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Global existence and energy decay for a coupled system of Kirchhoff beam equations with weakly damping and logarithmic source

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Abstract: This paper deals with the global solutions and exponential stability for a coupled system of Kirchhoff beam weakly damping and with a logarithmic source. We apply the potential well and establish the global well-posedness by using the Faedo–Galerkin approximations, taking into account that the initial data is located in a suitable set of stability created from the Nehari manifold. Moreover, by using Nakao's lemma, we prove the exponential stability of the solution.

Key words: Coupled Kirchhoff beam system, logarithmic source, global solutions, exponential stability

1. Introduction

In this paper, we consider a coupled system for a beam Kirchhoff system with a nonlinear logarithmic source and frictional damping given by

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2 + |\nabla v|^2)(-\Delta u) + u_t = |u|^{p-2}u \ln |u|^k, \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$v_{tt} + \Delta^2 v + M(|\nabla u|^2 + |\nabla v|^2)(-\Delta v) + v_t = |v|^{p-2}v \ln |v|^k, \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

$$(u(x,0), v(x,0)) = (u_0, v_0), \quad x \in \Omega, \tag{1.3}$$

$$(u_t(x,0), v_t(x,0)) = (u_1, v_1), \quad x \in \Omega, \tag{1.4}$$

$$u(x,t) = \frac{\partial u}{\partial \eta}(x,t) = v(x,t) = \frac{\partial v}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \tag{1.5}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $M(s)$ is a continuous function on $[0,\infty)$ and k is a positive real constant. Furthermore, we assume $(u_0, v_0) \in \left[H_0^2(\Omega)\right]^2$ and $(u_1, v_1) \in \left[L^2(\Omega)\right]^2$.

The nonlinear logarithmic source $|u|^{p-2}u \ln |u|^k$ arises in the inflation cosmology and supersymmetric fields, see [16]. This kind of problems are truly interesting. For wave equation with strong damping and source term $f(u)$,

$$u_{tt} - \Delta u - \Delta u_t = f(u), \text{ in } \Omega \times (0,\infty),$$

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when the source term is just polynomial $f(u) = |u|^{p-2}u$ with $2 < p < \infty$ for $n = 1, 2$ or $2 < p < \dfrac{2n}{n-2}$ for $n \geq 3$ under the appropriated conditions in [30] was proved that the weak solution blows up in finite time. However, when we consider a source logarithmic $f(u) = u \ln |u|^k$ in [26] was proved that the solution exists globally or occurs infinite blow-up under some appropriate conditions. This is a significant difference between the model with polynomial nonlinear source term and the model with logarithmic nonlinear source term.

The one-dimensional nonlinear equation of motion of an elastic string

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0}{m} + \frac{k}{2mL} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.6}$$

where τ_0 is the initial tension, m the mass of the string and k the Young's modulus of the material of the string in connection with some problems in nonlinear elasticity, was proposed by Kirchhoff (1883) [21] and rediscovered by Carrier (1945) [8]. This model describes small vibrations of a stretched string of the length L when only the transverse component of the tension is considered. For mathematical aspects of (1.6) see Bernstein (1940) [4].

The model (1.6) is a generalization of the linearized problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\tau_0}{m} \frac{\partial^2 u}{\partial x^2} = 0,$$

obtained by Euler (1707–1783) and d'Alembert (1714–1793). A particular case of (1.6) can be written, in general, as

$$\frac{\partial^2 u}{\partial t^2} - M \left(\int_\Omega |\nabla u(x,t)|^2 dx \right) \Delta u = 0, \tag{1.7}$$

or

$$\frac{\partial^2 u}{\partial t^2} + M \left(\|u(t)\|^2 \right) Au = 0, \tag{1.8}$$

in the operator notation, where we consider the Hilbert spaces $V \hookrightarrow H \hookrightarrow V'$, where V' is the dual of V with the immersions continuous and dense. By $\| \cdot \|$ we denote the norm in V and $A : V \to V'$ a bounded linear operator.

Berger [3] established the equation

$$u_{tt} - \left(Q + \int_\Omega |\nabla u|^2 dx \right) \Delta u = p(u, u_t, x) \tag{1.9}$$

which is called the Berger plate model [11], where the parameter Q describes in-plane forces applied to the plate and the function p represents transverse loads which may depend on the displacement u and the velocity u_t. If $n = 2$, Equation (1.9) represents the Berger approximation of the Von Kárman equations, modelling the nonlinear vibrations of a plate (see [45], pp. 501-507).

One of the first mathematical analysis for the Kirchhoff-type beam equation

$$u_{tt} + u_{xxxx} - M \left(\int_0^L |u_x|^2 \, dx \right) u_{xx} = 0 \tag{1.10}$$

was done by Ball [2]. Later Tucsnak [40] considered the beam equation for $\Omega \subset \mathbb{R}^n$

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) = 0. \tag{1.11}$$

The authors in [14] have studied strong solutions as well as the exponential decay of the energy to the mixed problem for the nonlinear beam equation in noncylindrical domain $Q_T \subset \mathbb{R}^2$

$$u_{tt} + u_{xxxx} - M\left(\int_0^L |u_x|^2\,dx\right)u_{xx} + \nu u_t = 0. \tag{1.12}$$

The authors considered $M(\,\cdot\,)$ a real function such that $M(\lambda) \geq -m_0$, for all $\lambda \geq 0$ and $m_0 > 0$.

The global existence and the longtime dynamics of solutions for an extensible beam equation with nonlinear damping and source terms

$$u_{tt} + \Delta^2 u + M\left(\int_\Omega |\nabla u|^2\,dx\right)(-\Delta u) + f(u) + g(u_t) = h(x) \quad \text{in } \Omega \times (0,\infty), \tag{1.13}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary, was considered in [41].

The nonlinear and damped extensible plate (or beam) equation below was considered in [9],

$$u_{tt} + \Delta^2 u + \alpha u + M\left(\int_\Omega |\nabla u|^2\,dx\right)(-\Delta u) + f(u) + g(u_t) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.14}$$

where Ω is any bounded or unbounded open set of $\mathbb{R}^n, \alpha > 0$ and f, g are power like functions. The existence of global solutions was proved by means of the fixed point theorem and continuity arguments.

Long-time behavior of solutions to the nonlinear plate equation with nonlocal weak damping given by

$$u_{tt} + \Delta^2 u + g(u) + M\left(\int_\Omega |\nabla u|^2\,dx\right)u_t = f \quad \text{in } \Omega \times (0,\infty), \tag{1.15}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary and f, g are external forcing terms, was presented in [20].

In the past years, several authors have been devoted to the study of qualitative properties of solutions for nonlinear wave and beam equations with damping and source terms. In the sequel, we mention some of them. The initial boundary problem for the two-dimensional Kirchhoff-type wave equation with an exponentially growing source term was presented in [25]. Existence, decay and blow up of solutions for the extensible beam equation with nonlinear damping and source terms was presented in [35]. Long-time behavior of extensible beam equation with the nonlocal weak damping on a bounded smooth domain of $\mathbb{R}^n$ with hinged (clamped) boundary condition was considered in [46]. The authors proved the well-posedness by employing the monotone operator theory and the existence of a global attractor when the growth exponent of the nonlinearity is up to the critical case in natural energy space. Global attractor is a basic concept in the study of longtime dynamics of nonlinear evolution equations with various dissipation. The existence of an attractor for the Kirchhoff equation with a strong dissipation was considered for instance in [12, 27, 42, 43]. Existence results for Kirchhoff-type boundary value problems have been established by using variational methods and critical point theory in [5, 6, 17–19]. The damped wave equation with a degenerate nonlocal weak damping and the nonlinear source was studied in [10]. Asymptotic stability of solutions of Kirchhoff systems, governed by the fractional p-Laplacian operator, with an external force and nonlinear damping terms was considered in [39]. Existence and the energy decay estimate of global solutions for an extensible beam equation with internal damping and source terms were studied in [33]. Blow-up of positive initial-energy solutions for the extensible beam equation with nonlinear damping and source terms were given in [37]. In [31], the existence of the solution to the mixed problem for coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory source term was proved. The global existence of global solution and asymptotic behavior of the energy for wave equations of Kirchhoff type with nonlocal boundary condition was proved in [15]. The existence, uniqueness, and uniform decay rates of the energy of solution for a nonlinear degenerate coupled beams system with weak damping were presented in [23]. The nonexistence of global solutions for positive initial energy was proved by Pişkin and Ekinci [36] for the coupled nonlinear Kirchhoff type equation with degenerate damping and source terms. Nonexistence of solutions for a Timoshenko equations with weak dissipation was give in [38]. The abstract formulation of the coupled system was given by Pereira et al. In [34], the existence of solutions by Faedo-Galerkin's method was studied and the exponential decay by using Nakao's Lemma was proved. Regarding problems with logarithmic nonlinearity, we cite [26, 47] and references therein. In view of the works mentioned above, to our knowledge, much less effort has been devoted to Coupled Kirchhoff beam equation with frictional damping and logarithmic source. Kirchhoff's function $M(\cdot)$ considered in this manuscript is more general than in those in the previous works, since that, we suppose just the continuity and moreover, it can even assume negative values. The outline of the paper is as follows. In Section 2 we introduce some notations and present some hypotheses and previous lemmas needed in the proof of our results. In Section 3 we introduce the stability set, associated with the nonlinear source term, created from the Nehari manifold. In Section 4, we prove the existence of solution through the Faedo–Galerkin method. By using a result of Nakao [29], the exponential stability is given in Section 5.

2. Assumptions and results

In this section, we give some assumptions and results, which play an essential role in proving our results. We denote $|\cdot|$ the Lebesgue Space $L^2(\Omega)$ norm and $|\cdot|_p$ we denote the space $L^p(\Omega)$ norm.

(H.1) $M \in C([0,\infty),\mathbb{R})$ such that $M(\lambda) \geq -\beta$, $\forall\ \lambda \geq 0$, $0 < \beta < \lambda_1$, λ_1 the first eigenvalue of the problem $\Delta^2 u - \lambda(-\Delta u) = 0$.

Remark 2.1 *Let λ_1 be the first eigenvalue of $\Delta^2 u - \lambda(-\Delta u) = 0$ with the campled boundary conditions $u\big|_{\partial\Omega} = \dfrac{\partial u}{\partial \eta}\Big|_{\partial\Omega} = 0$ then, (see Miklin [28])*

$$\lambda_1 = \inf_{u \in H_0^2(\Omega)} \frac{|\Delta u|^2}{|\nabla u|^2} \quad \textit{and} \quad |\nabla u|^2 \leq \frac{1}{\lambda_1}|\Delta u|^2.$$

(H.2) We suppose that $2 \leq p < \infty$ if $n \leq 2$ and $2 \leq p < \dfrac{n+2}{n-2}$ if $n \geq 3$.

Lemma 2.2 (Nakao's Lemma, [29]) *Suppose that $\varphi(t)$ is a bounded nonnegative function on $\mathbb{R}^+$, satisfying*

$$\operatorname*{ess\,sup}_{t \leq s \leq t+1} \varphi(s) \leq C_0[\varphi(t) - \varphi(t+1)]$$

for $t \geq 0$, *where* C_0 *is a positive constant. Then,*

$$\varphi(t) \leq Ce^{-\alpha t}, \text{ for all } t \geq 0,$$

where C_0 *and* α *are positive constants.*

Lemma 2.3 (Sobolev-Poincaré inequality, [13]) *Let* p *be a number with* $2 < p < \infty$ *if* $n = 1, 2$ *or* $2 \leq p \leq \dfrac{2n}{n-2}$ *if* $n \geq 3$, *then there exists a constant* $C > 0$ *such that*

$$|u|_p \leq C|\nabla u|, \quad \text{for all} \ \ u \in H_0^1(\Omega).$$

3. Potential well

In this section, we use the potential well theory, a powerful tool in the study of the global existence of solution to partial differential equation first developed by Payne and Sattinger [32].

It is well-known that the energy of a PDE system, in some sense, splits into kinetic and potential energy. We define the energy of the system (1.1)–(1.5) by

$$\begin{aligned} E(t) =& \frac{1}{2}\left[|u_t(t)|^2 + |v_t(t)|^2 + |\Delta u(t)|^2 + |\Delta v(t)|^2 + \hat{M}\left(|\nabla u(t)|^2 + |\nabla v(t)|^2\right)\right] \\ &+ \frac{k}{p^2}\left(|u(t)|_p^p + |v(t)|_p^p\right) - \frac{1}{p}\left(\int_\Omega |u(t)|^p \ln|u(t)|^k dx + \int_\Omega |v(t)|^p \ln|v(t)|^k dx\right). \end{aligned} \tag{3.1}$$

Multiplying (1.1), (1.2) by u_t, v_t respectively, after we integrate over Ω, performing integration by parts and using boundary conditions, we obtain

$$\frac{d}{dt}E(t) = -|u_t|^2 - |v_t|^2. \tag{3.2}$$

In general, it is possible that the energy from the source term causes the blow-up in a finite time. However, the potential energy creates a valley or a well of the depth d. We can split the well into two sets. For solutions with the initial data in the good part of the well, the potential energy of the solution can never escape and as a result, the total energy of the solution remains finite for all interval $[0, T)$, providing the global existence of the solution.

By following the idea of Ye [44], we will construct the invariant stability set corresponding to the logarithmic source term in the good part of the well. We proceed by defining the functionals I, $J : [H_0^2(\Omega)]^2 \to \mathbb{R}$.

By $(H.1)$ we get

$$\hat{M}\left(|\nabla u(t)|^2 + |\nabla v(t)|^2\right) \geq -\frac{\beta}{\lambda_1}\left(|\Delta u(t)|^2 + |\Delta v(t)|^2\right).$$

Then we introduce the functionals

$$\begin{aligned} I(t) =& \frac{1}{2}\left[|u_t(t)|^2 + |v_t(t)|^2 + \left(1 - \frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2 + |\Delta v|^2\right)\right] \\ &+ \frac{k}{p^2}\left(|u(t)|_p^p + |v(t)|_p^p\right) - \frac{1}{p}\left(\int_\Omega |u(t)|^p \ln|u(t)|^k dx + \int_\Omega |v(t)|^p \ln|v(t)|^k dx\right) \end{aligned} \tag{3.3}$$

and

$$J(u,v) \overset{\text{def}}{=} \frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right)+\frac{k}{p^2}\left(|u|_p^p+|v|_p^p\right)-\frac{1}{p}\left(\int_\Omega |u|^p \ln|u|^k dx+\int_\Omega |v|^p \ln|v|^k dx\right). \quad (3.4)$$

For $\lambda \geq 0$ and $(u,v) \in [H_0^2(\Omega)]^2$ we introduce the Nehari functional

$$J(\lambda u,\lambda v) = \frac{\lambda^2}{2}\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right)+\frac{k\lambda^p}{p}\left(|u|_p^p+|v|_p^p\right)-\frac{\lambda^p}{p}\left(\int_\Omega |u|^p \ln|u|^k dx+\int_\Omega |v|^p \ln|v|^k dx\right). \quad (3.5)$$

Associated with J, we have the Nehari manifold

$$\mathcal{N} \overset{\text{def}}{=} \left\{(u,v) \in [H_0^2(\Omega)]^2 \backslash \{(0,0)\}; \left[\frac{d}{dt}J(\lambda u,\lambda v)\right]_{\lambda=1}=0\right\}. \quad (3.6)$$

Equivalently

$$\mathcal{N} = \left\{(u,v) \in \left(H_0^2(\Omega)\right)^2 \backslash \{(0,0)\}; \frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right)=\frac{1}{p}\left(\int_\Omega |u|^p \ln|u|^k dx+\int_\Omega |v|^p \ln|v|^k dx\right)\right\}. \quad (3.7)$$

We define as in the mountain pass theorem due to Ambrosetti and Rabinowitz [1],

$$d \overset{\text{def}}{=} \inf_{(u,v)\in (H_0^2(\Omega))^2 \backslash \{(0,0)\}} \sup_{\lambda \geq 0} J(\lambda u,\lambda v). \quad (3.8)$$

Similar to the results in [45] one has

$$0 < d = \inf_{(u,v)\in \mathcal{N}} J(u,v). \quad (3.9)$$

Now, we introduce the potential well

$$W = \left\{(u,v) \in [H_0^2(\Omega)]^2; J(u,v) < d\right\} \cup \{(0,0)\},$$

and partition it into two sets as follows:

$$W_1 = \left\{(u,v) \in [H_0^2(\Omega)]^2; \frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right) > \frac{1}{p}\left(\int_\Omega \left(|u|^p \ln|u|^k+|v|^p \ln|v|^k\right) dx\right)\right\} \cup \{(0,0)\}, \quad (3.10)$$

$$W_2 = \left\{(u,v) \in [H_0^2(\Omega)]^2; \frac{1}{2}\left(1-\frac{\beta}{\lambda_1}\right)\left(|\Delta u|^2+|\Delta v|^2\right) < \frac{1}{p}\left(\int_\Omega \left(|u|^p \ln|u|^k dx+\ln|v|^k\right) dx\right)\right\}. \quad (3.11)$$

Now, we define by W_1 the set of stability for the problem (1.1)–(1.5). Before start the section of existence and uniqueness of solution, we will prove that W_1 is invariant set for subcritical initial energy $E(0) < d$.

Proposition 3.1 *Let* $(u_0, v_0) \in [W_1]^2$ *and* $(u_1, v_1) \in [L^2(\Omega)]^2$. *If* $E(0) < d$ *then* $u(t) \in W_1$.

Proof Let $T > 0$ be the maximum existence time. From (3.2) we get

$$E(t) \leq E(0) < d, \text{ for all } t \in [0, T).$$

and then,

$$\frac{1}{2}\left[|u_t(t)|^2 + |v_t(t)|^2\right] + J(u(t), v(t)) < d, \text{ for all } t \in [0, T). \tag{3.12}$$

Note that in W_1 we have $I(u(t)) > 0$ for all $t \in (0, T)$. Arguing by contradiction, we suppose that there exists a first $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$ and $I(u(t)) > 0$ for all $0 \leq t < t_0$, that is,

$$\frac{1}{2}\left[|u_t(t_0)|^2 + |v_t(t_0)|^2\right] + J(u(t_0), v(t_0)) = 0.$$

From the definition of $\mathcal{N}$, we have that $(u(t_0), v(t_0)) \in \mathcal{N}$, which leads to

$$J(u(t_0), v(t_0)) \geq \inf_{(u(t),v(t))\in\mathcal{N}} J(u(t), v(t)) = d.$$

We deduce

$$\frac{1}{2}\left[|u_t(t_0)|^2 + |v_t(t_0)|^2\right] + J(u(t_0), v(t_0)) \geq d,$$

which contradicts with (3.12). Then $(u(t), v(t)) \in W_1$ for all $t \in [0, T)$. □

4. Existence of global solutions

In this section we prove the existence of global weak solutions of system (1.1)–(1.5).

Theorem 4.1 *Let* $(u_0, v_0) \in [W_1]^2$, $(u_1, v_1) \in [L^2(\Omega)]^2$ *and* $E(0) < d$. *If the hypothesis (H.1) and (H.2) holds, then there exists functions* $u, v : [0, T) \to L^2(\Omega)$ *in the class*

$$(u, v) \in \left[L^\infty(0, T; H_0^2(\Omega))\right]^2, \tag{4.1}$$

$$(u_t, v_t) \in \left[L^\infty(0, T; L^2(\Omega))\right]^2, \tag{4.2}$$

such that for all $(w, z) \in [H_0^2(\Omega)]^2$

$$\frac{d}{dt}(u_t(t), w) + (\Delta u(t), \Delta w) + M\left(|\nabla u(t)|^2 + |\nabla v(t)|^2\right)(-\Delta u(t), w)$$
$$+(u_t(t), w) - (|u(t)|^{p-2}u(t)\ln|u(t)|^k, w) = 0, \quad \text{in } \mathcal{D}'(0, T), \tag{4.3}$$

$$\frac{d}{dt}(v_t(t), z) + (\Delta v(t), \Delta z) + M\left(|\nabla u(t)|^2 + |\nabla v(t)|^2\right)(-\Delta v(t), z)$$
$$+(v_t(t), z) - (|v(t)|^{p-2}v(t)\ln|v(t)|^k, z) = 0, \quad \text{in } \mathcal{D}'(0, T), \tag{4.4}$$

$$(u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1). \tag{4.5}$$

Proof We use the Faedo–Galerkin's method to prove the global existence of solutions. First the consider the approximated problem. Then we obtain the a priori estimates needed to passage to the limit in the approximated solutions. □

4.1. Approximated problem

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H_0^2(\Omega)$ by the eigenvectors of the operator $-\Delta$, that is $-\Delta w_\nu = \lambda_\nu w_\nu$, with $\lambda_\nu \to \infty$ when $\nu \to \infty$ and $w_\nu\big|_{\partial\Omega} = \dfrac{\partial w_\nu}{\partial \eta}\Big|_{\partial\Omega} = 0$.

Let $V_m = \text{Span}[w_1, w_2, \dots, w_m]$ and

$$u^m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j, \quad v^m(t) = \sum_{j=1}^{m} h_{jm}(t) w_j$$

be a solution of the approximated system

$$(u_{tt}^m(t), w) + (\Delta u^m(t), \Delta w) + M\left(|\nabla u^m(t)|^2 + |\nabla v^m(t)|^2\right)(-\Delta u^m(t), w)$$
$$+(u_t^m(t), w) - (|u^m(t)|^{p-2}u^m(t)\ln|u^m(t)|^k, w) = 0, \tag{4.6}$$
$$(v_{tt}^m(t), z) + (\Delta v^m(t), \Delta z) + M\left(|\nabla u^m(t)|^2 + |\nabla v^m(t)|^2\right)(-\Delta v^m(t), z)$$
$$+(v_t^m(t), z) - (|v^m(t)|^{p-2}v^m(t)\ln|v^m(t)|^k, z) = 0, \tag{4.7}$$
$$(u^m(0), v^m(0)) = (u_{0m}, v_{0m}) \to (u_0, v_0) \text{ strongly in } [H_0^2(\Omega)]^2, \tag{4.8}$$
$$(u_t^m(0), v_t^m(0)) = (u_{1m}, v_{1m}) \to (u_1, v_1) \text{ strongly in } [L^2(\Omega)]^2. \tag{4.9}$$

The system (4.6)–(4.9) has a local solution in $[0, t_m)$, $0 < t_m \leq T$, by virtue of Carathéodory's theorem, see [7]. The following a priori estimates allow us to extend this solutions to the interval $[0, T)$.

4.2. A priori estimates

Let $w = u_t^m(t)$ and $z = v_t^m(t)$ in (4.5) and (4.6), respectively, integrating from 0 to t, $0 \leq t \leq t_m$, and summing up the results, we obtain

$$\frac{1}{2}\left[|u_t^m(t)|^2 + |v_t^m(t)|^2 + |\Delta u^m(t)|^2 + |\Delta v^m(t)|^2 + \hat{M}\left(|\nabla u^m(t)|^2 + |\nabla v^m(t)|^2\right)\right] +$$
$$\frac{k}{p^2}\left(|u^m(t)|_p^p + |v^m(t)|_p^p\right) - \frac{1}{p}\left(\int_\Omega |u^m(t)|^p \ln|u^m(t)|^k dx + \int_\Omega |v^m(t)|^p \ln|v^m(t)|^k dx\right) +$$
$$\int_0^t \left(|u_t^m(s)|^2 + |v_t^m(s)|^2\right) ds =$$
$$\frac{1}{2}\left[|u_{1m}|^2 + |v_{1m}|^2 + |\Delta u_{0m}|^2 + |\Delta v_{0m}|^2 + \hat{M}\left(|\nabla u_{0m}|^2 + |\nabla v_{0m}|^2\right)\right] +$$
$$\frac{k}{p^2}\left(|u_{0m}|_p^p + |v_{0m}|_p^p\right) - \frac{1}{p}\left(\int_\Omega |u_{0m}|^p \ln|u_{0m}|^k dx + \int_\Omega |v_{0m}|^p \ln|v_{0m}|^k dx\right), \tag{4.10}$$

where

$$\hat{M}(s) = \int_0^s M(s)ds.$$

Let

$$E_m(t) = \frac{1}{2}\left[|u_t^m(t)|^2 + |v_t^m(t)|^2 + |\Delta u^m(t)|^2 + |\Delta v^m(t)|^2 + \hat{M}\left(|\nabla u^m(t)|^2 + |\nabla v^m(t)|^2\right)\right]$$

$$+ \frac{k}{p^2}\left(|u^m(t)|_p^p + |v^m(t)|_p^p\right) - \frac{1}{p}\left(\int_\Omega |u^m(t)|^p \ln |u^m(t)|^k dx - \int_\Omega |v^m(t)|^p \ln |v^m(t)|^k dx\right). \qquad (4.11)$$

Then,

$$E_m(t) + \int_0^t \left(|u_t^m(s)|^2 + |v_t^m(s)|^2\right) ds = E_m(0). \qquad (4.12)$$

Now, by (H.2) it follows that

$$\hat{M}\left(|\nabla u^m(t)|^2 + |\nabla v^m(t)|^2\right) \geq -\frac{\beta}{\lambda_1}\left(|\Delta u^m(t)|^2 + |\Delta v^m(t)|^2\right) \qquad (4.13)$$

and

$$\hat{M}\left(|\nabla u_{0m}|^2 + |\nabla v_{0m}|^2\right) \leq \frac{m_0}{\lambda_1}\left(|\Delta u_{0m}|^2 + |\Delta v_{0m}|^2\right) \qquad (4.14)$$

where $m_0 = \max_{0 \leq s \leq |\nabla u_{0m}|^2 + |\nabla v_{0m}|^2 \leq C} M(s)$.

Then by (4.10)–(4.14), we have

$$\frac{1}{2}\left[|u_t^m(t)|^2 + |v_t^m(t)|^2 + \left(1 - \frac{\beta}{\lambda_1}\right)\left(|\Delta u^m(t)|^2 + |\Delta v^m(t)|^2\right)\right] +$$

$$\frac{k}{p^2}\left(|u^m(t)|_p^p + |v^m(t)|_p^p\right) - \frac{1}{p}\left(\int_\Omega |u^m(t)|^p \ln |u^m(t)|^k dx + \int_\Omega |v^m(t)|^p \ln |v^m(t)|^k dx\right)$$

$$\leq E_m(t) \leq E_m(0) = \frac{1}{2}|u_{1m}|^2 + \frac{1}{2}|v_{1m}|^2 + C_1 J(u_{0m}, v_{0m}), \qquad (4.15)$$

where $C_1 = C_1(m_0, \lambda_1, \beta)$.

We have by (4.9) and $J(u_0, v_0) < d$, for sufficiently large m and $t \geq 0$, $J(u_{0m}, v_{0m}) < d$ and $\frac{1}{2}\left(|u_{1m}|^2 + |v_{1m}|^2\right) \leq C_1$, $C_1 > 0$ constant independent of m and t. Hence, by (4.15)

$$\frac{1}{2}\left[|u_t^m(t)|^2 + |v_t^m(t)|^2 + \left(1 - \frac{\beta}{\lambda_1}\right)\left(|\Delta u^m(t)|^2 + |\Delta v^m(t)|^2\right)\right] +$$

$$\frac{k}{p^2}\left(|u^m(t)|_p^p + |v^m(t)|_p^p\right) - \frac{1}{p}\left(\int_\Omega |u^m(t)|^p \ln |u^m(t)|^k dx + \int_\Omega |v^m(t)|^p \ln |v^m(t)|^k dx\right) \leq C_2, \qquad (4.16)$$

$C_2 > 0$ is a constant independent of m and t.

Then we can extended the approximated solutions $(u^m(t), v^m(t))$ to the interval $[0, T)$, $T > 0$.

So, by (4.12) and (4.16) it follows that

$$(u^m, v^m) \text{ are bounded in } \left[L^\infty\left(0,T;H_0^2(\Omega)\right)\cap L^\infty\left(0,T;L^p(\Omega)\right)\right]^2, \tag{4.17}$$

$$(u_t^m, v_t^m) \text{ are bounded in } \left[L^\infty\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;L^2(\Omega)\right)\right]^2. \tag{4.18}$$

4.3. Passage to the limit

From the estimates (4.17) and (4.18), there exists subsequences of $(u^m)_{m\in\mathbb{N}}$ and $(v^m)_{m\in\mathbb{N}}$, that we denote by $(u^m)_{m\in\mathbb{N}}$ and $(v^m)_{m\in\mathbb{N}}$ such that as $m\to\infty$

$$(u^m, v^m) \overset{*}{\rightharpoonup} (u,v) \text{ weakly star in } \left[L^\infty(0,T;H_0^2(\Omega))\right]^2, \tag{4.19}$$

$$(u_t^m, v_t^m) \overset{*}{\rightharpoonup} (u_t,v_t) \text{ weakly star in } \left[L^\infty(0,T;L^2(\Omega))\right]^2. \tag{4.20}$$

From (4.19) and (4.20) and Aubin–Lions's Lemma ([22] Theorem 5.1), we get

$$(u^m, v^m) \to (u,v) \text{ strongly in } \left[L^2(0,T;H_0^2(\Omega))\right]^2. \tag{4.21}$$

In particular,

$$(u^m, v^m) \to (u,v) \text{ strongly in } \left[L^2(0,T;L^2(\Omega))\right]^2 \tag{4.22}$$

and so

$$(u^m, v^m) \to (u,v) \text{ a.e. in } \Omega\times(0,T). \tag{4.23}$$

Since M is continuos, it follows that

$$M(|\nabla u^m|^2+|\nabla v^m|^2) \to M(|\nabla u|^2+|\nabla v|^2) \text{ strongly in } L^2(0,T). \tag{4.24}$$

Therefore

$$M\left(|\nabla u^m|^2+|\nabla v^m|^2\right)(-\Delta u^m,-\Delta v^m) \rightharpoonup M\left(|\nabla u|^2+|\nabla v|^2\right)(-\Delta u,-\Delta v)$$
$$\text{weakly in } [L^2\left(0,T;L^2(\Omega)\right)]^2. \tag{4.25}$$

Now, we observe that Sobolev inequality leads to

$$\int_\Omega \left||u^m(t)|^{p-2}u^m(t)\ln|u^m(t)|\right|^2dx \le |u^m(t)|_{2p}^{2p} \le C^{2p}|\nabla u^m|^{2p} \le \frac{C^{2p}}{\lambda_1^p}|\Delta u^m|^2 \le C_3.$$

Similarly,

$$\int_\Omega \left||v^m(t)|^{p-2}v^m(t)\ln|v^m(t)|\right|^2dx \le C_3,$$

with $C_3>0$ constant independent of m and t.

Whence,

$$\left(|u^m|^{p-2}u^m\ln|u^m|^k\right) \text{ and } \left(|v^m|^{p-2}v^m\ln|v^m|^k\right) \text{ are bounded in } L^2\left(0,T;L^2(\Omega)\right). \tag{4.26}$$

By (4.22) we have

$$\left(|u^m|^{p-2}u^m \ln |u^m|^k\right) \to \left(|u|^{p-2}u \ln |u|^k\right) \text{ a.e. in } \Omega \times (0,T), \tag{4.27}$$

$$\left(|v^m|^{p-2}v^m \ln |v^m|^k\right) \to \left(|v|^{p-2}v \ln |v|^k\right) \text{ a.e. in } \Omega \times (0,T). \tag{4.28}$$

So, by (4.26)–(4.28) and Lions's Lemma [22], we have

$$\left(|u^m|^{p-2}u^m \ln |u^m|^k, |v^m|^{p-2}v^m \ln |v^m|^k\right) \rightharpoonup \left(|u|^{p-2}u \ln |u|^k, |v|^{p-2}v \ln |v|^k\right)$$
$$\text{weakly in } \left[L^2\left(0,T;L^2(\Omega)\right)\right]^2. \tag{4.29}$$

By the convergence (4.19), (4.20), (4.24)–(4.29), we can pass to the limit approximated equations (4.6) and (4.7) and obtain, for all $w, z \in V_m$, in $\mathcal{D}'(0,T)$

$$\frac{d}{dt}(u_t(t), w) + (\Delta u(t), \Delta w) + M\left(|\nabla u(t)|^2 + |\nabla v(t)|^2\right)(-\Delta u(t), w) + (u_t(t), w) - (|u|^{p-2}u \ln |u|^k, w) = 0 \tag{4.30}$$

and

$$\frac{d}{dt}(v_t(t), z) + (\Delta v(t), \Delta z) + M\left(|\nabla u(t)|^2 + |\nabla v(t)|^2\right)(-\Delta v(t), z) + (v_t(t), z) - (|v|^{p-2}v \ln |v|^k, z) = 0. \tag{4.31}$$

As V_m is dense in $H_0^2(\Omega)$, the equations (4.30) and (4.31) are valid for all $w, z \in H_0^2(\Omega)$. The verification of the initial data is obtained in a standard way. The proof of Theorem 4.1 is complete.

5. Exponential decay

In this section we study the asymptotic behavior of solutions to the system (1.1)–(1.5). We show using the Nakao's method that the energy associated the system is exponentially stable. The main result of this paper is given by the following theorem.

Theorem 5.1 *Under the hypotheses of Theorem 4.1 the energy associated to system (1.1)–(1.5) satisfies*

$$E(t) \leq C_0 e^{-\alpha t}, \quad \textit{for all } t \geq 0,$$

where C_0 and α are positive constants.

Proof Let $w = u_t(t)$ and $z = v_t(t)$ in the equations (4.30) and (4.31), respectively, and summing up the results, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}|u_t(t)|^2 + \frac{1}{2}|v_t(t)|^2 + \frac{1}{2}|\Delta u(t)|^2 + \frac{1}{2}|\Delta u(t)|^2 + \frac{1}{2}\hat{M}\left(|\nabla u(t)|^2 + |\nabla v(t)|^2\right) + \frac{k}{p^2}\left(|u(t)|_p^p + |v(t)|_p^p\right) \right.$$
$$\left. - \frac{1}{p}\left(\int_\Omega |u(t)|^p \ln |u(t)|^k dx + \int_\Omega |v(t)|^p \ln |v(t)|^k dx\right)\right] + |u_t(t)|^2 + |v_t(t)|^2 = 0. \tag{5.1}$$

As the energy is given by

$$E(t) = \frac{1}{2}\left[|u_t(t)|^2 + |v_t(t)|^2 + |\Delta u(t)|^2 + |\Delta v(t)|^2 + \hat{M}\left(|\nabla u(t)|^2 + |\nabla v(t)|^2\right)\right] + \frac{k}{p^2}\left(|u(t)|_p^p + |v(t)|_p^p\right) - \frac{1}{p}\left(\int_\Omega |u(t)|^p \ln|u(t)|^k dx + \int_\Omega |v(t)|^p \ln|v(t)|^k dx\right), \tag{5.2}$$

then by (5.1) and (5.2), we have

$$\frac{d}{dt}E(t) + |u_t(t)|^2 + |v_t(t)|^2 \leq 0. \tag{5.3}$$

Integrating from t to $t+1$, $t \geq 0$, we obtain

$$\int_t^{t+1}\left(|u_t(s)|^2 + |v_t(s)|^2\right) ds \leq E(t) - E(t+1) \overset{\text{def}}{=} F^2(t). \tag{5.4}$$

Thus, there exists $t_1 \in \left[t, t+\frac{1}{4}\right]$, $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$|u_t(t_i)| + |v_t(t_i)| \leq 4F(t), \quad i = 1, 2. \tag{5.5}$$

Let $w = u(t)$ and $z = v(t)$ in the equations (4.30) and (4.31), respectively. Integrating from t_1 to t_2, summing up the results and using the hyphotesis (H.1), we have

$$\int_{t_1}^{t_2}\left[\left(1 - \frac{\beta}{\lambda_1}\right)\left(|\Delta u(s)|^2 + |\Delta v(s)|^2\right) - \int_\Omega |u(s)|^p \ln|u(s)|^k dx - \int_\Omega |v(s)|^p \ln|v(s)|^k dx\right] ds \leq |u_t(t_1)||u(t_1)| + |u_t(t_2)||u(t_2)| + |v_t(t_1)||v(t_1)| + |v_t(t_2)||v(t_2)| + \int_{t_1}^{t_2}\left(|u_t(s)||u(s)| + |v_t(s)||v(s)|\right) dx. \tag{5.6}$$

From (5.5) and Sobolev-Poincaré inequality, we get

$$|u_t(t_i)||u(t_i)| + |v_t(t_i)||v(t_i)| \leq 8\tilde{C}F(t)\left(|\Delta u(t_i)| + |\Delta v(t_i)|\right) \leq 8\tilde{C}F(t) \operatorname*{ess\,sup}_{t\leq s\leq t+1} E^{\frac{1}{2}}(s), \tag{5.7}$$

$i = 1, 2$ and $\tilde{C} > 0$ constant such that $|u| \leq \tilde{C}|\Delta u|$ and

$$\int_{t_1}^{t_2}\left(|u_t(s)||u(s)| + |v_t(s)||v(s)|\right) ds \leq \tilde{C}\int_{t_1}^{t_2}\left(|u_t(s)||\Delta u(s)| + |v_t(s)||\Delta v(s)|\right) ds \leq \frac{\tilde{C}^2}{\delta}F^2(t) + \delta\int_{t_1}^{t_2}\left(|\Delta u(s)|^2 + |\Delta v(s)|^2\right) ds, \tag{5.8}$$

where $0 < \delta < 1 - \dfrac{\beta}{\lambda_1}$.

By (5.6)–(5.8), we obtain

$$\int_{t_1}^{t_2} \left[\left(1 - \frac{\beta}{\lambda_1} - \delta\right) \left(|\Delta u(s)|^2 + |\Delta v(s)|^2\right) - \int_\Omega |u(s)|^p \ln |u(s)|^k dx - \int_\Omega |u(s)|^p \ln |u(s)|^k dx \right] ds \leq$$

$$C_1 \left[F(t) \operatorname*{ess\,sup}_{t \leq s \leq t+1} E^{\frac{1}{2}}(s) + F^2(t) \right] \stackrel{\text{def}}{=} G^2(t). \quad (5.9)$$

From (5.3) and (5.8) it follows that

$$\int_{t_1}^{t_2} E(s) ds \leq C_2 [F^2(t) + G^2(t)].$$

Then, there exists $t^* \in [t_1, t_2]$ such that

$$E(t^*) \leq C_3 [F^2(t) + G^2(t)]. \quad (5.10)$$

From (5.3)

$$\begin{aligned} \operatorname*{ess\,sup}_{t \leq s \leq t+1} E(s) &\leq E(t^*) + \int_t^{t+1} \left(|u_t(s)|^2 + |v_t(s)|^2\right) ds \\ &\leq C_4 [F^2(t) + G^2(t)] \leq C_5 \left[F^2(t) + F(t) \operatorname*{ess\,sup}_{t \leq s \leq t+1} E^{\frac{1}{2}}(s) \right] \\ &\leq C_6 F^2(t) + \frac{1}{2} \operatorname*{ess\,sup}_{t \leq s \leq t+1} E(s), \end{aligned}$$

therefore, by (5.4), we get

$$\operatorname*{esssup}_{t \leq s \leq t+1} E(s) \leq C_7 F^2(t) = C_7 [E(t) - E(t+1)],$$

where $Ci > 0,\ i = 1, 2, \ldots, 7$ are positive constants.

By Lemma 2.2, we have

$$E(t) \leq C e^{-\alpha t}, \quad \text{for all } t \geq 0,$$

where C and α are positive constant.

□

6. Conclusion and open problems

Our approach was made to subcritical initial energy $E(0) < d$. We strongly use the fact that the initial energy to be positive in the set of stability, created by the Nehari manifold. The cases for critical initial $E(0) = d$ and supercritical initial energy $E(0) > d$ also can be analyzed. It is an interesting question to analyze the finite time blow-up behavior in the instability set.

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