




A refinement of the Bergström inequality

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Abstract: In this paper, the Bergström inequality is studied, and a refinement of this inequality is obtained by performing the optimality conditions based on abstract concavity. Some numerical experiments are given to illustrate the efficacy of the refinement.

Key words: Abstract concavity, refinement, Bergström inequality, global optimization

1. Introduction

The refinements or extensions of inequalities take place widely in inequality theory. There are many approaches or techniques to obtain new versions of inequalities and studies on the equivalences of inequalities (See [3, 8, 10, 12]). Some of these approaches are based on the convexity of functions as in Jensen type, Hermite-Hadamard type inequalities. In recent years, new convexity types have been developed (See [4, 15, 19]). Many new inequalities and their versions, such as integral, fractional integral, and Hermite-Hadamard type inequalities, have been obtained for the function classes of various convexity types by different authors in [2, 5, 6, 11, 13, 15, 18, 20, 23–27]. Also, sharper versions of the well-known discrete inequalities have been derived by means of the results of abstract convexity in [1, 16, 20–22]. In [1] and [20], the sharper versions for weighted arithmetic, geometric, and harmonic mean inequalities and Hölder inequality are derived with the help of the results in [16]. In this study, we give a refinement of the Bergström inequality.

In [15], A.M. Rubinov gives the definition of abstract concave function as follows:

Let H be the class of functions on $\Omega \subset \mathbb{R}^n$. If there exists a function $h \in H$ such that $h \geq f$, it is said that $f : \Omega \rightarrow R \cup \{-\infty\}$ is majorized by H .

Definition 1.1 A function $f : \Omega \rightarrow R \cup \{-\infty\}$ majorized by H is called abstract concave with respect to H (or H – concave) if there exists a set $U \subset H$ such that

$$f(x) = \inf_{h \in U} h(x)$$

for all $x \in \Omega$.

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In [16], the optimality conditions for the function f that is abstract concave function with respect to H where

$$H = \left\{ h : X \rightarrow \mathbb{R} \mid a > 0, l \in X, c \in \mathbb{R}, h(x) = a \|x\|^2 + \langle l, x \rangle + c \right\}$$

are considered and the following result is obtained:

Theorem 1.2 [16] *Let $\|\cdot\|$ and $\|\cdot\|_o$ be two norms on \mathbb{R}^n . Let Ω be the subset of \mathbb{R}^n with nonempty interior and let $f \in C^1(\Omega)$. Suppose that $x \mapsto \nabla f(x)$ is a Lipschitz mapping on Ω and*

$$K := \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|} < \infty.$$

Let f have global minimum at $x^ \in \text{int } \Omega$. Consider the ball*

$$B_o(x^*, r) = \{x : \|x - x^*\|_o \leq r\} \subset \text{int}\Omega$$

and

$$M := \max \{\|\nabla f(x)\|_o : x \in B_o(x^*, r)\}.$$

Let q be a positive number such that $B_o(x^, r + q) \subset \Omega$ and let $a \geq \max\left(K, \frac{M}{2q}\right)$. Then*

$$\frac{1}{4a} \|\nabla f(x)\|^2 \leq f(x) - f(x^*), \quad x \in B_o(x^*, r).$$

Theorem 1.2 can be used to obtain sharper versions of well-known inequalities in such a way that each inequality can be written in the form of $f(x) \geq 0$, also if the points satisfying the equation $f(x) = 0$ are known, which are also global minimum points of f , then the theorem can be applied to suitable inequalities. In this paper, we will apply the theorem to obtain a refinement of the Bergström inequality. This inequality is given as follows:

For $x_k \in \mathbb{R}$ and $a_k > 0$, $k \in \{1, 2, \dots, n\}$,

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{a_1 + a_2 + \dots + a_n}$$

holds. Equality occurs if and only if $x_k = \lambda a_k$ for $k \in \{1, 2, \dots, n\}$ and $\lambda \in \mathbb{R}$.

Bergström inequality has stimulated several mathematicians' interest, and various extensions, refinements, and proofs of the inequality have been provided. We refer to [7, 9, 14, 17, 28, 29] and the references given therein.

In this paper, a refinement of this inequality for real numbers x_k is obtained.

2. Main results

Theorem 1.2 has been used to obtain the refinements of some inequalities in [1, 20]. Likewise, by means of the same theorem, the refined version of the Bergström inequality under certain conditions is given below.

Theorem 2.1 *If $x_k \in \mathbb{R}$ and $a_k > 0$, $k \in \{1, 2, \dots, n\}$, then*

$$\sum_{i=1}^n \frac{x_i^2}{a_i} \geq \frac{\left(\sum_{i=1}^n x_i\right)^2}{\sum_{i=1}^n a_i} + \frac{1}{2} \left[\sum_{k=1}^n a_k^{-2} - 2 \left(\sum_{i=1}^n a_i\right)^{-1} \sum_{k=1}^n a_k^{-1} + n^2 \left(\sum_{i=1}^n a_i\right)^{-2} \right]^{-\frac{1}{2}} \sum_{k=1}^n \left(\frac{x_k}{a_k} - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i} \right)^2.$$

Proof Let a_1, \dots, a_n and λ be positive numbers. Consider the function f on \mathbb{R}^n such that

$$f(x) = \sum_{i=1}^n \frac{x_i^2}{a_i} - \frac{\left(\sum_{i=1}^n x_i\right)^2}{\sum_{i=1}^n a_i}.$$

The function f has the global minimum at the points x such that $x_i = \lambda a_i$. Taking into account Theorem 1.2 for $f(x)$, we can sharpen the Bergström inequality. Some elementary calculations imply that

$$\nabla f(x) = 2 \left[\frac{x_1}{a_1} - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i}, \frac{x_2}{a_2} - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i}, \dots, \frac{x_n}{a_n} - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i} \right]$$

whence

$$\|\nabla f(x)\|^2 = 4 \sum_{k=1}^n \left[\frac{x_k}{a_k} - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i} \right]^2.$$

Assume $\|\cdot\|_\circ$ as maximum norm $\|\cdot\|_\infty$, consider the set

$$\begin{aligned} V_{\lambda,d} &= B_\infty(\lambda \mathbf{a}, d) = \{x \in \mathbb{R}^n : \|x - \lambda \mathbf{a}\|_\infty \leq d\} \\ &= \{x \in \mathbb{R}^n : \lambda a_i - d \leq x_i \leq \lambda a_i + d, i = 1, \dots, n\} \end{aligned}$$

where $\mathbf{a} = (a_1, a_2, \dots, a_n)$. Let us define $\rho_k(x) = \frac{x_k}{a_k} - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i}$ and estimate $\|\nabla \rho_k(x)\|$ for $x \in V_{\lambda,d}$ to show that

$\nabla f(x)$ is Lipschitz mapping on $V_{\lambda,d}$.

$$\begin{aligned} \frac{\partial \rho_k}{\partial x_k}(x) &= \frac{1}{a_k} - \frac{1}{\sum_{i=1}^n a_i}, \\ \frac{\partial \rho_k}{\partial x_j}(x) &= -\frac{1}{\sum_{i=1}^n a_i} \quad (k \neq j) \end{aligned}$$

so

$$\|\nabla \rho_k(x)\| = \left[\frac{1}{a_k^2} - \frac{2}{a_k \sum_{i=1}^n a_i} + n \left(\sum_{i=1}^n a_i\right)^{-2} \right]^{\frac{1}{2}}. \tag{2.1}$$

Assume that $x, z \in V_{\lambda, d}$. By means of the mean value theorem and Cauchy-Schwarz inequality, it can be concluded that there exist numbers $\theta_i \in (0, 1)$ for $i = 1, 2, \dots, n$ such that

$$\begin{aligned} \|\nabla f(x) - \nabla f(z)\| &= 2 \|\rho_1(x) - \rho_1(z), [\rho_2(x) - \rho_2(z)], \dots, [\rho_n(x) - \rho_n(z)]\| \\ &= 2 \left(\sum_{k=1}^n [\rho_k(x) - \rho_k(z)]^2 \right)^{\frac{1}{2}} \\ &= 2 \left(\sum_{k=1}^n [\nabla \rho_k(x + \theta_k(z - x))(x - z)]^2 \right)^{\frac{1}{2}} \\ &\leq 2 \left(\sum_{k=1}^n \|\nabla \rho_k(x + \theta_k(z - x))\|^2 \right)^{\frac{1}{2}} \|x - z\| \\ &= 2 \left(\sum_{k=1}^n \left[\frac{1}{a_k^2} - \frac{2}{a_k \sum_{i=1}^n a_i} + n \left(\sum_{i=1}^n a_i \right)^{-2} \right] \right)^{\frac{1}{2}} \|x - z\| \\ &= 2 \left(\sum_{k=1}^n \frac{1}{a_k^2} - \frac{2}{\sum_{i=1}^n a_i} \sum_{k=1}^n \frac{1}{a_k} + n^2 \left(\sum_{i=1}^n a_i \right)^{-2} \right)^{\frac{1}{2}} \|x - z\|. \end{aligned}$$

Now we have

$$\|\nabla f(x) - \nabla f(z)\| \leq A \|x - z\|, \quad x, z \in V_{\lambda, d}$$

where

$$A = 2 \left(\sum_{k=1}^n a_k^{-2} - 2 \left(\sum_{i=1}^n a_i \right)^{-1} \sum_{k=1}^n a_k^{-1} + n^2 \left(\sum_{i=1}^n a_i \right)^{-2} \right)^{\frac{1}{2}}.$$

In conclusion, the mapping $x \rightarrow \nabla f(x)$ is Lipschitz with the constant $K \leq A$.

Let $r \in (0, d)$ and $q = d - r$. We can estimate $M = \max \{\|\nabla f(x)\|_{\infty} : x \in V_{\lambda, r}\}$ as follows:

$$\begin{aligned} M &= \max_{x \in V_{\lambda, r}} \{\|\nabla f(x)\|_{\infty}\} = 2 \max_{x \in V_{\lambda, r}} \left\{ \max_{1 \leq k \leq n} \left| \frac{x_k}{a_k} - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i} \right| \right\} \\ &\leq 2r \max_{1 \leq k \leq n} \left\{ \frac{1}{a_k} + \frac{n}{\sum_{i=1}^n a_i} \right\} \\ &\equiv 2r \left\{ \frac{1}{\min_{1 \leq k \leq n} a_k} + \frac{n}{\sum_{i=1}^n a_i} \right\} = M_0. \end{aligned}$$

Now we can determine the number a such that $a \geq \max\left(K, \frac{M}{2q}\right)$. Since $\lim_{d \rightarrow r^+} \frac{M_0}{2(d-r)} = +\infty$ and the function $d \mapsto \frac{M_0}{2(d-r)}$ is continuous on (r, λ) , we can set the number a in Theorem 1.2 as

$$a = \min_{r < d < \lambda} \max\left\{A, \frac{M_0}{2(d-r)}\right\} = A.$$

Applying Theorem 1.2, we conclude that

$$\sum_{i=1}^n \frac{x_i^2}{a_i} \geq \frac{\left(\sum_{i=1}^n x_i\right)^2}{\sum_{i=1}^n a_i} + \frac{1}{A} \sum_{k=1}^n \left[\frac{x_k}{a_k} - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i} \right]^2 \quad \text{for } x \in V_{\lambda,r}. \tag{2.2}$$

It is seen that (2.1) is independent of the values of x , whence A is independent of x . On the other hand, M_0 changes with respect to r , but it does not affect the determination of a since $\frac{M_0}{2(d-r)} \rightarrow \infty$ as $d \rightarrow r^+$. Thus, we can deduce that (2.2) is valid for all $x \in \mathbb{R}$. □

Remark 2.2 Assuming $\lambda = 1$ and $r = \max_k |x_k - a_k| + c$ for positive numbers a_k, c and real numbers x_k , one can determine the appropriate region Ω to which theorem is applied.

3. Numerical examples

In this section, we present some numerical examples to see the numerical efficacy of sharper version of the Bergström inequality. Let f be a function as denoted in the proof and let

$$u(x) = \frac{1}{2} \left[\sum_{k=1}^n a_k^{-2} - 2 \left(\sum_{i=1}^n a_i \right)^{-1} \sum_{k=1}^n a_k^{-1} + n^2 \left(\sum_{i=1}^n a_i \right)^{-2} \right]^{-\frac{1}{2}} \sum_{k=1}^n \left(\frac{x_k}{a_k} - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i} \right)^2.$$

Since the amount of $u(x)$ can increase with respect to a_k values and n , we present $\frac{u(x)}{f(x)}$ as relative sharpening ratio to see the efficacy of the proposed inequality.

In Table 1, we choose a_i and x_k as positive numbers in increasing order for different n values.

Table 1. $a_i = i, x_i = i^3$

n	$\approx f(x)$	$\approx u(x)$	$\approx \frac{u(x)}{f(x)}$ (%)
5	1.050×10^3	2.0141×10^2	19.14
10	5.445×10^4	5.445×10^3	10
20	3.072×10^6	1.593×10^5	5.18
40	1.836×10^8	4.870×10^6	2.65
50	6.903×10^8	1.473×10^7	2.12

In Table 2, we choose a_i in increasing order as in Table 1 and x_k with alternating sign.

Table 2. $a_i = i, x_i = (-1)^i i^3$

n	$\approx f(x)$	$\approx u(x)$	$\approx \frac{u(x)}{f(x)}$ (%)
5	3.988×10^3	4.271×10^2	10.71
10	2.148×10^5	1.040×10^4	4.84
20	1.224×10^7	2.882×10^5	2.35
40	7.336×10^8	8.577×10^6	1.16
50	2.760×10^9	2.580×10^7	0.93

In Table 3, we choose a_i and x_i nonmonotonic.

Table 3. $a_i = 100 |\sin i|, x_i = i^3 \sin i$

n	$\approx f(x)$	$\approx u(x)$	$\approx \frac{u(x)}{f(x)}$ (%)
5	1.147×10^2	1.165×10^1	10.01
10	1.120×10^4	1.130×10^3	10
20	1.402×10^6	8.762×10^4	6.25
40	5.208×10^7	3.493×10^5	0.67
50	7.675×10^7	4.679×10^5	0.61

According to the numerical examples, one can empirically deduce that when n increases, the sharpening ratio decreases. One reason for this may be the place of n in $u(x)$. The more n increases, the more $u(x)$ decreases, so does sharpening ratio.

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