

## A discussion on the existence and uniqueness analysis for the coupled two-term fractional differential equations

Sachin Kumar VERMA<sup>a</sup>, Ramesh Kumar VATS<sup>b</sup>, Avadhesh KUMAR<sup>c</sup>,  
Velusamy VIJAYAKUMAR<sup>d</sup> Anurag SHUKLA<sup>e,1\*</sup>

<sup>a</sup>Department of Mathematics, Govt College for Women Bawani Khera Bhiwani, Haryana, India

<sup>b</sup>Department of Mathematics and Scientific Computing, NIT Hamirpur HP-177005, India

<sup>c</sup>Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning, AP-515134, India

<sup>d</sup>Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology,  
Vellore - 632 014, Tamil Nadu, India

<sup>e</sup>Department of Applied Science, Rajkiya Engineering College Kannauj, Kannauj-209732, India

Received: 10.07.2021

Accepted/Published Online: 02.09.2021

Final Version: ..2021

**Abstract:** This paper mainly concentrates on the study of a new boundary value problem of coupled nonlinear two-term fractional differential system. We make use of the theories on fractional calculus and fixed point approach to derive the existence and uniqueness results of the considered two-term fractional systems. To confirm the application of the stated outcomes, two examples are provided.

**Key words:** Multiterm fractional differential systems, coupled system, fixed point theorem

### 1. Introduction

In the last few decades, the principles of fractional calculus have played the main role in Mathematics. Differential equations of integer order cannot suit some physical problems, but these kind of problems fit in the differential equations of fractional order. Recently, many researchers have done valuable performances in electromagnetic, control theory, signal, porous media, viscoelasticity, biological, engineering problems, image processing, fluid flow, diffusion, theology, etc. For more specifics, [1–5, 10–47] and references therein. In some circumstances, we have to solve the equation containing more than one derivative term. This type of equation is known as a multiterm fractional differential equation. A standard example of a two-term fractional differential equation is the Bagley–Torvik equation [6]:

$$R\Psi''(t) + S {}^c D^{\frac{3}{2}}\Psi(t) + T\Psi(t) = \hbar(t),$$

where  $\hbar$  is a given function and  $R, S$  and  $T$  are some constants. The above equation makes an appearance in motion modeling of a thin plate in a Newtonian fluid. Bagley–Torvik equation was discussed from a numerical point of view by Diethelm and Ford [9]. Another example of multiterm fractional differential equation is the Basset equation [7, 8].

$$\Psi'(t) + Q {}^c D^{\mu}\Psi(t) + R\Psi(t) = w(t), \Psi(0) = \Psi_0, 0 < \mu < 1.$$

\*Correspondence: anuragshukla259@gmail.com

2010 AMS Mathematics Subject Classification: 26A33, 34B15.

Basset equation is often applied with  $\mu = \frac{1}{2}$ . It represents the forces that arise if a spherical item is dropped into an incompressible viscous fluid.

Kaufmann and Yao [19] proved the existence of the two-term boundary value problem (BVP) having fractional order of the form

$$\begin{cases} (D^\mu - aD^\nu)u(t) + f(t, u(t)) = 0, & 1 < \mu < 2, 0 < \nu < \mu, t \in [0, 1], \\ u(0) = 0, u(1) = 0, \end{cases}$$

where  $D^\mu$  is the Riemann–Liouville (R-L) fractional derivative and  $a \in \mathbb{R}$ .

Ibrahim et al. [16] study the existence of solutions for a BVP of the form

$$\begin{cases} {}^cD^\mu u(t) - a {}^cD^\nu u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u_0, u(1) = u_1, \end{cases}$$

where  ${}^cD^\mu$  and  ${}^cD^\nu$  are the Caputo fractional derivative with  $1 < \mu \leq 2$  and  $1 \leq \nu < \mu$ ,  $a \in \mathbb{R}$  is a constant.

This paper concerns the existence and uniqueness of solutions of the following coupled system of two-term fractional differential systems of the form

$$\begin{cases} {}^cD^{\xi_1} \Psi_1(t) + \alpha_1 {}^cD^{\sigma_1} \Psi_1(t) + \beta_1 h_1(t, \Psi_1(t), \Psi_2(t)) = 0, \\ {}^cD^{\xi_2} \Psi_2(t) + \alpha_2 {}^cD^{\sigma_2} \Psi_2(t) + \beta_2 h_2(t, \Psi_1(t), \Psi_2(t)) = 0, \\ \Psi_1(0) = \Psi_1''(0) = \Psi_2(0) = \Psi_2''(0) = 0, \\ \rho_1(I^{\nu_1} \Psi_1)(\varepsilon_1) = \Psi_1(1), \rho_2(I^{\nu_2} \Psi_2)(\varepsilon_2) = \Psi_2(1), \end{cases} \quad (1.1)$$

where  $t \in [0, 1]$ ,  $0 < \varepsilon_i < 1$ ,  $2 < \xi_i \leq 3$ ,  $2 \leq \sigma_i < \xi_i$ ,  $\nu_i > 0$ ,  $\alpha_i, \beta_i \in \mathbb{R}$ ,  $i = 1, 2$ , and  ${}^cD^{\xi_i}$  stands for the Caputo fractional derivative of order  $\xi_i$ ,  $i = 1, 2$ ,  $I^{\nu_i}$  are the R-L fractional integral of order  $\nu_i$ ,  $h_i : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  are continuous functions and  $\rho_i \in \mathbb{R}$ , then

$$\rho_i \neq \frac{\Gamma(\nu_i + \xi_i - \sigma_i + 2) + a_i \frac{\Gamma(\nu_i + \xi_i - \sigma_i + 2)}{\Gamma(\xi_i - \sigma_i + 2)}}{a_i \varepsilon_i^{\nu_i + \xi_i - \sigma_i + 1} + \varepsilon_i^{\nu_i + 1} \frac{\Gamma(\nu_i + \xi_i - \sigma_i + 2)}{\Gamma(\nu_i + 2)}}, \quad i = 1, 2.$$

Let  $V = C([0, 1], \mathbb{R})$ , then obviously  $(V, \|\cdot\|_V)$  is a Banach space provided

$$\|\Psi\|_V = \{\sup |\Psi(t)| : t \in [0, 1]\}.$$

## 2. Preliminaries

We provide fundamental realities, thoughts and lemmas wanted to arrange the primary consequences of our paper.

**Definition 2.1** [35] *The Caputo fractional derivative of order  $q$  for a continuous function  $f : [0, \infty) \rightarrow \mathbb{R}$  is given by*

$${}^cD^q f(t) = \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{n - q - 1} f^{(n)}(s) ds, \quad n = [q] + 1,$$

if  $f^{(n)}(t)$  exists, where  $[q]$  denotes the integer part of the real number  $q$ .

**Definition 2.2** [35] The Riemann–Liouville fractional integral of order  $q$  for  $f(t)$  is presented by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad q > 0,$$

if such integral exists.

**Lemma 2.3** [10] Let  $m, n > 0$  and  $h \in P_1[a, b]$  then  $I^m I^n h = I^{m+n} h$ .

**Lemma 2.4** [10] Let  $\nu > -1$  and  $n > 0$ . Then

$$I^n x^\nu = \frac{\Gamma(\nu+1)}{\Gamma(n+\nu+1)} x^{n+\nu}.$$

**Lemma 2.5** [23] Let  $\mu > 0$ , then

$$I^\mu {}^c D^\mu v(t) = v(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

for some  $c_i \in \mathbb{R}$ ,  $i = 0, 1, 2, \dots, n-1$ , where  $n = [\mu]$ .

### 3. Auxiliary result

For proving the existence of (1.1), we have to convert the system (1.1) into the equivalent integral equations.

**Lemma 3.1** Let  $\rho \neq \frac{\Gamma(\nu+\xi-\sigma+2)+a\frac{\Gamma(\nu+\xi-\sigma+2)}{\Gamma(\xi-\sigma+2)}}{a\varepsilon^{\nu+\xi-\sigma+1}+\varepsilon^{\nu+1}\frac{\Gamma(\nu+\xi-\sigma+2)}{\Gamma(\nu+2)}}$ . Then for  $\zeta \in V$ , the solution of

$$\begin{cases} {}^c D^\xi \Psi(t) + a {}^c D^\sigma \Psi(t) + b \zeta(t) = 0, \\ \Psi(0) = \Psi''(0) = 0, \\ \rho(I^\nu \Psi)(\varepsilon) = \Psi(1), 2 < \xi \leq 3, 2 \leq \sigma < \xi, \end{cases} \quad (3.1)$$

satisfies the integral equation

$$\begin{aligned} \Psi(t) &= \frac{-a}{\Gamma(\xi-\sigma)} \int_0^t (t-s)^{\xi-\sigma-1} \Psi(s) ds - \frac{b}{\Gamma(\xi)} \int_0^t (t-s)^{\xi-1} \zeta(s) ds \\ &+ Q \left( \frac{t\Gamma(\xi-\sigma+2) + at^{\xi-\sigma+1}}{\Gamma(\xi-\sigma+2)} \right) \left[ \frac{\rho a}{\Gamma(\nu+\xi-\sigma)} \right. \\ &\times \int_0^\varepsilon (\varepsilon-s)^{\nu+\xi-\sigma-1} \Psi(s) ds + \frac{\rho b}{\Gamma(\nu+\xi)} \int_0^\varepsilon (\varepsilon-s)^{\nu+\xi-1} \zeta(s) ds \\ &\left. - \frac{a}{\Gamma(\xi-\sigma)} \int_0^1 (1-s)^{\xi-\sigma-1} \Psi(s) ds - \frac{b}{\Gamma(\xi)} \int_0^1 (1-s)^{\xi-1} \zeta(s) ds \right], \end{aligned} \quad (3.2)$$

where

$$Q = \frac{\gamma}{\rho a \varepsilon^{\nu+\xi-\sigma+1} + \rho \varepsilon^{\nu+1} \frac{\gamma}{\Gamma(\nu+2)} - a \frac{\gamma}{\Gamma(\xi-\sigma+2)} - \gamma},$$

and  $\gamma = \Gamma(\nu+\xi-\sigma+2)$ .

**Proof** Since  $2 < \xi \leq 3$ , by Lemma 2.5, we have

$$I^{\xi} {}^c D^{\xi} \Psi(t) = \Psi(t) + c_0 + c_1 t + c_2 t^2$$

for some constants  $c_0, c_1, c_2$  and  $t \in [0, 1]$ .

Now

$$\Psi(0) = 0 \Rightarrow c_0 = 0,$$

and

$$\Psi''(0) = 0 \Rightarrow c_2 = 0.$$

So, we have

$$I^{\xi} {}^c D^{\xi} \Psi(t) = \Psi(t) + c_1 t.$$

Implement  $I^{\xi}$  on both sides of the given equation, we have

$$\begin{aligned} I^{\xi} {}^c D^{\xi} \Psi(t) &= -a I^{\xi} {}^c D^{\sigma} \Psi(t) - b I^{\xi} \zeta(t) \\ \Rightarrow \Psi(t) + c_1 t &= -a I^{\xi} {}^c D^{\sigma} \Psi(t) - b I^{\xi} \zeta(t). \end{aligned}$$

Now

$$\begin{aligned} I^{\xi} {}^c D^{\sigma} \Psi(t) &= I^{\xi-\sigma} I^{\sigma} {}^c D^{\sigma} \Psi(t) \\ &= I^{\xi-\sigma} (\Psi(t) + c_1 t) \\ &= I^{\xi-\sigma} \Psi(t) + \frac{c_1 t^{\xi-\sigma+1}}{\Gamma(\xi-\sigma+2)}. \\ \Rightarrow \Psi(t) + c_1 t &= -a I^{\xi-\sigma} \Psi(t) - \frac{ac_1 t^{\xi-\sigma+1}}{\Gamma(\xi-\sigma+2)} - b I^{\xi} \zeta(t). \end{aligned} \tag{3.3}$$

Further

$$\Psi(1) = -a I^{\xi-\sigma} \Psi(1) - c_1 - \frac{ac_1}{\Gamma(\xi-\sigma+2)} - b I^{\xi} \zeta(1).$$

Applying  $I^{\nu}$  to both sides of (3.3), we get

$$\begin{aligned} I^{\nu} \Psi(t) + \frac{c_1 t^{1+\nu}}{\Gamma(\nu+2)} &= -a I^{\nu+\xi-\sigma} \Psi(t) - \frac{ac_1 t^{\nu+\xi-\sigma+1}}{\Gamma(\nu+\xi-\sigma+2)} - b I^{\nu+\xi} \zeta(t) \\ \Rightarrow \rho I^{\nu} \Psi(\varepsilon) + \frac{\rho c_1 \varepsilon^{1+\nu}}{\Gamma(\nu+2)} &= -\rho a I^{\nu+\xi-\sigma} \Psi(\varepsilon) - \frac{\rho a c_1 \varepsilon^{\nu+\xi-\sigma+1}}{\Gamma(\nu+\xi-\sigma+2)} - \rho b I^{\nu+\xi} \zeta(\varepsilon). \end{aligned}$$

Using the boundary condition  $\rho(I^{\nu} \Psi)(\varepsilon) = \Psi(1)$ , we have

$$\begin{aligned} -a I^{\xi-\sigma} \Psi(1) - c_1 - \frac{ac_1}{\Gamma(\xi-\sigma+2)} - b I^{\xi} \zeta(1) + \frac{\rho c_1 \varepsilon^{\nu+1}}{\Gamma(\nu+2)} &= -\rho a I^{\nu+\xi-\sigma} \Psi(\varepsilon) - \frac{\rho a c_1 \varepsilon^{\nu+\xi-\sigma+1}}{\Gamma(\nu+\xi-\sigma+2)} - \rho b I^{\nu+\xi} \zeta(\varepsilon). \end{aligned}$$

Solving for  $c_1$ , we get

$$\begin{aligned} c_1 = & -\frac{\rho a Q}{\Gamma(\nu + \xi - \sigma)} \int_0^\varepsilon (\varepsilon - s)^{\nu + \xi - \sigma - 1} \Psi(s) ds \\ & - \frac{\rho b Q}{\Gamma(\nu + \xi)} \int_0^\varepsilon (\varepsilon - s)^{\nu + \xi - 1} \zeta(s) ds \\ & + \frac{b Q}{\Gamma(\xi)} \int_0^1 (1 - s)^{\xi - 1} \zeta(s) ds + \frac{a Q}{\Gamma(\xi - \sigma)} \int_0^1 (1 - s)^{\xi - \sigma - 1} \Psi(s) ds. \end{aligned}$$

Using  $c_1$  in (3.3), one can attain the solution (3.2). □

By referring Lemma 3.1, the solution of (1.1) is presented as the following integral equations:

$$\begin{aligned} \Psi_1(t) = & -\frac{\alpha_1}{\Gamma(\xi_1 - \sigma_1)} \int_0^t (t - s)^{\xi_1 - \sigma_1 - 1} \Psi_1(s) ds \\ & - \frac{\beta_1}{\Gamma(\xi_1)} \int_0^t (t - s)^{\xi_1 - 1} h_1(s, \Psi_1(s), \Psi_2(s)) ds \\ & + Q_1 \left( \frac{t \Gamma(\xi_1 - \sigma_1 + 2) + \alpha_1 t^{\xi_1 - \sigma_1 + 1}}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\ & \left[ \frac{\rho_1 \alpha_1}{\Gamma(\nu_1 + \xi_1 - \sigma_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - \sigma_1 - 1} \Psi_1(s) ds \right. \\ & + \frac{\rho_1 \beta_1}{\Gamma(\nu_1 + \xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - 1} h_1(s, \Psi_1(s), \Psi_2(s)) ds \\ & - \frac{\alpha_1}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1 - s)^{\xi_1 - \sigma_1 - 1} \Psi_1(s) ds \\ & \left. - \frac{\beta_1}{\Gamma(\xi_1)} \int_0^1 (1 - s)^{\xi_1 - 1} h_1(s, \Psi_1(s), \Psi_2(s)) ds \right], \end{aligned}$$

and

$$\begin{aligned} \Psi_2(t) = & -\frac{\alpha_2}{\Gamma(\xi_2 - \sigma_2)} \int_0^t (t - s)^{\xi_2 - \sigma_2 - 1} \Psi_2(s) ds \\ & - \frac{\beta_2}{\Gamma(\xi_2)} \int_0^t (t - s)^{\xi_2 - 1} h_2(s, \Psi_1(s), \Psi_2(s)) ds \\ & + Q_2 \left( \frac{t \Gamma(\xi_2 - \sigma_2 + 2) + a_2 t^{\xi_2 - \sigma_2 + 1}}{\Gamma(\xi_2 - \sigma_2 + 2)} \right) \\ & \left[ \frac{\rho_2 \alpha_2}{\Gamma(\nu_2 + \xi_2 - \sigma_2)} \int_0^{\varepsilon_2} (\varepsilon_2 - s)^{\nu_2 + \xi_2 - \sigma_2 - 1} \Psi_2(s) ds \right. \\ & + \frac{\rho_2 \beta_2}{\Gamma(\nu_2 + \xi_2)} \int_0^{\varepsilon_2} (\varepsilon_2 - s)^{\nu_2 + \xi_2 - 1} h_2(s, \Psi_1(s), \Psi_2(s)) ds \\ & - \frac{\alpha_2}{\Gamma(\xi_2 - \sigma_2)} \int_0^1 (1 - s)^{\xi_2 - \sigma_2 - 1} \Psi_2(s) ds \\ & \left. - \frac{\beta_2}{\Gamma(\xi_2)} \int_0^1 (1 - s)^{\xi_2 - 1} h_2(s, \Psi_1(s), \Psi_2(s)) ds \right], \end{aligned}$$

where

$$Q_i = \frac{\gamma_i}{\rho_i \alpha_i \varepsilon_i^{\nu_i + \xi_i - \sigma_i + 1} + \rho_i \varepsilon_i^{\nu_i + 1} \frac{\gamma_i}{\Gamma(\nu_i + 2)} - \alpha_i \frac{\gamma_i}{\Gamma(\xi_i - \sigma_i + 2)} - \gamma_i}$$

and  $\gamma_i = \Gamma(\nu_i + \xi_i - \sigma_i + 2)$  for  $i = 1, 2$ .

Let  $A = V \times V$ . Then,  $(A, \|\cdot\|_A)$  is also a Banach space along with

$$\|(\Psi_1, \Psi_2)\|_A = \|\Psi_1\|_V + \|\Psi_2\|_V.$$

Let us define an operator  $\chi : A \rightarrow A$  as

$$\chi(\Psi_1, \Psi_2)(t) = (\chi_1(\Psi_1, \Psi_2)(t), \chi_2(\Psi_1, \Psi_2)(t)), \tag{3.4}$$

where

$$\begin{aligned} \chi_1(\Psi_1, \Psi_2)(t) = & -\frac{\alpha_1}{\Gamma(\xi_1 - \sigma_1)} \int_0^t (t-s)^{\xi_1 - \sigma_1 - 1} \Psi_1(s) ds \\ & - \frac{\beta_1}{\Gamma(\xi_1)} \int_0^t (t-s)^{\xi_1 - 1} h_1(s, \Psi_1(s), \Psi_2(s)) ds \\ & + Q_1 \left( \frac{t\Gamma(\xi_1 - \sigma_1 + 2) + \alpha_1 t^{\xi_1 - \sigma_1 + 1}}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\ & \left[ \frac{\rho_1 \alpha_1}{\Gamma(\nu_1 + \xi_1 - \sigma_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - \sigma_1 - 1} \Psi_1(s) ds \right. \\ & + \frac{\rho_1 \beta_1}{\Gamma(\nu_1 + \xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - 1} h_1(s, \Psi_1(s), \Psi_2(s)) ds \\ & - \frac{\alpha_1}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1-s)^{\xi_1 - \sigma_1 - 1} \Psi_1(s) ds \\ & \left. - \frac{\beta_1}{\Gamma(\xi_1)} \int_0^1 (1-s)^{\xi_1 - 1} h_1(s, \Psi_1(s), \Psi_2(s)) ds \right], \end{aligned}$$

and

$$\begin{aligned} \chi_2(\Psi_1, \Psi_2)(t) = & -\frac{\alpha_2}{\Gamma(\xi_2 - \sigma_2)} \int_0^t (t-s)^{\xi_2 - \sigma_2 - 1} \Psi_2(s) ds \\ & - \frac{\beta_2}{\Gamma(\xi_2)} \int_0^t (t-s)^{\xi_2 - 1} h_2(s, \Psi_1(s), \Psi_2(s)) ds \\ & + Q_2 \left( \frac{t\Gamma(\xi_2 - \sigma_2 + 2) + a_2 t^{\xi_2 - \sigma_2 + 1}}{\Gamma(\xi_2 - \sigma_2 + 2)} \right) \\ & \left[ \frac{\rho_2 \alpha_2}{\Gamma(\nu_2 + \xi_2 - \sigma_2)} \int_0^{\varepsilon_2} (\varepsilon_2 - s)^{\nu_2 + \xi_2 - \sigma_2 - 1} \Psi_2(s) ds \right. \\ & + \frac{\rho_2 \beta_2}{\Gamma(\nu_2 + \xi_2)} \int_0^{\varepsilon_2} (\varepsilon_2 - s)^{\nu_2 + \xi_2 - 1} h_2(s, \Psi_1(s), \Psi_2(s)) ds \\ & - \frac{\alpha_2}{\Gamma(\xi_2 - \sigma_2)} \int_0^1 (1-s)^{\xi_2 - \sigma_2 - 1} \Psi_2(s) ds \\ & \left. - \frac{\beta_2}{\Gamma(\xi_2)} \int_0^1 (1-s)^{\xi_2 - 1} h_2(s, \Psi_1(s), \Psi_2(s)) ds \right]. \end{aligned}$$

Observe that the fixed point of  $\chi$  is the solution of (1.1).

For our comfort, we assume

$$\begin{aligned} P_i = & \frac{|\alpha_i|}{\Gamma(\xi_i - \sigma_i + 1)} + |Q_i| \left( \frac{\Gamma(\xi_i - \sigma_i + 2) + |\alpha_i|}{\Gamma(\xi_i - \sigma_i + 2)} \right) \\ & \left( \frac{|\rho_i \alpha_i|}{\Gamma(\nu_i + \xi_i - \sigma_i + 1)} + \frac{|\alpha_i|}{\Gamma(\xi_i - \sigma_i + 1)} \right), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} J_i = & \frac{|\beta_i|}{\Gamma(\xi_i + 1)} + |Q_i| \left( \frac{\Gamma(\xi_i - \sigma_i + 2) + |\alpha_i|}{\Gamma(\xi_i - \sigma_i + 2)} \right) \\ & \left( \frac{|\rho_i \beta_i|}{\Gamma(\nu_i + \xi_i + 1)} + \frac{|\beta_i|}{\Gamma(\xi_i + 1)} \right), \end{aligned} \tag{3.6}$$

for  $i = 1, 2$ .

#### 4. Main results

**Theorem 4.1** Assume that  $\exists p_i, j_i \geq 0$  ( $i = 1, 2$ ) and  $p_0, j_0 > 0$  such that  $\forall \Psi_i \in \mathbb{R}$ , we have

$$|h_1(t, \Psi_1, \Psi_2)| \leq p_0 + p_1 \|\Psi_1\| + p_2 \|\Psi_2\|,$$

$$|h_2(t, \Psi_1, \Psi_2)| \leq j_0 + j_1 \|\Psi_1\| + j_2 \|\Psi_2\|.$$

with

$$P_1 + P_2 + J_1 p_1 + J_2 j_1 < 1,$$

$$\text{and } J_1 p_2 + J_2 j_2 < 1,$$

therefore, the system (1.1) must have a solution.

**Proof** Since  $h_1$  and  $h_2$  are continuous, therefore  $\chi$  is also continuous. We claim  $\chi$  is a completely continuous or compact operator.

Let  $(\Psi_1, \Psi_2) \in \beta_\epsilon = \{(\Psi_1, \Psi_2) \in A; \|(\Psi_1, \Psi_2)\|_A \leq \epsilon\}$  where  $\epsilon > 0$

Then  $\|\Psi_i\|_V \leq \mu_i$  for some  $\mu_i, i = 1, 2$ .

Let  $t \in [0, 1]$ , we have

$$\begin{aligned}
 |\chi_1(\Psi_1, \Psi_2)(t)| &\leq \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^t (t-s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \\
 &+ \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^t (t-s)^{\xi_1 - 1} |h_1(s, \Psi_1(s), \Psi_2(s))| ds \\
 &+ |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \left[ \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1)} \right. \\
 &\times \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \\
 &+ \left. \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - 1} |h_1(s, \Psi_1(s), \Psi_2(s))| ds \right. \\
 &+ \left. \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1-s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \right. \\
 &+ \left. \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^1 (1-s)^{\xi_1 - 1} |h_1(s, \Psi_1(s), \Psi_2(s))| ds \right] \\
 &\leq \left[ \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \right. \\
 &\left( \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1 + 1)} + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} \right) \left. \right] \|\Psi_1\|_V \\
 &+ \left[ \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \right. \\
 &\left( \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} \right) \left. \right] [p_0 + p_1 \|\Psi_1\|_V + p_2 \|\Psi_2\|_V] \\
 &\leq J_1 p_0 + (P_1 + J_1 p_1) \mu_1 + J_1 p_2 \mu_2.
 \end{aligned}$$

Thus, we have

$$\|\chi_1(\Psi_1, \Psi_2)\|_V \leq J_1 p_0 + (P_1 + J_1 p_1) \mu_1 + J_1 p_2 \mu_2.$$

Similarly, we get

$$\|\chi_2(\Psi_1, \Psi_2)\|_V \leq J_2 j_0 + (P_2 + J_2 j_1) \mu_1 + J_2 j_2 \mu_2.$$

From the above inequalities, it follows that bounded sets of  $A$  are mapped into bounded sets of  $A$  under the mapping  $\chi$ .



Next, we show that  $\chi(\beta_\epsilon)$  is equicontinuous. Let  $0 \leq t_1 < t_2 \leq 1$  and  $(\Psi_1, \Psi_2) \in \beta_\epsilon$ . Then we have

$$\begin{aligned}
 & |\chi_1(\Psi_1, \Psi_2)(t_2) - \chi_1(\Psi_1, \Psi_2)(t_1)| \\
 & \leq \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \left| \int_0^{t_1} [(t_2 - s)^{\xi_1 - \sigma_1 - 1} - (t_1 - s)^{\xi_1 - \sigma_1 - 1}] \Psi_1(s) ds \right. \\
 & \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\xi_1 - \sigma_1 - 1} \Psi_1(s) ds \right| \\
 & \quad + \frac{|\beta_1|}{\Gamma(\xi_1)} \left| \int_0^{t_1} [(t_2 - s)^{\xi_1 - 1} - (t_1 - s)^{\xi_1 - 1}] \hbar_1(s, \Psi_1(s), \Psi_2(s)) ds \right. \\
 & \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\xi_1 - 1} \hbar_1(s, \Psi_1(s), \Psi_2(s)) ds \right| \\
 & \quad + |Q_1| \left( \frac{|t_2 - t_1| \Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1| |t_2^{\xi_1 - \sigma_1 + 1} - t_1^{\xi_1 - \sigma_1 + 1}|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\
 & \quad \left[ \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1)} \int_0^{\epsilon_1} (\epsilon_1 - s)^{\nu_1 + \xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \right. \\
 & \quad \left. + \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1)} \int_0^{\epsilon_1} (\epsilon_1 - s)^{\nu_1 + \xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s))| ds \right. \\
 & \quad \left. + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1 - s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \right. \\
 & \quad \left. + \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^1 (1 - s)^{\xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s))| ds \right] \\
 & \leq \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} |t_2^{\xi_1 - \sigma_1} - t_1^{\xi_1 - \sigma_1}| \mu_1 \\
 & \quad + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} |t_2^{\xi_1} - t_1^{\xi_1}| [p_0 + p_1 \mu_1 + p_2 \mu_2] \\
 & \quad + |Q_1| \left( \frac{|t_2 - t_1| \Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1| |t_2^{\xi_1 - \sigma_1 + 1} - t_1^{\xi_1 - \sigma_1 + 1}|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\
 & \quad \left[ \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1 + 1)} \mu_1 + \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} [p_0 + p_1 \mu_1 + p_2 \mu_2] \right. \\
 & \quad \left. + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} \mu_1 + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} [p_0 + p_1 \mu_1 + p_2 \mu_2] \right].
 \end{aligned}$$

Now the right-hand side approaches to zero when  $t_1$  approaches to  $t_2$ . Thus

$$\|\chi_1(\Psi_1, \Psi_2)(t_2) - \chi_1(\Psi_1, \Psi_2)(t_1)\|_V \rightarrow 0 \text{ as } t_1 \rightarrow t_2.$$

Similarly

$$\|\chi_2(\Psi_1, \Psi_2)(t_2) - \chi_2(\Psi_1, \Psi_2)(t_1)\|_V \rightarrow 0 \text{ as } t_1 \rightarrow t_2.$$

Thus

$$\|\chi(\Psi_1, \Psi_2)(t_2) - \chi(\Psi_1, \Psi_2)(t_1)\|_A \rightarrow 0, \text{ as } t_1 \rightarrow t_2.$$

Combining steps I to III and using the consequence of Arzelá–Ascoli theorem,  $\chi : A \rightarrow A$  is a completely continuous operator.

Let  $\Theta = \{(\Psi_1, \Psi_2) \in A : (\Psi_1, \Psi_2) = \theta\chi(\Psi_1, \Psi_2) \text{ for some } \theta \in (0, 1)\}$ .

We will show that the set  $\Theta$  is bounded.

Let  $(\Psi_1, \Psi_2) \in \Theta \Rightarrow (\Psi_1, \Psi_2)(t) = \theta\chi(\Psi_1, \Psi_2)(t)$  for some  $\theta \in (0, 1)$ .

Then we have

$$\Psi_1(t) = \theta\chi_1(\Psi_1, \Psi_2)(t), \quad \Psi_2(t) = \theta\chi_2(\Psi_1, \Psi_2)(t), \quad \forall t \in [0, 1].$$

$$\begin{aligned} |\Psi_1(t)| &= |\theta\chi_1(\Psi_1, \Psi_2)(t)| \\ &\leq \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^t (t-s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \\ &\quad + \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^t (t-s)^{\xi_1 - 1} |h_1(s, \Psi_1(s), \Psi_2(s))| ds \\ &\quad + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \left[ \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1)} \right. \\ &\quad \times \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \\ &\quad + \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - 1} |h_1(s, \Psi_1(s), \Psi_2(s))| ds \\ &\quad + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1-s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s)| ds \\ &\quad \left. + \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^1 (1-s)^{\xi_1 - 1} |h_1(s, \Psi_1(s), \Psi_2(s))| ds \right] \\ &\leq \left[ \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \right. \\ &\quad \left. \left( \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1 + 1)} + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} \right) \right] \|\Psi_1\|_V \\ &\quad + \left[ \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \right. \\ &\quad \left. \left( \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} \right) \right] [p_0 + p_1 \|\Psi_1\|_V + p_2 \|\Psi_2\|_V]. \end{aligned}$$

Thus, we have

$$\|\Psi_1\|_V \leq J_1 p_0 + (P_1 + J_1 p_1) \|\Psi_1\|_V + J_1 p_2 \|\Psi_2\|_V.$$

Similarly, we get

$$\|\Psi_2\|_V \leq J_2 j_0 + (P_2 + J_2 j_1) \|\Psi_1\|_V + J_2 j_2 \|\Psi_2\|_V.$$

i.e.

$$\begin{aligned} \|(\Psi_1, \Psi_2)\|_A &\leq J_1 p_0 + J_2 j_0 + (P_1 + J_1 p_1 + P_2 + J_2 j_1) \|\Psi_1\|_V \\ &\quad + (J_1 p_2 + J_2 j_2) \|\Psi_2\|_V. \end{aligned}$$

which implies that

$$\|(\Psi_1, \Psi_2)\|_A \leq \frac{J_1 p_0 + J_2 j_0}{N},$$

where

$$N = \min\{1 - P_1 - J_1 p_1 - P_2 - J_2 j_1, 1 - J_1 p_2 - J_2 j_2\},$$

$\Rightarrow \Theta$  is a bounded set.

In view of Schaefer's fixed point theorem,  $\chi$  should have at least one fixed point and that is a solution of (1.1).  $\square$

**Theorem 4.2** Assume that  $\exists$  constants  $\Upsilon_i > 0$ ,  $i = 1, 2$  such that  $\forall t \in [0, 1]$  and  $\forall \Psi_i, \Psi'_i \in \mathbb{R}$ ,  $i = 1, 2$ ,

$$|\hbar_i(t, \Psi_1, \Psi_2) - \hbar_i(t, \Psi'_1, \Psi'_2)| \leq \Upsilon_i(|\Psi_1 - \Psi'_1| + |\Psi_2 - \Psi'_2|),$$

and

$$P_1 + P_2 + \Upsilon_1 J_1 + \Upsilon_2 J_2 < 1,$$

therefore, the system (1.1) has a unique solution defined on  $[0, 1]$ .

**Proof** Let  $(\Psi_1, \Psi_2), (\Psi'_1, \Psi'_2) \in A$  and  $t \in [0, 1]$ , then we have

$$\begin{aligned} & |\chi_1(\Psi_1, \Psi_2)(t) - \chi_1(\Psi'_1, \Psi'_2)(t)| \\ & \leq \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^t (t-s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s) - \Psi'_1(s)| ds \\ & + \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^t (t-s)^{\xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s)) - \hbar_1(s, \Psi'_1(s), \Psi'_2(s))| ds \\ & + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \left[ \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1)} \right. \\ & \times \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - \sigma_1 - 1} |\Psi_1(s) - \Psi'_1(s)| ds \\ & + \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\nu_1 + \xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s)) - \hbar_1(s, \Psi'_1(s), \Psi'_2(s))| ds \\ & + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1)} \int_0^1 (1-s)^{\xi_1 - \sigma_1 - 1} |\Psi_1(s) - \Psi'_1(s)| ds \\ & \left. + \frac{|\beta_1|}{\Gamma(\xi_1)} \int_0^1 (1-s)^{\xi_1 - 1} |\hbar_1(s, \Psi_1(s), \Psi_2(s)) - \hbar_1(s, \Psi'_1(s), \Psi'_2(s))| ds \right] \\ & \leq [P_1 + \Upsilon_1 J_1][\|\Psi_1 - \Psi'_1\|_V + \|\Psi_2 - \Psi'_2\|_V], \end{aligned}$$

and consequently, we obtain

$$\begin{aligned} & \|\chi_1(\Psi_1, \Psi_2) - \chi_1(\Psi'_1, \Psi'_2)\|_V \\ & \leq [P_1 + \Upsilon_1 J_1](\|\Psi_1 - \Psi'_1\|_V + \|\Psi_2 - \Psi'_2\|_V). \end{aligned} \tag{4.1}$$

Similarly

$$\begin{aligned} & \|\chi_2(\Psi_1, \Psi_2) - \chi_2(\Psi'_1, \Psi'_2)\|_V \\ & \leq [P_2 + \Upsilon_2 J_2](\|\Psi_1 - \Psi'_1\|_V + \|\Psi_2 - \Psi'_2\|_V). \end{aligned} \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\begin{aligned} & \|\chi(\Psi_1, \Psi_2) - \chi(\Psi'_1, \Psi'_2)\|_A \\ & \leq (P_1 + P_2 + \Upsilon_1 J_1 + \Upsilon_2 J_2)(\|\Psi_1 - \Psi'_1\|_V + \|\Psi_2 - \Psi'_2\|_V). \end{aligned}$$

Since  $P_1 + P_2 + \Upsilon_1 J_1 + \Upsilon_2 J_2 < 1$ , then  $\chi$  is a contraction. Therefore, by referring Banach fixed point theorem,  $\chi$  must have a unique fixed point i.e. (1.1) has a unique solution.  $\square$

### 5. Examples

#### Example 1

$$\begin{cases} {}^c D^{\frac{13}{5}} \Psi_1(t) + \frac{{}^c D^{\frac{15}{5}} \Psi_1(t)}{1993} + 5 \left( \frac{(t+8)^2 |\Psi_1(t) + \Psi_2(t)|}{(1 + |\Psi_1(t) + \Psi_2(t)|)} + \sin\left(\frac{\Psi_1(t)}{65} + \frac{\Psi_2(t)}{78}\right) \right) = 0, \\ {}^c D^{\frac{17}{7}} \Psi_2(t) + \frac{{}^c D^{\frac{15}{7}} \Psi_2(t)}{1989} + 6 \left( (t+6)^2 + \ln\left(1 + \left|\frac{\Psi_1(t)}{56} + \frac{\Psi_2(t)}{87}\right|\right) \right) = 0, \\ \Psi_1(0) = \Psi_1''(0) = \Psi_2(0) = \Psi_2''(0) = 0, \\ \frac{1}{11}(I^{\frac{8}{5}} \Psi_1)\left(\frac{1}{4}\right) = \Psi_1(1), \quad \frac{1}{13}(I^{\frac{19}{7}} \Psi_2)\left(\frac{1}{5}\right) = \Psi_2(1). \end{cases} \tag{5.1}$$

Here  $\xi_1 = \frac{13}{5}, \alpha_1 = \frac{1}{1993}, \sigma_1 = \frac{11}{5}, \beta_1 = 5, \xi_2 = \frac{17}{7}, \alpha_2 = \frac{1}{1989}, \sigma_2 = \frac{15}{7}, \beta_2 = 6, \rho_1 = \frac{1}{11}, \nu_1 = \frac{8}{5}, \varepsilon_1 = \frac{1}{4}, \rho_2 = \frac{1}{13}, \nu_2 = \frac{19}{7}, \varepsilon_2 = \frac{1}{5}, h_1(t, \Psi_1, \Psi_2) = \frac{(t+8)^2 |\Psi_1 + \Psi_2|}{(1 + |\Psi_1 + \Psi_2|)} + \sin\left(\frac{\Psi_1}{65} + \frac{\Psi_2}{78}\right)$  and  $h_2(t, \Psi_1, \Psi_2) = (t+6)^2 + \ln\left(1 + \left|\frac{\Psi_1}{56} + \frac{\Psi_2}{87}\right|\right)$ .

For all  $\Psi_1, \Psi_2 \in \mathbb{R}$  and  $t \in [0, 1]$ , we have

$$\begin{aligned} |h_1(t, \Psi_1, \Psi_2)| & \leq 81 + \left| \frac{\Psi_1}{65} + \frac{\Psi_2}{78} \right| \\ & \leq 81 + \frac{1}{65} |\Psi_1| + \frac{1}{78} |\Psi_2|, \end{aligned}$$

and

$$\begin{aligned} |h_2(t, \Psi_1, \Psi_2)| & \leq 49 + \left| \frac{\Psi_1}{56} + \frac{\Psi_2}{87} \right| \\ & \leq 49 + \frac{1}{56} |\Psi_1| + \frac{1}{87} |\Psi_2|. \end{aligned}$$

Here  $p_0 = 81, p_1 = \frac{1}{65}, p_2 = \frac{1}{78}, j_0 = 49, j_1 = \frac{1}{56}$  and  $j_2 = \frac{1}{87}$ .

Also

$$Q_1 = -1.00026, \quad Q_2 = -0.999578.$$

Further

$$\begin{aligned}
 & P_1 + P_2 + p_1 J_1 + j_1 J_2 \\
 &= \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\
 &\left( \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1 + 1)} + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} \right) \\
 &+ \frac{|\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 1)} + |Q_2| \left( \frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)} \right) \\
 &\left( \frac{|\rho_2 \alpha_2|}{\Gamma(\nu_2 + \xi_2 - \sigma_2 + 1)} + \frac{|\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 1)} \right) \\
 &+ \frac{p_1 |\beta_1|}{\Gamma(\xi_1 + 1)} + p_1 |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\
 &\left( \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} \right) \\
 &+ \frac{j_1 |\beta_2|}{\Gamma(\xi_2 + 1)} + j_1 |Q_2| \left( \frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)} \right) \\
 &\left( \frac{|\rho_2 \beta_2|}{\Gamma(\nu_2 + \xi_2 + 1)} + \frac{|\beta_2|}{\Gamma(\xi_2 + 1)} \right) \\
 &= 0.113658 < 1,
 \end{aligned}$$

and

$$\begin{aligned}
 & p_2 J_1 + j_2 J_2 \\
 &= \frac{p_2 |\beta_1|}{\Gamma(\xi_1 + 1)} + p_2 |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\
 &\left( \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} \right) \\
 &+ \frac{j_2 |\beta_2|}{\Gamma(\xi_2 + 1)} + j_2 |Q_2| \left( \frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)} \right) \\
 &\left( \frac{|\rho_2 \beta_2|}{\Gamma(\nu_2 + \xi_2 + 1)} + \frac{|\beta_2|}{\Gamma(\xi_2 + 1)} \right) \\
 &= 0.079586 < 1.
 \end{aligned}$$

Therefore, in view of Theorem 4.1, there exists at least one solution for (5.1).

**Example 2**

$$\begin{cases}
 {}^c D^{\frac{5}{2}} \Psi_1(t) + {}^c D^{\frac{7}{3}} \Psi_1(t) + 3 \left( \frac{|\Psi_1(t)|}{(t+9)^2(1+|\Psi_1(t)|)} + \frac{|\Psi_2(t)|}{(t+79)(1+|\Psi_2(t)|)} \right) = 0, \\
 {}^c D^{\frac{9}{4}} \Psi_2(t) + {}^c D^{\frac{11}{5}} \Psi_2(t) + 4 \left( \frac{\tan^{-1} \Psi_1(t)}{89+t} + \frac{\sin(2\pi \Psi_2(t))}{192\pi} \right) = 0, \\
 \Psi_1(0) = \Psi_1''(0) = \Psi_2(0) = \Psi_2''(0) = 0, \\
 \frac{1}{7} (I^{\frac{11}{6}} \Psi_1)\left(\frac{1}{3}\right) = \Psi_1(1), \quad \frac{1}{9} (I^{\frac{19}{20}} \Psi_2)\left(\frac{1}{2}\right) = \Psi_2(1).
 \end{cases} \tag{5.2}$$

In this case,  $\xi_1 = \frac{5}{2}, \alpha_1 = \frac{1}{499}, \sigma_1 = \frac{7}{3}, \beta_1 = 3, \xi_2 = \frac{9}{4}, \alpha_2 = \frac{1}{1947}, \sigma_2 = \frac{11}{5}, \beta_2 = 4, \rho_1 = \frac{1}{7}, \nu_1 = \frac{11}{6}, \varepsilon_1 = \frac{1}{3}, \rho_2 = \frac{1}{9}, \nu_2 = \frac{19}{20}, \varepsilon_2 = \frac{1}{2}, \hbar_1(t, \Psi_1, \Psi_2) = \frac{|\Psi_1|}{(t+9)^2(1+|\Psi_1|)} + \frac{|\Psi_2|}{(t+79)(1+|\Psi_2|)}$  and  $\hbar_2(t, \Psi_1, \Psi_2) = \frac{\tan^{-1} \Psi_1}{89+t} + \frac{\sin(2\pi\Psi_2)}{192\pi}$ .

For all  $\Psi_1, \Psi_2, \Psi'_1, \Psi'_2 \in \mathbb{R}$  and  $t \in [0, 1]$ , we have

$$\begin{aligned} |\hbar_1(t, \Psi_1, \Psi_2) - \hbar_1(t, \Psi'_1, \Psi'_2)| &\leq \frac{1}{81}|\Psi_1 - \Psi'_1| + \frac{1}{79}|\Psi_2 - \Psi'_2| \\ &\leq \frac{1}{79}(|\Psi_1 - \Psi'_1| + |\Psi_2 - \Psi'_2|), \end{aligned}$$

and

$$\begin{aligned} |\hbar_2(t, \Psi_1, \Psi_2) - \hbar_2(t, \Psi'_1, \Psi'_2)| &\leq \frac{1}{89}|\Psi_1 - \Psi'_1| + \frac{1}{96}|\Psi_2 - \Psi'_2| \\ &\leq \frac{1}{89}(|\Psi_1 - \Psi'_1| + |\Psi_2 - \Psi'_2|). \end{aligned}$$

Here  $\Upsilon_1 = \frac{1}{79}$  and  $\Upsilon_2 = \frac{1}{89}$ .

Also

$$Q_1 = -0.9995, \quad Q_2 = -1.0148.$$

Further

$$\begin{aligned} &P_1 + P_2 + \Upsilon_1 J_1 + \Upsilon_2 J_2 \\ &= \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} + |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\ &\quad \left( \frac{|\rho_1 \alpha_1|}{\Gamma(\nu_1 + \xi_1 - \sigma_1 + 1)} + \frac{|\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 1)} \right) \\ &+ \frac{|\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 1)} + |Q_2| \left( \frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)} \right) \\ &\quad \left( \frac{|\rho_2 \alpha_2|}{\Gamma(\nu_2 + \xi_2 - \sigma_2 + 1)} + \frac{|\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 1)} \right) \\ &+ \frac{\Upsilon_1 |\beta_1|}{\Gamma(\xi_1 + 1)} + \Upsilon_1 |Q_1| \left( \frac{\Gamma(\xi_1 - \sigma_1 + 2) + |\alpha_1|}{\Gamma(\xi_1 - \sigma_1 + 2)} \right) \\ &\quad \left( \frac{|\rho_1 \beta_1|}{\Gamma(\nu_1 + \xi_1 + 1)} + \frac{|\beta_1|}{\Gamma(\xi_1 + 1)} \right) \\ &+ \frac{\Upsilon_2 |\beta_2|}{\Gamma(\xi_2 + 1)} + \Upsilon_2 |Q_2| \left( \frac{\Gamma(\xi_2 - \sigma_2 + 2) + |\alpha_2|}{\Gamma(\xi_2 - \sigma_2 + 2)} \right) \\ &\quad \left( \frac{|\rho_2 \beta_2|}{\Gamma(\nu_2 + \xi_2 + 1)} + \frac{|\beta_2|}{\Gamma(\xi_2 + 1)} \right) \\ &= 0.0647757 < 1. \end{aligned}$$

Thus, in view of Theorem 4.2, we deduce that (5.2) has a unique solution.

## Conclusion

Our paper mainly concentrated on the study of a new BVP of coupled nonlinear two-term fractional differential systems. We employed the theories on fractional calculus and fixed point approach to derive the existence and uniqueness results of the considered two-term fractional systems. To confirm the application of the stated outcomes, two examples are provided.

In future work, we focus on the existence results for the BVP of coupled nonlinear two-term fractional impulsive differential systems.

## Conflict of interest

The authors have no conflict of interest.

## References

- [1] Afshari H, Marasi H, Aydi H. Existence and uniqueness of positive solutions for boundary value problems of fractional differential equations. *Filomat* 2017; 31 (9): 2675-2682.
- [2] Agarwal RP, Benchohra M, Hamani S. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. 2010; 109: 973-1033.
- [3] Agarwal RP, Benchohra M, Hamani S. Boundary value problems for fractional differential equations. *Georgian Mathematical Journal* 2009; 16 (3): 401-411.
- [4] Ahmad B, Alsaedi A, Assolami A. Existence results for Caputo type fractional differential equations with four-point nonlocal fractional integral boundary conditions. *Electronic Journal of Qualitative Theory of Differential Equations* 2012; 93: 1-11.
- [5] Ahmad B. Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations. *Applied Mathematics Letters* 2010; 23 (4): 390-394.
- [6] Bagley RL, Torvik PJ. On the appearance of the fractional derivative in the behavior of real materials. *Journal of Applied Mechanics* 1984; 51: 294-298.
- [7] Basset AB. On the descent of a sphere in a viscous liquid. *Quarterly Journal of Pure and Applied Mathematics* 1910; 41: 369-381.
- [8] Basset AB. On the motion of a sphere in a viscous liquid. *Philosophical Transactions of the Royal Society B* 1888; 179: 43-63.
- [9] Diethelm K, Ford NJ. Numerical solution of the Bagley-Torvik equation. *BIT Numerical Mathematics* 2002; 42 (3): 490-507.
- [10] Diethelm K. *The Analysis of Fractional Differential Equations*. Springer, Berlin, 2004.
- [11] Dineshkumar C, Udhayakumar R. New results concerning to approximate controllability of Hilfer fractional neutral stochastic delay integro-differential systems. *Numerical Methods for Partial Differential Equations* 2021; 37 (2): 1072-1090.
- [12] Dineshkumar C, Udhayakumar R. Results on approximate controllability of nondensely defined fractional neutral stochastic differential systems. *Numerical Methods for Partial Differential Equations* 202; 1-27. doi: 10.1002/num.22687.
- [13] Dineshkumar C, Udhayakumar R, Vijayakumar V, Nisar KS. A discussion on the approximate controllability of Hilfer fractional neutral stochastic integro-differential systems. *Chaos, Solitons & Fractals* 2021; 142: 1-12. 110472.
- [14] Dineshkumar C, Udhayakumar R, Vijayakumar V, Nisar KS, Shukla A. A note on the approximate controllability of Sobolev type fractional stochastic integro-differential delay inclusions with order  $1 < r < 2$ . *Mathematics and Computers in Simulation* 2021; 190: 1003-1026.

- [15] Guezane-Lakoud A, Khaldi R. Solvability of a three-point fractional nonlinear boundary value problem. 2012; 20: 395-403.
- [16] Ibrahim BHE, Dong Q, Fan Z, Existence for boundary value problems of two-term Caputo fractional differential equations. *Journal of Nonlinear Sciences and Applications* 2017; 10: 511-520.
- [17] Ibrahim RW, Momani S, On the existence and uniqueness of solutions of a class of fractional differential equations. *Journal of Mathematical Analysis and Applications* 2008; 334 (1): 1-10.
- [18] Ibrahim RW, Solutions of fractional diffusion problems. *Electronic Journal of Differential Equations* 2010; 47: 1-11.
- [19] Kaufmann ER, Yao KD. Existence of solutions for a nonlinear fractional order differential equation. *Electronic Journal of Qualitative Theory of Differential Equations* 2009; 71: 1-9.
- [20] Kavitha K, Vijayakumar V, Udhayakumar R, Ravichandran C. Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness. *Asian Journal of control* (2021), 1-10. doi: 10.1002/asjc.2549.
- [21] Kavitha K, Vijayakumar V, Udhayakumar R, Sakthivel N, Nisar KS. A note on approximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay. *Mathematical Methods in the Applied Sciences* 2020; 44 (6): 4428-4447.
- [22] Kavitha K, Vijayakumar V, Shukla A, Nisar KS, Udhayakumar R, Results on approximate controllability of Sobolev-type fractional neutral differential inclusions of Clarke subdifferential type. *Chaos, Solitons and Fractals*, 2021; 151: 1-8, 111264.
- [23] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.
- [24] Kumar S, Vats RK, Nashine HK. Existence and uniqueness results for three-point nonlinear fractional (arbitrary order) boundary value problem. *Matematicki Vesnik Journal* 2018; 70 (4): 314-325.
- [25] Liang Y, Yang H. Controllability of fractional integro-differential evolution equations with nonlocal conditions. *Applied Mathematics and Computation* 2015; 254: 20-29.
- [26] Marasi H, Piri H, Aydi H. Existence and multiplicity of solutions for nonlinear fractional differential equations. *Journal of Nonlinear Sciences and Applications* 2016; 9 (6): 4639-4646.
- [27] Matar MM, Existence and uniqueness of solutions to fractional semilinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions. *Electronic Journal of Differential Equations* 2009; 155: 1-7.
- [28] Matar MM, On existence of solution to nonlinear fractional differential equations for  $0 \leq \mu \leq 3$ . *Fractional Calculus and Applied Analysis* 2012; 3 (14): 1-8.
- [29] Mohan Raja M, Vijayakumar V, Udhayakumar R. Results on the existence and controllability of fractional integro-differential system of order  $1 < r < 2$  via measure of noncompactness. *Chaos, Solitons & Fractals*, 2020; 139: 1-11. 110299.
- [30] Mohan Raja M, Vijayakumar V, Udhayakumar R, Zhou, Y. A new approach on the approximate controllability of fractional differential evolution equations of order  $1 < r < 2$  in Hilbert spaces, *Chaos, Solitons & Fractals* 2020; 141: 1-10. 110310.
- [31] Mohan Raja M, Vijayakumar V, Udhayakumar R, A new approach on approximate controllability of fractional evolution inclusions of order  $1 < r < 2$  with infinite delay. *Chaos, Solitons & Fractals* 2020; 141: 1-13. 110343.
- [32] Nisar KS, Vijayakumar V. Results concerning to approximate controllability of non-densely defined Sobolev-type Hilfer fractional neutral delay differential system. *Mathematical Methods in the Applied Sciences* 2021; 1-22. DOI: 10.1002/mma.7647
- [33] Ntouyas SK, Existence results for nonlocal boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions. *Discussiones Mathematicae Differential Inclusions, Control and Optimization* 2013; 33: 17-39.



- [34] Patel R, Shukla A, Jadon SS. Existence and optimal control problem for semilinear fractional order  $(1,2]$  control system. *Mathematical Methods in the Applied Sciences* 2020; 1-12. DOI:10.1002/mma.6662.
- [35] Podlubny I. *Fractional Differential Equations*, Academic Press, New York, (1999).
- [36] Rehman MU, Khan RA. Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations. *Applied Mathematics Letters* 2010; 23: 1038-1044.
- [37] Shukla A, Sukavanam N, Pandey DN. Approximate controllability of semilinear fractional control systems of order  $\alpha \in (1, 2]$  with infinite delay. *Mediterranean Journal of Mathematics* 2016; 13 (5): 2539-2550.
- [38] Shukla A, Sukavanam N, Pandey DN. Approximate controllability of fractional semilinear stochastic system of order  $\alpha \in (1, 2]$ . *Journal of Dynamical and Control Systems* 2017; 23 (4): 679-691.
- [39] Shukla A, Patel R. Controllability results for fractional semilinear delay control systems. *Journal of Applied Mathematics and Computing* 2021; 65 (1-2): 861-875.
- [40] Singh A, Shukla A, Vijayakumar V, Udhayakumar R. Asymptotic stability of fractional order  $(1, 2]$  stochastic delay differential equations in Banach spaces. *Chaos, Solitons and Fractals* 2021; 150: 1-9. 111095.
- [41] Vijayakumar V, Udhayakumar R. A new exploration on existence of Sobolev-type Hilfer fractional neutral integro-differential equations with infinite delay. *Numerical Methods for Partial Differential Equations* 2021; 37 (1): 750-766.
- [42] Vijayakumar V, Udhayakumar R. Results on approximate controllability for non-densely defined Hilfer fractional differential system with infinite delay. *Chaos, Solitons & Fractals* 2020; 139: 1-11. 110019.
- [43] Wang L, Xu G. Existence results for nonlinear fractional differential equations with integral boundary value problems. *Theoretical Mathematics & Applications* 2013; 3 (3): 63-73.
- [44] Williams WK, Vijayakumar V, Udhayakumar R, Nisar KS. A new study on existence and uniqueness of nonlocal fractional delay differential systems of order  $1 < r < 2$  in Banach spaces, *Numerical Methods for Partial Differential Equations*, 2021; 37 (2): 949-961.
- [45] Williams WK, Vijayakumar V, Udhayakumar R, Panda SK, Nisar KS. Existence and controllability of nonlocal mixed Volterra-Fredholm type fractional delay integro-differential equations of order  $1 < r < 2$ . *Numerical Methods for Partial Differential Equations* 2020; 1-21. doi: 10.1002/num.22697.
- [46] Zhou Y, He JW. New results on controllability of fractional evolution systems with order  $\alpha \in (1, 2)$ . *Evolution Equations Control Theory* 2021; 10 (3): 491-509.
- [47] Zhou Y, Jiao F. Existence of mild solutions for fractional neutral evolution equations. *Computers and Mathematics with Applications* 2010; 59 (3): 1063-1077.