\ c(z)\varphi_n(z) + d(z)\overline{\varphi_n(z)}, & n \in \mathbb{Z}^+, \end{cases} \tag{2.16}$$

for $z \in \mathbb{R}\backslash\{(2k+1)\pi : k \in \mathbb{Z}\}$.

It is clear that $\Phi(z) = \{\Phi_n(z)\}_{n\in\mathbb{Z}^*}$ and $\Psi(z) = \{\Psi_n(z)\}_{n\in\mathbb{Z}^*}$ are the solutions of the system (2.1) for $\lambda = 2\sin\frac{z}{2}$, and these solutions defined by (2.15) and (2.16) are called *Jost solutions of the system* (2.1)-(2.2).

Theorem 2.2 *In the case that the solution* Φ *provides the condition* (2.2)*, we can obtain* $a(z)$ *and* $b(z)$ *as*

$$a(z) = \frac{a_{-2}}{2icos\dfrac{z}{2}\det \boldsymbol{B}}\left[\overline{v(z)}\varphi_1^{(1)}(z) - \overline{u(z)}\varphi_2^{(2)}(z)\right], \tag{2.17}$$

and

$$b(z) = -\frac{a_{-2}}{2icos\dfrac{z}{2}\det \boldsymbol{B}}\left[v(z)\varphi_1^{(1)}(z) - u(z)\varphi_2^{(2)}(z)\right], \tag{2.18}$$

where

$$\begin{cases} u(z) := \gamma_{11}\psi_{-1}^{(2)}(z) + \gamma_{12}\psi_{-2}^{(1)}(z), \\ v(z) := \gamma_{21}\psi_{-1}^{(2)}(z) + \gamma_{22}\psi_{-2}^{(1)}(z), \end{cases} \tag{2.19}$$

and also the following asymptotic equations are satisfied by these functions, $u(z)$ *and* $v(z)$*, given by* (2.19)*:*

$$u(z) = e^{iz}\left[-i\gamma_{11}\beta_{-1}^{21} + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad |z| \to +\infty, \tag{2.20}$$

$$v(z) = e^{iz}\left[-i\gamma_{21}\beta_{-1}^{21} + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad |z| \to +\infty. \tag{2.21}$$

Proof Let the solution Φ provide the condition (2.2). In this case, we obtain

$$\begin{cases} \Phi_1^{(1)}(z) = \gamma_{11}\Phi_{-1}^{(2)}(z) + \gamma_{12}\Phi_{-2}^{(1)}(z), \\ \Phi_2^{(2)}(z) = \gamma_{21}\Phi_{-1}^{(2)}(z) + \gamma_{22}\Phi_{-2}^{(1)}(z), \end{cases}$$

and if we take into account of the definition of Φ given by (2.15), we can easily get the following system:

$$\begin{cases} \varphi_1^{(1)}(z) = u(z)a(z) + \overline{u(z)}b(z), \\ \varphi_2^{(2)}(z) = v(z)a(z) + \overline{v(z)}b(z), \end{cases}$$

where $u(z)$ and $v(z)$ are defined by (2.19).

We solve $a(z)$ and $b(z)$ from the last system as

$$a(z) = \frac{a_{-2}}{2icos\dfrac{z}{2}\det \mathbf{B}}\left[\overline{v(z)}\varphi_1^{(1)}(z) - \overline{u(z)}\varphi_2^{(2)}(z)\right],$$

and

$$b(z) = -\frac{a_{-2}}{2icos\dfrac{z}{2}\det \mathbf{B}}\left[v(z)\varphi_1^{(1)}(z) - u(z)\varphi_2^{(2)}(z)\right].$$

On the other hand, it's clear that the asymptotic equations (2.20) and (2.21) are obtained from (2.10). □

Similarly, taking into account (2.8) and (2.10), the asymptotic equations

$$\overline{u(z)} = e^{iz}\left[i\gamma_{11}\beta_{-1}^{21} + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad |z| \to +\infty, \tag{2.22}$$

$$\overline{v(z)} = e^{iz}\left[i\gamma_{21}\beta_{-1}^{21} + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad |z| \to +\infty, \tag{2.23}$$

are satisfied and next, we can give the following theorem without proof.

Theorem 2.3 *In the event that the solution* Ψ *provides the condition* (2.2)*, we can obtain* $c(z)$ *and* $d(z)$ *as*

$$c(z) = -\frac{a_1}{2icos\frac{z}{2}}\left[v(z)\overline{\varphi_1^{(1)}(z)} - u(z)\overline{\varphi_2^{(2)}(z)}\right], \tag{2.24}$$

and

$$d(z) = \frac{a_1}{2icos\frac{z}{2}}\left[v(z)\varphi_1^{(1)}(z) - u(z)\varphi_2^{(2)}(z)\right], \tag{2.25}$$

where $u(z)$ *and* $v(z)$ *are given by* (2.19)*.*

From the functions given by (2.17), (2.18), (2.24) and (2.25), it's easy to see that the following relations are valid:

$$c(z) = \frac{a_1}{a_{-2}}\overline{a(z)}\det\mathbf{B}, \qquad d(z) = -\frac{a_1}{a_{-2}}b(z)\det\mathbf{B}. \tag{2.26}$$

On the other hand, using the expressions of $a(z)$ and $b(z)$ given by (2.17) and (2.18), we get

$$M\left|b(z)\right|^2 = \left|v(z)\varphi_1^{(1)}(z)\right|^2 - v(z)\overline{u(z)}\varphi_1^{(1)}(z)\overline{\varphi_2^{(2)}(z)} - u(z)\overline{v(z)}\varphi_2^{(2)}(z)\overline{\varphi_1^{(1)}(z)} + \left|u(z)\varphi_2^{(2)}(z)\right|^2,$$

$$M\left|a(z)\right|^2 = \left|v(z)\varphi_1^{(1)}(z)\right|^2 - u(z)\overline{v(z)}\varphi_1^{(1)}(z)\overline{\varphi_2^{(2)}(z)} - v(z)\overline{u(z)}\varphi_2^{(2)}(z)\overline{\varphi_1^{(1)}(z)} + \left|u(z)\varphi_2^{(2)}(z)\right|^2,$$

where $M := \left(\frac{2}{a_{-2}}cos\frac{z}{2}\det\mathbf{B}\right)^2$, and, from here, we obtain the following relation:

$$\left|b(z)\right|^2 = \left|a(z)\right|^2 - \frac{a_{-2}}{a_1\det\mathbf{B}}, \tag{2.27}$$

and also, using (2.26) and (2.27) we obtain

$$\left|d(z)\right|^2 = \left|c(z)\right|^2 - \frac{a_1\det\mathbf{B}}{a_{-2}}. \tag{2.28}$$

Using the equations given by (2.14), (2.15) and (2.16) and also taking into account the relations (2.27) and (2.28), we can easily obtain the following theorem.

Theorem 2.4 *For* $z \in \mathbb{R}\setminus\{(2k+1)\pi : k \in \mathbb{Z}\}$*, the following wronskians hold:*

(i) $W\left[\Phi(z), \overline{\Phi(z)}\right] = \begin{cases} -\frac{a_{-2}}{a_1\det\boldsymbol{B}}2icos\frac{z}{2}, & n \in \mathbb{Z}^- \\ 2icos\frac{z}{2}, & n \in \mathbb{Z}^+, \end{cases}$

(ii) $W\left[\Psi(z),\overline{\Psi(z)}\right]=\begin{cases}-2icos\dfrac{z}{2}, & n\in\mathbb{Z}^{-}\\ \dfrac{a_1\det\boldsymbol{B}}{a_{-2}}2icos\dfrac{z}{2}, & n\in\mathbb{Z}^{+},\end{cases}$

(iii) $W\left[\Phi(z),\Psi(z)\right]=\begin{cases}2ib(z)cos\dfrac{z}{2}, & n\in\mathbb{Z}^{-}\\ -\dfrac{a_1\det\boldsymbol{B}}{a_{-2}}2ib(z)cos\dfrac{z}{2}, & n\in\mathbb{Z}^{+},\end{cases}$

(iv) $W\left[\Phi(z),\overline{\Psi(z)}\right]=\begin{cases}-2ia(z)cos\dfrac{z}{2}, & n\in\mathbb{Z}^{-}\\ \dfrac{a_1\det\boldsymbol{B}}{a_{-2}}2ia(z)cos\dfrac{z}{2}, & n\in\mathbb{Z}^{+}.\end{cases}$

3. The scattering matrix of L and its properties

We will denote the set of all spectral singularities and eigenvalues of the system (2.1)-(2.2) by $\sigma_{ss}(\mathrm{L})$ and $\sigma_d(\mathrm{L})$, respectively. It's clear that the sets $\sigma_{ss}(\mathrm{L})$ and $\sigma_d(\mathrm{L})$ are given as follows:

$$\sigma_{ss}(\mathrm{L})=\{z:\quad z\in\mathbb{R},\quad b(z)=0\},$$
$$\sigma_d(\mathrm{L})=\{z:\quad z\in\mathbb{C}_+,\quad b(z)=0\}.$$

Theorem 3.1 *$b(z)\neq 0$ for all $z\in\mathbb{R}\backslash\{(2k+1)\pi:k\in\mathbb{Z}\}$.*

Proof On the contrary, there exists a real number z_0 such that $b(z_0)=0$. In this case, we get

$$W\left[\Phi(z_0),\Psi(z_0)\right]=0.$$

Since $W\left[\Phi(z_0),\Psi(z_0)\right]\neq 0$ for all $z\in\mathbb{R}\backslash\{(2k+1)\pi:k\in\mathbb{Z}\}$ from Theorem 2.4, the assumption can't be true. □

Remark 3.2 *Since $b(z)\neq 0$ for all $z\in\mathbb{R}\backslash\{(2k+1)\pi:k\in\mathbb{Z}\}$, we can obtain $\sigma_{ss}(\mathrm{L})=\emptyset$.*

It is clear that the set $\left\{\Psi,\overline{\Psi}\right\}$ forms the fundamental system of solutions of (2.1) for all $z\in\mathbb{R}\backslash\{(2k+1)\pi:k\in\mathbb{Z}\}$. Also, we introduce the following matrix function to investigate the scattering matrix of L

$$\mathrm{S}(z)=\begin{pmatrix}s_{11}(z) & s_{12}(z)\\ s_{21}(z) & s_{22}(z)\end{pmatrix},$$

and thus, we can give the relation of the solutions of L as follows:

$$\begin{pmatrix}\overline{\Phi(z)}\\ \Phi(z)\end{pmatrix}=\mathrm{S}(z)\begin{pmatrix}\overline{\Psi(z)}\\ \Psi(z)\end{pmatrix}.$$

If we consider Theorem 2.4, the components of the scattering matrix of L have the following forms:

$$s_{22}(z)=\overline{s_{11}(z)}=b(z),\quad z\in\mathbb{R}\backslash\{(2k+1)\pi:k\in\mathbb{Z}\},$$

$$s_{21}(z)=\overline{s_{12}(z)}=a(z),\quad z\in\mathbb{R}\backslash\{(2k+1)\pi:k\in\mathbb{Z}\},$$

and also, from (2.27), the determinant of $\mathrm{S}(z)$ is obtained as

$$\det\mathrm{S}(z)=\frac{a_{-2}}{a_1\det\mathbf{B}}.$$

The function $b(z)$ encountered as the component of the scattering matrix of L is called *transmission coefficient*. If the function $b(z)$ is given, the other components of the scattering matrix of L are also easily found by using (2.26) and (2.27).

Finally, we must show that the scattering matrix $\mathrm{S}(z)$ is bounded in all $z\in\overline{\mathbb{C}}_+$. For this purpose, we must obtain the asymptotic equtions of $a(z)$ and $b(z)$.

Theorem 3.3 *Transmission coefficient of the scattering matrix of* L *satisfies:*

(i) $a(z)=e^{5iz/2}\left[K+o(1)\right],\quad z\in\overline{\mathbb{C}}_+,\quad |z|\to+\infty,$

(ii) $b(z)=e^{5iz/2}\left[-K+o(1)\right],\quad z\in\overline{\mathbb{C}}_+,\quad |z|\to+\infty,$
where K *is a complex constant and* $K\neq 0$.

Proof Using the asymptotics (2.22), (2.23) and (2.9) in the expression (2.17), we can easily write

$$a(z)=e^{5iz/2}\left[K+o(1)\right],\quad z\in\overline{\mathbb{C}}_+,\quad |z|\to+\infty,$$

where $K=\dfrac{\alpha_1^{12}\beta_{-1}^{21}a_{-2}\gamma_{22}}{i\det\mathbf{B}}$ is a complex constant and K must be nonzero as the numbers $\alpha_1^{12},\beta_{-1}^{21},a_{-2}$ and γ_{22} are different from zero. Similarly, from (2.20), (2.21) and (2.9) in the expression (2.18), we can easily show the proof of the (ii). □

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