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Scattering theory of the quadratic eigenparameter depending impulsive Sturm–Liouville equations

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Abstract: We handle an impulsive Sturm–Liouville boundary value problem. We find the Jost solution, Jost function, and scattering function of this problem and examine the properties of scattering function. We also study eigenvalues and resolvent operator of this problem. Finally, we exemplify our work by taking a different problem.

Key words: Differential equation, Jost solution, Jost function, impulsive operator, asymptotic, spectral parameter

1. Introduction

First of all, let us give a brief information about the existing literature of Sturm–Liouville operators. The investigation of the spectral theory of Sturm–Liouville operators was first started by Naimark [14] in 1960. Naimark explored one part of the continuous spectrum. This part is a mathematical barrier to the integrity of eigenvectors known as spectral singularity. Therefore, Guseinov [8], Levitan [9], Marchenko [11], and Schwartz [16] studied the spectral singularities of Sturm–Liouville boundary value problem. Then, spectral analysis and scattering analysis of some boundary value problems were investigated using the Sturm–Liouville equations, Hamilton systems, and Schrödinger equations. We refer to these references for detailed information [2, 4, 7, 10, 15, 18]. On the other hand, with the development of science, such equations could not mathematically explain situations involving discontinuity. To deal with discontinuities, new conditions called jump points, point interaction, impulsive conditions, interface points, and transmission points are added on discontinuous points. Impulsive differential equations have great importance in physical and chemical phenomena and quantum mechanics. The theory of impulsive differential equations is discussed in a detailed way in applied mathematics [1, 12, 13, 17, 19]. Impulsive condition is well developed in case of continuity. The scattering analysis of the impulsive Sturm–Liouville equation was examined in the studies of Bairamov et al. [3, 5, 6]. The difference in our study is that the spectral parameter ζ exists both in differential equation and in boundary condition. Moreover, the boundary condition is in the quadratic form with the ζ spectral parameter. This gives the problem a new perspective.

We shall define the impulsive Sturm–Liouville boundary value problem (ISBVP)

$$-u'' + q(z)u = \zeta^2 u, \quad z \in [0,1) \cup (1,\infty), \tag{1.1}$$

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$$\left(\hbar_0 + \hbar_1\zeta + \hbar_2\zeta^2\right) u'(0) + \left(\eta_0 + \eta_1\zeta + \eta_2\zeta^2\right) u(0) = 0, \tag{1.2}$$

$$u(1^+) = \delta_{11}u(1^-) + \delta_{12}u'(1^-), \quad u'(1^+) = \delta_{21}u(1^-) + \delta_{22}u'(1^-), \tag{1.3}$$

where ζ is a spectral parameter, $\delta_{11},\ \delta_{12},\ \delta_{21},\ \delta_{22},\ \hbar_i,\ \eta_i,\ i = 0, 1, 2$ are real numbers, $\hbar_2\eta_2 \neq 0,\ \delta_{12} \neq 0,$ $\delta_{11}\delta_{22} - \delta_{21}\delta_{12} > 0$ and q is a real valued function that satisfies the following condition

$$\int_0^\infty (1+z)|q(z)|dz < \infty. \tag{1.4}$$

Under the condition (1.4), $e(z,\zeta)$ is a solution of the equation (1.1) that satisfies the following condition

$$\lim_{z\to\infty} e(z,\zeta)e^{-i\zeta} = 1, \quad \zeta \in \overline{\mathbb{C}}_+ := \{\zeta \in \mathbb{C} : Im\zeta \geq 0\}.$$

The solution which is referred to as the Jost solution can be expressed as

$$e(z,\zeta) = e^{i\zeta z} + \int_z^\infty K(z,t)e^{i\zeta t}dt, \quad \zeta \in \overline{\mathbb{C}}_+, \tag{1.5}$$

where $K(z,t)$ is defined by the potential function q in Levitan [9] and Marchenko [11]. The solution $e(z,\zeta)$ is analytic with respect to ζ in $\mathbb{C}_+ := \{\zeta \in \mathbb{C} : Im\zeta > 0\}$ and continuous up to the real axis.

The plan of this paper is as follows: In Section 2, we deal with the impulsive Sturm–Liouville boundary value problem on the semiaxis. We first obtain Jost solutions and Jost function of (1.1)–(1.3), then we get the scattering function by using the Jost function. We also investigate charactaristic properties of the scattering function of (1.1)–(1.3). In Section 3, we define the set of eigenvalues of ISBVP (1.1)–(1.3). Furthermore, we get an asymptotic equation for Jost function and resolvent operator of this problem. In Section 4, we are interested in unperturbated impulsive boundary value problem of (1.1)–(1.3). Finally, we express some conclusions in Section 5.

2. Scattering solutions and scattering function

In this section, we will be interested in equation (1.1) with the conditions (1.2) and (1.3). We obtain Jost function and scattering function of (1.1)–(1.3). Then, we investigate the properties of scattering function.

Let $S(z,\zeta^2)$ and $C(z,\zeta^2)$ be the fundamental solutions of (1.1) in the interval $[0,1)$ fulfilling the initial conditions

$$S(0,\zeta^2) = 0, \quad S'(0,\zeta^2) = 1$$

and

$$C(0,\zeta^2) = 1, \quad C'(0,\zeta^2) = 0.$$

In addition, integral representations of the solutions $S(z,\zeta^2)$ and $C(z,\zeta^2)$ are expressed as follows

$$S(z,\zeta^2) = \frac{\sin\zeta z}{\zeta} + \int_0^z A(z,t)\frac{sin\zeta t}{\zeta}dt \tag{2.1}$$

and

$$C(z,\zeta^2)=\cos\zeta z+\int_0^z B(z,t)\cos\zeta t dt, \tag{2.2}$$

where the kernel functions $A(z,t)$ and $B(z,t)$ are defined by the potential function q in Levitan [9]. It is clear that the solutions $S(z,\zeta^2)$ and $C(z,\zeta^2)$ are entire functions of ζ and

$$W[S(z,\zeta^2),C(z,\zeta^2)]=-1, \quad \zeta\in\mathbb{C},$$

where $W[u_1,u_2]$ denotes the wronskian of the solutions u_1 and u_2 of the equation (1.1). From $S(z,\zeta^2)$, $C(z,\zeta^2)$, and $e(z,\zeta)$, let us define the following function for $\zeta\in\overline{\mathbb{C}}_+$

$$E(z,\zeta)=\begin{cases}\rho(\zeta)S(z,\zeta^2)+\tau(\zeta)C(z,\zeta^2); & z\in[0,1)\\ e(z,\zeta); & z\in(1,\infty).\end{cases} \tag{2.3}$$

By using (1.3), we write for $\zeta\in\overline{\mathbb{C}}_+$

$$e(1,\zeta)=\rho(\zeta)\left\{\delta_{11}S(1,\zeta^2)+\delta_{12}S'(1,\zeta^2)\right\}+\tau(\zeta)\left\{\delta_{11}C(1,\zeta^2)+\delta_{12}C'(1,\zeta^2)\right\}$$

$$e'(1,\zeta)=\rho(\zeta)\left\{\delta_{21}S(1,\zeta^2)+\delta_{22}S'(1,\zeta^2)\right\}+\tau(\zeta)\left\{\delta_{21}C(1,\zeta^2)+\delta_{22}C'(1,\zeta^2)\right\}$$

and from these equations, we obtain

$$\begin{aligned}\rho(\zeta)=\frac{1}{\delta_{12}\delta_{21}-\delta_{11}\delta_{22}}&[e(1,\zeta)\left\{\delta_{21}C(1,\zeta^2)+\delta_{22}C'(1,\zeta^2)\right\}\\ &-e'(1,\zeta)\left\{\delta_{11}C(1,\zeta^2)+\delta_{12}C'(1,\zeta^2)\right\}]\end{aligned} \tag{2.4}$$

$$\begin{aligned}\tau(\zeta)=\frac{1}{\delta_{11}\delta_{22}-\delta_{12}\delta_{21}}&[e(1,\zeta)\left\{\delta_{21}S(1,\zeta^2)+\delta_{22}S'(1,\zeta^2)\right\}\\ &-e'(1,\zeta)\left\{\delta_{11}S(1,\zeta^2)+\delta_{12}S'(1,\zeta^2)\right\}].\end{aligned} \tag{2.5}$$

The function $E(z,\zeta)$ is the Jost solution of ISBVP (1.1)-(1.3), where $\rho(\zeta)$ and $\tau(\zeta)$ are defined in (2.4) and (2.5), respectively. By using the boundary condition (1.2), we obtain

$$E(\zeta)=(\hbar_0+\hbar_1\zeta+\hbar_2\zeta^2)\rho(\zeta)+(\eta_0+\eta_1\zeta+\eta_2\zeta^2)\tau(\zeta). \tag{2.6}$$

The function $E(\zeta)$ is the Jost function of (1.1)–(1.3), it is analytic in $\mathbb{C}_+$ and continuous up to the real axis. It is known from [11]

$$W[e(z,\zeta),e(z,-\zeta)]=-2i\zeta, \quad \zeta\in\mathbb{R}\backslash\{0\}.$$

Now we can give the following solution of (1.1)–(1.3) for $\zeta\in\mathbb{R}\backslash\{0\}$

$$F(z,\zeta)=\begin{cases}\psi(z,\zeta); & z\in[0,1)\\ \upsilon(\zeta)e(z,\zeta)+\varphi(\zeta)e(z,-\zeta); & z\in(1,\infty),\end{cases}$$

where $\psi(z,\xi)$ is the solution of (1.1) given by

$$\psi(z,\zeta) = -(\eta_0 + \eta_1\zeta + \eta_2\zeta^2)S(z,\zeta^2) + (\hbar_0 + \hbar_1\zeta + \hbar_2\zeta^2)C(z,\zeta^2).$$

It is obvious that the function $\psi(z,\zeta)$ is an entire function of ζ. From the impulsive condition (1.3), we get

$$\upsilon(\zeta)e(1,\zeta) + \varphi(\zeta)e(1,-\zeta) = \delta_{11}\psi(1,\zeta) + \delta_{12}\psi'(1,\zeta)$$

$$\upsilon(\zeta)e'(1,\zeta) + \varphi(\zeta)e'(1,-\zeta) = \delta_{21}\psi(1,\zeta) + \delta_{22}\psi'(1,\zeta).$$

Using (2.4) and (2.5), we find that

$$\upsilon(\zeta) = \frac{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}{2i\zeta}\left[\left\{\hbar_0 + \hbar_1\zeta + \hbar_2\zeta^2\right\}\overline{\rho(\zeta)} + \left\{\eta_0 + \eta_1\zeta + \eta_2\zeta^2\right\}\overline{\tau(\zeta)}\right] \tag{2.7}$$

and

$$\varphi(\zeta) = \frac{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}{2i\zeta}\left[\left\{\hbar_0 + \hbar_1\zeta + \hbar_2\zeta^2\right\}\rho(\zeta) + \left\{\eta_0 + \eta_1\zeta + \eta_2\zeta^2\right\}\tau(\zeta)\right]. \tag{2.8}$$

Theorem 2.1 *For all $\zeta \in \mathbb{R}\setminus\{0\}, E(\zeta) \neq 0$.*

Proof Let us assume that there exists a ζ_0 in $\mathbb{R}\setminus\{0\}$, such that $E(\zeta_0) = 0$. From (2.7) and (2.8), we find $\upsilon(\zeta_0) = \varphi(\zeta_0) = 0$. Then the solution $F(z,\zeta_0)$ is equal to zero identically, so it is a trivial solution of (1.1)–(1.3) which gives a contradiction with our assumption, i.e. $E(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}\setminus\{0\}$. □

The scattering function of (1.1)–(1.3) is defined by

$$S(\zeta) = \frac{\overline{E(\zeta)}}{E(\zeta)}, \quad \zeta \in \mathbb{R}\setminus\{0\}.$$

Using (2.6), we can rewrite the scattering function as follows

$$S(\zeta) = \frac{(\hbar_0 + \hbar_1\zeta + \hbar_2\zeta^2)\overline{\rho(\zeta)} + (\eta_0 + \eta_1\zeta + \eta_2\zeta^2)\overline{\tau(\zeta)}}{(\hbar_0 + \hbar_1\zeta + \hbar_2\zeta^2)\rho(\zeta) + (\eta_0 + \eta_1\zeta + \eta_2\zeta^2)\tau(\zeta)} \tag{2.9}$$

for all $\zeta \in \mathbb{R}\setminus\{0\}$.

Theorem 2.2 *For all $\zeta \in \mathbb{R}\setminus\{0\}$, the scattering function satisfies*

$$\overline{S(\zeta)} = S^{-1}(\zeta) \text{ and } S(-\zeta) \neq \overline{S(\zeta)}.$$

Proof Using (2.9), we get

$$\overline{S(\zeta)} = \frac{(\hbar_0 + \hbar_1\zeta + \hbar_2\zeta^2)\rho(\zeta) + (\eta_0 + \eta_1\zeta + \eta_2\zeta^2)\tau(\zeta)}{(\hbar_0 + \hbar_1\zeta + \hbar_2\zeta^2)\overline{\rho(\zeta)} + (\eta_0 + \eta_1\zeta + \eta_2\zeta^2)\overline{\tau(\zeta)}}. \tag{2.10}$$

From (2.9) and (2.10), we find

$$\overline{S(\zeta)} = S^{-1}(\zeta).$$

Similarly, using equation (2.9), we write

$$S(-\zeta)=\frac{(\hbar_0-\hbar_1\zeta+\hbar_2\zeta^2)\overline{\rho(-\zeta)}+(\eta_0-\eta_1\zeta+\eta_2\zeta^2)\overline{\tau(-\zeta)}}{(\hbar_0-\hbar_1\zeta+\hbar_2\zeta^2)\rho(-\zeta)+(\eta_0-\eta_1\zeta+\eta_2\zeta^2)\tau(-\zeta)}. \tag{2.11}$$

From (2.10) and (2.11), we get

$$S(-\zeta)\neq\overline{S(\zeta)}.$$

□

Lemma 2.3 *For $\zeta\in\mathbb{R}\backslash\{0\}$, the following wronskian holds*

$$W[E(z,\zeta),F(z,\zeta)]=\begin{cases}-E(\zeta); & z\in[0,1)\\ -(\delta_{11}\delta_{22}-\delta_{12}\delta_{21})E(\zeta); & z\in(1,\infty).\end{cases}$$

Proof If we write the wronskian of $E(z,\zeta)$ and $F(z,\zeta)$ for $z\in[0,1)$, we find

$$W[E(z,\zeta),F(z,\zeta)]=-(\eta_0+\eta_1\zeta+\eta_2\zeta^2)\tau(\zeta)-(\eta_0+\eta_1\zeta+\eta_2\zeta^2)\rho(\zeta)$$
$$=-E(\zeta).$$

Similarly, for $z\in(1,\infty)$, we get

$$W[E(z,\zeta),F(z,\zeta)]=\varphi(\zeta)W[e(z,\zeta),e(z,-\zeta)].$$

After applying the definition $\varphi(\zeta)$ given by (2.8), we find

$$W[E(z,\zeta),F(z,\zeta)]=-(\delta_{11}\delta_{22}-\delta_{12}\delta_{21})E(\zeta)$$

for $z\in(1,\infty)$. This completes the proof. □

3. Resolvent operator and discrete spectrum

In this part, we give unbounded solution of (1.1)–(1.3). Afterward, we obtain resolvent operator and Green function of ISBVP (1.1)-(1.3) by using this unbounded solution.

Theorem 3.1 *The set of eigenvalues of (1.1)–(1.3) is*

$$\sigma_d=\{\mu:\mu=\zeta^2,\zeta\in\mathbb{C}_+,E(\zeta)=0\}.$$

Proof

$$\lim_{z\to\infty}\breve{e}(z,\zeta)e^{i\zeta z}=1,\quad \lim_{z\to\infty}\breve{e}'(z,\zeta)e^{i\zeta z}=-i\zeta,\quad \zeta\in\overline{\mathbb{C}}_+$$

$\breve{e}(z,\zeta)$ is unbounded solution in $(1,\infty)$ of equation (1.1) satisfying above conditions [15]. It is clear that

$$W[e(z,\zeta),\breve{e}(z,\zeta)]=-2i\zeta,\quad z\in(1,\infty),\quad \zeta\in\overline{\mathbb{C}}_+.$$

On the other hand, for all $\zeta \in \overline{\mathbb{C}}_+ \backslash \{0\}$, equation (1.1) admits another solution as follows

$$G(z,\zeta) = \begin{cases} \psi(z,\zeta); & z \in [0,1) \\ \gamma(\zeta) e(z,\zeta) + \kappa(\zeta) \breve{e}(z,\zeta), & z \in (1,\infty). \end{cases} \tag{3.1}$$

By using the impulsive condition (1.3), we get the coefficients

$$\kappa(\zeta) = \frac{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}}{2i\zeta} E(\zeta) \tag{3.2}$$

and

$$\gamma(\zeta) = \frac{1}{2i\zeta} \left[\breve{e}(1,\zeta) \{ \delta_{21}\psi(1,\zeta) + \delta_{22}\psi'(1,\zeta) \} - \breve{e}'(1,\zeta) \{ \delta_{11}\psi(1,\zeta) + \delta_{12}\psi'(1,\zeta) \} \right]. \tag{3.3}$$

It is evident from (3.1) that the first part of $G(z,\zeta)$ is in $L_2(0,1)$. Moreover, if $\kappa(\zeta) = 0$, then the second part of the $G(z,\zeta)$ is in $L_2(1,\infty)$. It follows from (3.1), (3.2), and the definition of eigenvalues in Naimark [15] that

$$\sigma_d = \{ \mu : \mu = \zeta^2, \zeta \in \mathbb{C}_+, E(\zeta) = 0 \}$$

or

$$\sigma_d = \{ \mu : \mu = \zeta^2, \zeta \in \mathbb{C}_+, \kappa(\zeta) = 0 \}.$$

It completes the proof. □

By using (2.3) and (3.1), we also find

$$W[E(z,\zeta), G(z,\zeta)] = \begin{cases} -E(\zeta); & z \in [0,1) \\ -(\delta_{11}\delta_{22} - \delta_{12}\delta_{21}) E(\zeta); & z \in (1,\infty) \end{cases}$$

for all $\zeta \in \overline{\mathbb{C}}_+ \backslash \{0\}$.

Theorem 3.2 *The Jost function of ISBVP (1.1)–(1.3) satisfies*

$$E(\zeta) = \frac{\zeta^4}{\delta_{11}\delta_{22} - \delta_{12}\delta_{21}} \left(\frac{\delta_{12}\hbar_2}{2} + o(1) \right), \quad \zeta \in \overline{\mathbb{C}}_+, \quad |\zeta| \to \infty. \tag{3.4}$$

Proof Using (2.1) and (2.2), we obtain for $\zeta \in \overline{\mathbb{C}}_+$ and $|\zeta| \to \infty$

$$S\left(1,\zeta^2\right) = \frac{e^{-i\zeta}}{\zeta} \left(\frac{i}{2} + o(1) \right)$$

$$S'\left(1,\zeta^2\right) = e^{-i\zeta} \left(\frac{1}{2} + O\left(\frac{1}{\zeta} \right) \right)$$

$$C\left(1,\zeta^2\right) = e^{-i\zeta} \left(\frac{1}{2} + o(1) \right) \tag{3.5}$$

$$C'\left(1,\zeta^2\right)=\zeta e^{-i\zeta}\left(-\frac{i}{2}+O\left(\frac{1}{\zeta}\right)\right).$$

Similarly from (1.5), we find for $\zeta\in\overline{\mathbb{C}}_+$ and $|\zeta|\to\infty$

$$e(1,\zeta)=e^{i\zeta}(1+o(1))$$

$$e'(1,\zeta)=\zeta e^{i\zeta}\left(i+O\left(\frac{1}{\zeta}\right)\right). \tag{3.6}$$

By the help of (3.5) and (3.6), proof of the theorem is completed. □

Theorem 3.3 *Under condition (1.4), we get the resolvent operator of ISBVP (1.1)–(1.3)*

$$R_\zeta f=\int_0^\infty G(z,t;\zeta)f(t)dt,$$

where f is an arbitrary function in $L_2(0,\infty)$ and $G(z,t;\zeta)$ is the Green function of (1.1)–(1.3) defined as

$$G(z,t;\zeta)=\begin{cases}\dfrac{E(z,\zeta)G(t,\zeta)}{W[E(z,\zeta),G(z,\zeta)]}; & 0\le t<z\\[2ex] \dfrac{G(z,\zeta)E(t,\zeta)}{W[E(z,\zeta),G(z,\zeta)]}; & z\le t<\infty\end{cases}$$

for all $z\neq 1,\ t\neq 1$.

Proof In order to get the resolvent operator, we will consider the following equation

$$-u''+q(z)u-\zeta^2u=f(z),\quad z\in[0,1)\cup(1,\infty). \tag{3.7}$$

From (2.3) and (3.1), we write the solution of (3.7)

$$\chi(z,\zeta)=a_1(z)E(z,\zeta)+a_2(z)G(z,\zeta).$$

Using the method of variation of parameters, $a_1(z)$ and $a_2(z)$ values are obtained as follows

$$a_1(z)=b+\int_0^z\frac{f(t)G(t,\zeta)}{W[E(z,\zeta),G(z,\zeta)]}dt$$

$$a_2(z)=c+\int_z^\infty\frac{f(t)E(t,\zeta)}{W[E(z,\zeta),G(z,\zeta)]}dt,$$

where b and c are real numbers. Since the solution $\chi(z,\zeta)$ is in $L_2(0,\infty)$, the coefficient b also becomes zero from the boundary condition (1.2). The proof is completed. □

4. An example

In this section, we give a detailed example to illustrate our results. We find the Jost solution, Jost function, and scattering function of this example.

Let us investigate the following ISBVP

$$-u'' = \zeta^2 u, \quad z \in [0,1) \cup (1,\infty) \tag{4.1}$$

with the boundary condition

$$u'(0) + \zeta^2 u(0) = 0 \tag{4.2}$$

and the impulsive conditions

$$u(1^+) = \delta_{11} u(1^-) + \delta_{12} u'(1^-)$$

$$u'(1^+) = \delta_{21} u(1^-) + \delta_{22} u'(1^-), \tag{4.3}$$

where ζ is a spectral parameter, δ_{11}, δ_{12}, δ_{21}, δ_{22} are real numbers and $\delta_{11}\delta_{22} - \delta_{21}\delta_{22} > 0$. It is clear that

$$e(z,\zeta) = e^{i\zeta z}, \quad S\left(z,\zeta^2\right) = \frac{\sin \zeta z}{\zeta}, \quad C\left(z,\zeta^2\right) = \cos \zeta z.$$

For $\zeta \in \overline{\mathbb{C}}_+$, we also write the solution of (4.1)–(4.3)

$$E(z,\zeta) = \begin{cases} \rho(\zeta)\dfrac{\sin(\zeta z)}{\zeta} + \tau(\zeta)\cos(\zeta z); & z \in [0,1) \\ e^{i\zeta z}; & z \in (1,\infty). \end{cases} \tag{4.4}$$

From (4.3), we find

$$\rho(\zeta) = \frac{e^{i\zeta}}{\delta_{12}\delta_{21} - \delta_{11}\delta_{22}}\left(\delta_{21}\cos\zeta - \delta_{22}\zeta\sin\zeta - \delta_{11} i\zeta\cos\zeta + \delta_{12} i\zeta^2 \sin\zeta\right)$$

and

$$\tau(\zeta) = \frac{e^{i\zeta}}{\delta_{11}\delta_{22} - \delta_{21}\delta_{12}}\left(\delta_{21}\frac{\sin\zeta}{\zeta} + \delta_{22}\cos\zeta - \delta_{11} i\sin\zeta - \delta_{12} i\zeta\cos\zeta\right).$$

By using (4.2) and (4.4), we get Jost function of (4.1)–(4.3) as

$$E(\zeta) = \frac{e^{i\zeta}}{\delta_{11}\delta_{22} - \delta_{21}\delta_{12}}[\delta_{11} i(\zeta^2\sin\zeta - \zeta\cos\zeta) + \delta_{12} i(\zeta^2\sin\zeta + \zeta^3\cos\zeta)$$
$$+\delta_{21}(\cos\zeta - \zeta\sin\zeta) - \delta_{22}(\zeta\sin\zeta + \zeta^2\cos\zeta)].$$

The scattering function of ISBVP (4.1)–(4.3) is written as

$$S(\zeta) = \frac{\overline{E(\zeta)}}{E(\zeta)},$$

where

$$\overline{E(\zeta)} = \frac{e^{-i\zeta}}{\delta_{11}\delta_{22} - \delta_{21}\delta_{12}}[\delta_{11} i(\zeta\cos\zeta - \zeta^2\sin\zeta) - \delta_{12} i(\zeta^2\sin\zeta + \zeta^3\cos\zeta)$$
$$+\delta_{21}(\cos\zeta - \zeta\sin\zeta) - \delta_{22}(\zeta\sin\zeta + \zeta^2\cos\zeta)].$$

5. Conclusion

Impulsive equations have recently been developed in applied mathematics which has extensive physical and realistic mathematical models. This study is important because it is the first study to examine the scattering solutions of the impulsive Sturm–Liouville equation whose boundary condition is in quadratic form with the spectral parameter. With the help of these solutions, we obtain the Jost function and scattering function of (1.1)–(1.3). Then, we give information about the properties of the scattering function. Furthermore, we find the resolvent operator, discrete spectrum of the problem. This study can be a reference for researchers studying scattering theory.

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