

b -property of sublattices in vector lattices

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Abstract: We study b -property of a sublattice (or an order ideal) F of a vector lattice E . In particular, b -property of E in E^δ , the Dedekind completion of E , b -property of E in E^u , the universal completion of E , and b -property of E in $\hat{E}(\hat{\tau})$, the completion of E .

Key words: Vector lattice, universal completion, Dedekind completion, b -property, local solid vector lattice

1. Introduction and preliminaries

Vector lattices considered here are all real and Archimedean. Vector topologies are assumed to be Hausdorff.

Definition 1.1 A sublattice F of a vector lattice E is said to have b -property in E , if x_α is a net in F^+ and $0 \leq x_\alpha \uparrow e$ for some $e \in E$, then there exists $f \in F$ with $0 \leq x_\alpha \uparrow f$.

Recall that a subset F of E is said to be majorizing in E if, for each $0 < e \in E$, there exists $f \in F$ with $0 \leq e \leq f$.

A subset U of a vector lattice (VL) is called solid if $|u| \leq |v|$, $v \in U$, imply $u \in U$. A linear topology τ on a VL E is called locally solid if τ has a base of zero consisting of solid sets.

A locally solid VL E (LSVL) satisfies the Lebesgue property if $x_\alpha \downarrow 0$ in E implies $x_\alpha \xrightarrow{\tau} 0$.

A LSVL $E(\tau)$ satisfies the Fatou property if τ has a base of zero consisting of solid and order closed sets.

A sublattice F in a VL E is regular if $\inf A$ is the same as in F and E whenever $A \subset F$ whose infimum exists in F . Ideals are regular in E .

E is called laterally σ -complete if the supremum of every disjoint sequence exists in E^+ and laterally complete if supremum of every disjoint subset in E^+ exists in E .

A vector lattice E which is both Dedekind (σ -) complete and laterally (σ -) complete is called universally (σ -) complete.

Example 1.2 [1, p.198] Let X be a topological space. A function $f : X \rightarrow \mathbb{R}$ is called a step function if there exists a collection of mutually disjoint subsets $\{V_i\}$ of X such that $\bigcup_i V_i = X$, f is constant on each V_i , and $f \in C^\infty(X)$. Let

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$S^\infty(X)$ be the space of step functions on an extremally disconnected topological space X . Then $S^\infty(X)$ is a laterally complete VL.

Universal (σ -) completion of a VL E is a laterally (σ -) complete and Dedekind (σ -) complete vector lattice E^u which contains E as an order dense sublattice. Every VL E has a unique universal completion [1, Theorem 7.21].

Lateral completion E^λ of a VL E is defined to be the intersection of all laterally complete vector lattices between E and E^u .

Example 1.3 Let X be an extremally disconnected topological space. $C^\infty(X)$, the space of all extended continuous functions on X with the usual algebraic and lattice operations is a universally complete VL.

A net $(x_\alpha)_{\alpha \in A}$ in a VL E is order convergent to $x \in E$ if there exists a net $(x_\beta)_{\beta \in B}$, possibly over a different index set, such that $x_\beta \downarrow 0$ and, for each $\beta \in B$, there exists $\alpha_0 \in A$ with $|x_\alpha - x| \leq x_\beta$ for all $\alpha \geq \alpha_0$. In this case we write $x_\alpha \xrightarrow{o} x$.

A net x_α in E uo -converges to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$. In this case we write $x_\alpha \xrightarrow{uo} x$.

Let $E(\tau)$ be a LSVL. A net x_α in E is $u\tau$ -convergent to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{\tau} 0$ for all $u \in E^+$. A net x_α in E is called order Cauchy (uo -Cauchy) if the doubly indexed net $(x_\alpha - x_{\alpha'})_{(\alpha, \alpha')}$ is order convergent (uo -convergent) to zero. $E(\tau)$ is called uo -complete if every uo -Cauchy net is uo -convergent in E .

The b -property of a VL E was defined in [2] as: a VL E has b -property if every subset A in E , which is order bounded in $(E^\sim)^\sim$, remains to be order bounded in E . We say that a vector sublattice F of a VL E has (countable) b -property in E whenever each (sequence) net f_α in F , with $0 \leq f_\alpha \uparrow \leq e$ for some $e \in E$, is order bounded in F (cf. e.g. [2], [3, p.766]).

Example 1.4 Every perfect VL, and therefore every order dual, have the b -property. Every reflexive BL and every KB-space have b -property [2-5]. On the other hand, by considering the basis vectors e_n in c_0 , we see that c_0 does not have the b -property in l_∞ .

Let us note that Fremlin had considered subsets of a VL E that are order bounded in the universal completion E^u of E . He proved that if E is a Dedekind σ -complete VL then E is laterally σ -complete iff E has the countable b -property in E^u [1, Theorem 7.38]. That is, each sequence x_n in E with $0 \leq x_n \uparrow \leq e$ for some $e \in E^u$ has an upper bound in E ,

Example 1.5 Each projection band F in a vector lattice E has b -property in E . In particular, every band in a Dedekind complete vector lattice has b -property. An element u in a VL E is called an atom if whenever $v \wedge w = 0$, $0 \leq v \leq u$, and $0 \leq w \leq u$ imply either $v = 0$ or $w = 0$. If x is an atom in E , the principal band B_x generated by x is a projection band and therefore has b -property in E .

Example 1.6 Every majorizing sublattice F has b -property in E . Let $0 \leq x_\alpha \uparrow \leq e$ for some net $x_\alpha \subseteq F$, $e \in E$. As F is majorizing, there exists $f \in F$ with $e \leq f$. Then $0 \leq x_\alpha \leq f$. Since it is well known that E is majorizing in the Dedekind completion E^δ , the lattice E has b -property in E^δ .

Example 1.7 Every order ideal F in a vector lattice E with b -property in E is a band of E . Indeed, let x_α be a net in F such that $0 \leq x_\alpha \uparrow e \in E$, then the net x_α is order bounded in F , say $0 \leq x_\alpha \uparrow \leq f \in F$, by the b -property of F in E . Hence, $0 \leq e \leq f$ and as F is an ideal, $e \in F$.

Example 1.8 Let $E \subseteq F$ be a sublattice of F and $I(E)$ be the ideal generated by E in F . Then E has b -property in $I(E)$. Having b -property is transitive: if $E \subseteq F \subseteq G$ are sublattices of a VL X such that E has b -property in F and F has b -property in G , then E has b -property in G . If E has b -property in G , then E has b -property in every sublattice of G containing E as a sublattice.

Example 1.9 Let F be a norm-closed sublattice of a Banach lattice $(E, \|\cdot\|)$ with order continuous norm. Let x_n be a sequence in F such that $0 \leq x_n \uparrow \leq e$ for some $e \in E$. Then x_n is norm-convergent to some $x \in E$. As F is norm-closed, $x \in F$. Since $x_n \leq x$ for all n , then F has countable b -property in E . Order continuity of the ambient space is essential in this example, if one takes $E = l^\infty$ and $F = c_0$. Then, by considering the sequence e_n in c_0 , we see that c_0 has no b -property in l^∞ .

Example 1.10 Generalizing Example 1.9, let $E(\tau)$ be an LSVL with Lebesgue property. Then every τ -closed order ideal F has b -property in $E(\tau)$. This is because every τ -closed ideal is a band and, as $E(\tau)$ is Dedekind complete, it is a projection band.

Example 1.11 Given a VL E , let us denote by E^λ its lateral completion and E^u its universal completion. Since X is majorizing in X^δ by Example 1.6, the equality $(E^\lambda)^\delta = (E^\delta)^\lambda = E^u$ (see [1, Exer.10 on p.213]) shows that E^λ is majorizing in E^u ; therefore, each laterally complete VL E has b -property in its universal completion E^u .

Example 1.12 If E is a laterally complete VL, then it has the band projection property and every band on E has b -property. Furthermore, a subset $A \subset E^+$ of a laterally complete VL E is order bounded in E^u iff A is order bounded in E by [1, Theorems 7.14 and 7.37].

Let us observe that all Lebesgue topologies on a LSVL $E(\tau)$ induce the same topology on order bounded subsets of E . Therefore, if F is a sublattice of E then on all subsets of F with b -property in E all Lebesgue topologies on E induce the same topology.

Example 1.13 Let F be an order dense sublattice of a vector lattice E . If F is laterally complete in its own right, then F majorizes E and therefore has b -property in E .

We refer to [1, 10] for all undefined terms.

2. Main results

Lemma 2.1 *Let F be a sublattice of a LSVL $E(\tau)$. Then each b -bounded in E subset B of F is τ -bounded with respect to induced topology on F .*

Proof To say that B is b -bounded in E is to say that B is order bounded in E . Therefore, if U is a neighborhood of 0 in τ then $B \subseteq \lambda U$ for some $\lambda > 0$. Then $B \subseteq \lambda U \cap F = \lambda(U \cap F)$. \square

Lemma 2.2 *Let E be a vector lattice and F be an order dense sublattice of E . Then TFAE:*

- i) F has b -property in E ;*
- ii) F is majorizing in E .*

Proof *i) \implies ii):* Let $0 \leq x \in E$ be arbitrary, as F is order dense in E , there exists a net x_α in F such that $0 \leq x_\alpha \uparrow x$. As x_α is b -bounded in E by assumption, there exists $x_0 \in F^+$ with $0 \leq x_\alpha \leq x_0$ for all α , as $x_\alpha \uparrow x$, we have $x \leq x_0$ and F is majorizing.

ii) \implies i): Let x_α be a net in F with $0 \leq x_\alpha \uparrow x$ for some $x \in E$. Since F is assumed to be majorizing E , there exists $y \in F$ with $x \leq y$. Consequently, $0 \leq x_\alpha \uparrow \leq y \in F$; hence, F has b -property in E . \square

This yields: E has b -property in E^u iff E is majorizing in E^u . We also have, if $E(\tau)$ is a LSVL where E is an ideal of $\hat{E}(\hat{\tau})$ and \hat{E} is the completion, that E has b -property in $\hat{E}(\hat{\tau})$.

On the other hand, if $E(\tau)$ is a LSVL with Fatou property, then every increasing τ -bounded net of E^+ is order bounded in E^u , i.e. every increasing τ -bounded net of E^+ is b -bounded in E^u by [1, Theorem 7.51].

The following property was introduced in [8] and [9].

Definition 2.3 *A locally solid vector lattice $E(\tau)$ is called boundedly order bounded (BOB) if every τ -bounded net in E^+ is order bounded in E .*

We show BOB is equivalent to b -property if the LSVL $E(\tau)$ has Fatou property.

Lemma 2.4 *Let $E(\tau)$ be a LSVL with Fatou property. Then E has b -property in E^u iff E is BOB.*

Proof Suppose E is BOB and x_α be a net in E with $0 \leq x_\alpha \uparrow \leq x_0$ for some $x_0 \in E^u$. Then, by Lemma 2.1, x_α is τ -bounded in E and, by assumption that E is BOB, $0 \leq x_\alpha \leq x$ for some $x \in E$.

Conversely, suppose that x_α is τ -bounded increasing net in E^+ , then by [1, Theorem 7.50], x_α is order bounded in E^u . Thus by b -property of E in E^u , there exists $x \in E$ with $0 \leq x_\alpha \leq x$ and $E(\tau)$ is BOB. \square

[1, Theorem 7.49] shows that, in a laterally σ -complete LSVL $E(\tau)$, every disjoint sequence in E^+ converges to zero with respect to any LS topology on E . We show a similar result. The proof is similar.

Proposition 2.5 *Let $E(\tau)$ be a LSVL which has countable b -property in its lateral σ -completion. Then every disjoint sequence in E^+ converges to zero with respect to any locally solid topology on E . In particular, every locally solid topology on E has the pre-Lebesgue property.*

Proof Let x_n be a disjoint sequence in E^+ . Then nx_n is also a disjoint sequence in E^+ . Then $x = \bigvee_{n=1}^{\infty} nx_n$ exists in the lateral σ -completion, and we have $0 \leq x_n \leq \frac{1}{n}x$ for all n . Countable b -property of E in its lateral completion yields a vector $e \in E$ with $0 \leq x_n \leq \frac{1}{n}e$ for all n . Thus, x_n converges to zero with respect to any locally solid topology on E . \square

Recall that E has a countable b -property in its lateral completion E^λ if, for each x_n with $0 \leq x_n \uparrow e$ for some $e \in E^\lambda$, there holds $x_n \uparrow x \in E$.

Corollary 2.6 *Let $E(\tau)$ be an LSVL with Lebesgue property. If E has countable b -property in its lateral σ -completion, then the topological completion \hat{E} of $E(\tau)$ is E^u .*

Proof Under the given conditions, every disjoint sequence in E^+ is τ -convergent to zero by Proposition 2.5. Thus, the corollary follows from [1, Theorem 7.51]. \square

Proposition 2.7 *A laterally complete vector lattice E has b -property in every vector lattice which contains E as an order dense sublattice.*

Proof In this case, E majorizes the vector lattice that contains it. The result now follows from [1, Theorem 7.15]. \square

In [11, Proposition 2.22] it is proved that if $E(\tau)$ is a LSVL with Lebesgue topology, then a sublattice F of E is $u\tau$ -closed in E iff it is τ -closed. It was asked in [11, Question 2.24] whether Lebesgue assumption could be removed. The next result yields an answer utilizing b -property.

Theorem 2.8 *Let F be an order ideal of an LSVL $E(\tau)$. If F has b -property in E , then F is $u\tau$ -closed iff it is τ -closed in E .*

Proof As $u\tau$ is coarser than τ , the forward implication is clear.

We will show $x \in F$. Suppose that F is τ -closed and y_α is a net in F with $y_\alpha \xrightarrow{u\tau} x$ for some $x \in E$. The lattice operations are $u\tau$ -continuous so that $y_\alpha^\pm \xrightarrow{u\tau} x$. Therefore, WLOG we may assume $0 \leq y_\alpha$ for all α . Let $z \in E^+$ be arbitrary, then

$$|y_\alpha \wedge z - x \wedge z| \leq |y_\alpha - x| \wedge z \xrightarrow{\tau} 0.$$

Since $0 \leq y_\alpha \wedge x \leq y_\alpha$ for all α , and F is an order ideal, we have $y_\alpha \wedge x \in F$ for all α and $y_\alpha \wedge x \xrightarrow{\tau} x \wedge x$.

Take $y \in F$, then $y_\alpha \wedge y \xrightarrow{\tau} x \wedge y$, since F is τ -closed, we have $x \wedge y \in F$ for each $y \in F^+$. If $z \in F^d$, then $y_\alpha \wedge z = 0$ for all α and we have $x \wedge z = 0$. Thus, $x \in F^{dd}$. That is, x is in the band generated by F in E . Hence, there exists a net z_β in F^+ such that $0 \leq z_\beta \uparrow |x|$. Therefore, z_β is b -bounded in E , by b -property of F in E , $0 \leq z_\beta \leq x_0$ for some $x_0 \in F$ and $|x| \leq x_0$. Hence, $x \in F$ as F is an ideal. \square

It is shown in [1, Theorem 7.39] that a Dedekind complete vector lattice is universally complete iff it is universally σ -complete and has a weak unit. In the next result, we replace universally σ -completeness with countable b -property of E in E^u .

Theorem 2.9 *Let E be a Dedekind complete vector lattice. Then E has a weak order unit and possesses countable b -property in E^u iff $E = E^u$.*

Proof If $E = E^u$ then E has b -property in E^u and has a weak order unit (cf. [1, Theorem 7.2]). Now we prove the converse. Let $0 < e$ be a weak order unit for E . Then E is an order ideal in E^u by [1, Theorem 1.40]. Let $0 < u \in E^u$ be arbitrary. Since e is also a weak unit for E^u (E is order dense in E^u), we have $0 < u \wedge ne \uparrow u$. As $u \wedge ne \in E$ for each n , we see that the sequence $u \wedge ne$ is b -bounded in E^u . Therefore, the sequence $u \wedge ne$ has an upper bound in E by the assumption. Thus, $0 \leq u \wedge ne \leq x$ for some $x \in E$; hence, $0 \leq u \leq x$. As E is an order ideal in E^u , we have $u \in E$. \square

It is well known that if $E(\tau)$ is a LSVL with Levi property and τ -complete order intervals, then E is Dedekind complete. In the following, we reach the same conclusion by replacing Levi property with weaker condition that E having b -property in $\hat{E}(\hat{\tau})$.

Proposition 2.10 *Let $E(\tau)$ be an LSVL with τ -complete order intervals. If $E(\tau)$ has b -property in the τ -completion \hat{E} of $E(\tau)$, then $E(\tau)$ is τ -complete.*

Proof The assumption on order intervals implies that $E(\tau)$ is an order dense ideal of \hat{E} by [1, Theorem 2.42]. Let $0 < \hat{x} \in \hat{E}$ be arbitrary. Since $E(\tau)$ is order dense in \hat{E} , there exists a net x_α such that $0 \leq x_\alpha \uparrow \hat{x}$. By the b -property of $E(\tau)$ in \hat{E} , we can find $x_0 \in E$ with $0 \leq x_\alpha \leq x_0$, but then since $x_\alpha \uparrow \hat{x}$, we have $\hat{x} \leq x_0$ and $\hat{x} \in E$ because E is an ideal in \hat{E} . Therefore, $E(\tau) = \hat{E}$ as required. \square

Proposition 2.11 *Let F be a regular sublattice of a Dedekind complete VL E . Then each increasing net of elements of F which is order bounded in E is uo -Cauchy in F .*

Proof Let x_α be a net in F such that $0 \leq x_\alpha \uparrow \leq e$ for some $e \in E^+$. Since E is Dedekind complete, $x_\alpha \uparrow x$ for some $x \in E^+$. Then x_α is o -Cauchy in E ; hence, it is uo -Cauchy in E . Therefore, x_α is uo -Cauchy in F by [7, Theorem 3.2]. \square

It was observed in [7, Theorem 3.2] for a net x_α in a regular sublattice F of a vector lattice E , $x_\alpha \xrightarrow{uo} 0$ in F iff $x_\alpha \xrightarrow{uo} 0$ in E . However, this may fail for $u\tau$ -convergence. $u\tau$ -Convergence in a sublattice may not imply $u\tau$ -convergence in the entire space. For example, the standard unit vectors e_n in l^∞ is easily seen to be a null sequence in the unbounded norm topology of c_0 but not so in l^∞ .

Proposition 2.12 *Let F be a sublattice of an LSVL $E(\tau)$. Suppose that F has b -property in E . For a net x_α in F for which $x_\alpha \xrightarrow{u\tau} 0$ in F , we have $x_\alpha \xrightarrow{u\tau} 0$ in $E(\tau)$.*

Proof Suppose $x_\alpha \xrightarrow{u\tau} 0$ in F . WLOG we may suppose $0 \leq x_\alpha$ for all α . Then $0 \leq x_\alpha \wedge y \xrightarrow{\tau} 0$ for each $y \in F^+$. On the other hand, for each $x \in E^+$, $0 \leq x_\alpha \wedge x \leq x$ and the net $0 \leq (x_\alpha \wedge x)$ is b -bounded in F , by the hypothesis, there exists $y \in F^+$ such that $0 \leq x_\alpha \wedge x \leq y$ for all α . Then

$$0 \leq x_\alpha \wedge x \leq x_\alpha \wedge y \xrightarrow{\tau} 0$$

from which we obtain $x_\alpha \wedge x \xrightarrow{\tau} 0$. As x is arbitrary $x_\alpha \xrightarrow{u\tau} 0$ in $E(\tau)$. \square

Proposition 2.13 *Let $E(\tau)$ be a laterally complete vector lattice, then E has b -property in $(E^\sim)_n^\sim$.*

Proof Recall that E is order dense in $(E^\sim)_n^\sim$. Then E is majorizing in $(E^\sim)_n^\sim$ by [1, Theorem 7.15]. Therefore, E has b -property in $(E^\sim)_n^\sim$ \square

Theorem 2.14 *Let $E(\tau)$ be an LSVL with Lebesgue property. Then every order closed sublattice F of $E(\tau)$ has countable b -property in $\hat{E}(\hat{\tau})$.*

Proof Let $F^+ \ni x_n \uparrow \leq \hat{x} \in \hat{E}(\hat{\tau})$. Since the topology $\hat{\tau}$ of $\hat{E}(\hat{\tau})$ is also Lebesgue [1, Theorem 3.26] and hence is pre-Lebesgue, the sequence x_n is $\hat{\tau}$ -Cauchy in $\hat{E}(\hat{\tau})$; therefore, $x_n \xrightarrow{\hat{\tau}} z$ for some $z \in \hat{E}(\hat{\tau})$. Since $\hat{\tau}$ is Fatou by [1, Lemma 4.2], and F being order closed is $\hat{\tau}$ -closed by [1, Theorem 4.20], $z \in F$. As $x_n \uparrow$, $x_n \xrightarrow{\hat{\tau}} z$, hence $z = \sup x_n$ by [1, Theorem 2.21], and F has countable b -property in $\hat{E}(\hat{\tau})$. \square

Proposition 2.15 *Let F be a uo -closed sublattice of a Dedekind complete vector lattice E . Then F has b -property in E .*

Proof Let x_α be a net in F with $0 \leq x_\alpha \uparrow \leq x$ for some $x \in E$. As E is Dedekind complete, $x_\alpha \uparrow \hat{x}$ for some $\hat{x} \in E$. Then $x_\alpha \xrightarrow{o} \hat{x}$, consequently $x_\alpha \xrightarrow{uo} \hat{x}$ in E as F is uo -complete, $\hat{x} \in F$. \square

Notice that Theorem 2.14 follows from Proposition 2.15 under an additional assumption that $\hat{E}(\hat{\tau})$ is Dedekind complete.

Theorem 2.16 *Let E be a vector lattice admitting a minimal topology τ . Let x_n be an increasing sequence of elements of E order bounded in E^u . Then x_n is τ -Cauchy in E .*

Proof Let x_n be such that $0 \leq x_n \uparrow \leq x^u$ for some $x^u \in E^u$. Since E^u is Dedekind complete, x_n being order bounded in E^u , has a supremum in E^u , let it be x . Therefore $x_n \xrightarrow{o} x$, it follows that x_n is uo -Cauchy in E^u . Since E is order dense in E^u , and order dense sublattices are regular, E is regular in E^u and by [7, Theorem 3.2], x_n is uo -Cauchy in E . As every minimal topology is Lebesgue, τ is Lebesgue and x_n is $u\tau$ -Cauchy. As τ is unbounded, it follows that x_n is τ -Cauchy on E . \square

Definition 2.17 *A locally solid vector lattice $E(\tau)$ is called boundedly uo -complete if every τ -bounded uo -Cauchy net in $E(\tau)$ is uo -convergent.*

Proposition 2.18 *A boundedly uo -complete LSVL $E(\tau)$ has b -property in E^u .*

Proof Let $0 \leq x_\alpha \uparrow \leq x^u$, where $x^u \in E^u$, be a net in E . As x_α is a b -bounded subset of E , it is τ -bounded by Lemma 2.1. We show x_α has an upper bound in E . As E^u is Dedekind complete, $\sup x_\alpha$ exists in E^u . Let this supremum be x . Then $0 \leq x_\alpha \uparrow x$ in E^u . Thus, $x_\alpha \xrightarrow{o} x$. It follows that x_α is uo -Cauchy in E as E is order dense and a regular sublattice of E^u . Thus, x_α being uo -Cauchy and τ -bounded, x_α uo -converges to some $x' \in E$, but as $x_\alpha \xrightarrow{o} x$ we have $x = x'$. \square

Definition 2.19 A Banach lattice is *monotonically complete* (has the *Levy property*) if every norm bounded increasing net in E^+ has supremum.

We now show that every boundedly *uo*-complete Banach lattice E has *b*-property in $(E_n^\sim)_n^\sim$. The proof uses an idea of [6] in that $(E_n^\sim)_n^\sim$ is monotonically complete and the canonical map $J : E \rightarrow (E_n^\sim)_n^\sim$ maps a bounded increasing net in E^+ to a net in $(E_n^\sim)_n^\sim$ with similar properties.

Theorem 2.20 Let E be a boundedly *uo*-complete Banach lattice with E_n^\sim separating points of E . If x_α is an increasing net in E^+ which is order bounded in $(E_n^\sim)_n^\sim$, then x_α is order bounded in E .

Proof Since the net x_α is order bounded in $(E_n^\sim)_n^\sim$, it is norm bounded in $(E_n^\sim)_n^\sim$ and hence norm bounded in E by Lemma 2.1.

Let $J : E \rightarrow (E_n^\sim)_n^\sim$ be the natural embedding, where $J(x)(f) = f(x)$ for each $x \in E$ and $f \in E_n^\sim$. The map J is a vector lattice isomorphism and the range $J(E)$ in $(E_n^\sim)_n^\sim$ is order dense in $(E_n^\sim)_n^\sim$ by [1, Theorem 1.43]. Therefore, $J(E)$ is a regular sublattice of $(E_n^\sim)_n^\sim$.

By [10, 2.4.19], $(E_n^\sim)_n^\sim$ is a monotonically complete Banach lattice. Thus, the increasing net $J(x_\alpha)$ has a supremum in $(E_n^\sim)_n^\sim$ say x . Therefore, $J(x_\alpha) \uparrow x$ and $J(x_\alpha)$ is order Cauchy in $(E_n^\sim)_n^\sim$. It follows that $J(x_\alpha)$ is *uo*-Cauchy in $(E_n^\sim)_n^\sim$ and in the regular sublattice $J(E)$. As J is 1-1 and onto $J(E)$ is lattice isomorphism, x_α is *uo*-Cauchy in E . Since E is boundedly *uo*-complete, $x_\alpha \xrightarrow{uo} x_1$ for some $x_1 \in E$. On the other hand, $0 \leq x_\alpha \uparrow$ implies $x_\alpha \uparrow x_1$; hence, the net x_α is order bounded in E . \square

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