


On b -generalized skew derivations in Banach algebras

Abdul Nadim KHAN* 

Department of Mathematics, Faculty of Science & Arts, King Abdulaziz University, Rabigh, Saudi Arabia

Received: 15.11.2020

Accepted/Published Online: 23.03.2021

Final Version: .2021

Abstract: Let \mathcal{A} be a Banach algebra over \mathbb{R} or \mathbb{C} . In this paper, we describe the behavior of recently defined b -generalized skew derivations which satisfy certain differential identities on some specific subsets of \mathcal{A} .

Key words: Banach algebra, b -generalized skew derivation

1. Introduction

In this paper, unless otherwise mentioned, \mathcal{A} always denotes an unital prime Banach algebra over \mathbb{R} or \mathbb{C} . A linear map $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is said to be a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. A linear map $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ is called a generalized derivation if there exists a derivation δ of \mathcal{A} such that $\mathcal{F}(xy) = \mathcal{F}(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. Let \mathcal{A} be an associative ring and α be an automorphism of \mathcal{A} . A linear mapping $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is said to be a skew derivation of \mathcal{A} if $\delta(xy) = \delta(x)y + \alpha(x)\delta(y)$ for all $x, y \in \mathcal{A}$. The derivation δ is uniquely determined by \mathcal{F} , which is called an associated derivation of \mathcal{F} . The definition of generalized skew derivation is a unified notion of skew derivation and generalized derivation, which are considered as classical linear mappings of noncommutative algebras. Let \mathcal{A} be an associative algebra and α be an automorphism of \mathcal{A} . A linear mapping $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ is said to be a generalized skew derivation of \mathcal{A} if $\mathcal{F}(xy) = \mathcal{F}(x)y + \alpha(x)\delta(y)$ for all $x, y \in \mathcal{A}$. In this case, δ is called an associated skew derivation of \mathcal{F} and α is called an associated automorphism of \mathcal{F} . In a recent paper [10], Koşan and Lee proposed that an additive map $\mathcal{F} : \mathcal{R} \rightarrow \mathcal{Q}$ is called a left b -generalized derivation, with associated additive mapping δ from \mathcal{R} to \mathcal{Q} , if $\mathcal{F}(xy) = \mathcal{F}(x)y + b\delta(y)$ for all $x, y \in \mathcal{R}$ and $b \in \mathcal{Q}$, where \mathcal{R} is a prime ring and \mathcal{Q} is the right Martindale quotient ring of \mathcal{R} . In the same paper, it is proved that, if \mathcal{R} is a prime ring, then δ is a derivation of \mathcal{R} . For simplicity of notation, this mapping \mathcal{F} will be called a b -generalized derivation with associated pair (b, δ) . Clearly, any generalized derivation with associated derivation δ is a b -generalized derivation with associated pair $(1, \delta)$. Similarly, the mapping $x \rightarrow ax + b[x, c]$, for $a, b, c \in \mathcal{Q}$, is a b -generalized derivation with associated pair $(b, ad(c))$, where $ad(c)(x) = [x, c]$ denotes the inner derivation of \mathcal{R} induced by the element c . More generally, the mapping $x \rightarrow ax + qxc$, for $a, c, q \in \mathcal{Q}$, is a b -generalized derivation with associated pair $(q, ad(c))$. This mapping is called inner b -generalized derivation. Moreover, if $\alpha \in Aut(\mathcal{R})$, with $\alpha(x) = qxq^{-1}$ for q an invertible element of \mathcal{Q} , and \mathcal{F} is the inner generalized skew derivation with associated automorphism α , then \mathcal{F} is a b -generalized derivation with associated pair $(q, ad(q^{-1}b))$, for a suitable element $b \in \mathcal{Q}$.

*Correspondence: abdulnadimkhan@gmail.com

2010 AMS Mathematics Subject Classification: 16W25, 46J10

Motivated by the aforementioned definitions, recently the author and his team proved the following result in [9] for skew generalized derivations:

Theorem 1.1 *Let \mathcal{A} be a unital prime Banach algebra with centre $\mathcal{Z}(\mathcal{A})$ and Θ_1, Θ_2 be open subsets of \mathcal{A} , $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous linear generalized skew derivation, and $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous linear map. If, for each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x, y) > 1$ such that either $\mathcal{F}((xy)^m) - \mathcal{G}(x^m y^m) \in \mathcal{Z}(\mathcal{A})$ or $\mathcal{F}((xy)^m) - \mathcal{G}(y^m x^m) \in \mathcal{Z}(\mathcal{A})$, then \mathcal{A} is commutative.*

Theorem 1.2 *Let \mathcal{A} be a unital prime Banach algebra with centre $\mathcal{Z}(\mathcal{A})$ and Θ_1, Θ_2 be open subsets of \mathcal{A} , $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous linear generalized skew derivation, and $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous linear map. If, for each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x, y) > 1$ such that either $\mathcal{F}((xy)^m) + \mathcal{G}(x^m y^m) \in \mathcal{Z}(\mathcal{A})$ or $\mathcal{F}((xy)^m) - \mathcal{G}(y^m x^m) \in \mathcal{Z}(\mathcal{A})$, then \mathcal{A} is commutative.*

In 2018, De Filippis and Wei [5] extended the notion of b -generalized derivation to b -generalized skew derivation as follows. Let \mathcal{R} be a prime ring, $b \in \mathcal{Q}$, the right Martindale quotient ring of \mathcal{R} , $\delta : \mathcal{R} \rightarrow \mathcal{R}$ a linear mapping and α be an automorphism of \mathcal{R} . An additive mapping $\mathcal{F} : \mathcal{R} \rightarrow \mathcal{R}$ is called a b -generalized skew derivation of \mathcal{R} , with an associated term (b, α, d) if $\mathcal{F}(xy) = \mathcal{F}(x)y + b\alpha(x)\delta(y)$ for all $x, y \in \mathcal{R}$. Moreover, they proved that the linear map δ is a skew derivation with associated automorphism α .

According to the definition of b -generalized skew derivation, we can conclude that general results about b -generalized skew derivations may give useful and powerful corollaries about derivations, generalized derivations, skew derivations, and generalized skew derivations. The definition of b -generalized skew derivations is a unified notion of skew derivation and b -generalized derivation, which are considered as classical linear mappings of associative algebras. Interestingly, every b -generalized skew derivation neither b -generalized derivation nor generalized skew derivation (for example see [4, Section 4]). In light of these interesting facts, we shall establish the following results:

Theorem 1.3 *Let \mathcal{A} be a unital noncommutative prime Banach algebra with centre $\mathcal{Z}(\mathcal{A})$ and Θ_1, Θ_2 be open subsets of \mathcal{A} , $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous linear b -generalized skew derivation, and $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous linear map. If, for each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x, y) > 1$ such that either $\mathcal{F}((xy)^m) - \mathcal{G}(x^m y^m) \in \mathcal{Z}(\mathcal{A})$ or $\mathcal{F}((xy)^m) - \mathcal{G}(y^m x^m) \in \mathcal{Z}(\mathcal{A})$, then \mathcal{A} satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree four.*

Theorem 1.4 *Let \mathcal{A} be a unital noncommutative prime Banach algebra with centre $\mathcal{Z}(\mathcal{A})$ and Θ_1, Θ_2 be open subsets of \mathcal{A} , $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous linear b -generalized skew derivation, and $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{A}$ be a continuous linear map. If, for each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x, y) > 1$ such that either $\mathcal{F}((xy)^m) + \mathcal{G}(x^m y^m) \in \mathcal{Z}(\mathcal{A})$ or $\mathcal{F}((xy)^m) - \mathcal{G}(y^m x^m) \in \mathcal{Z}(\mathcal{A})$, then \mathcal{A} satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree four.*

2. Preliminaries results

En route to establishing the above mentioned theorems, we recall some well known facts without proof:

Fact 1 ([1]) Let $p(t) = \sum_{r=0}^n b_r t^r$ be a polynomial in real variable t for infinite values of t and each $b_r \in \mathcal{A}$. If $p(t) \in \mathcal{M}$ for infinite real values t , then each b_r lies in \mathcal{M} .

Fact 2 ([2]) It is well known that automorphisms, derivations and skew derivations of a prime ring \mathcal{A} can be extended to the right Martindale quotient ring of \mathcal{A} , \mathcal{Q}_r .

Fact 3 ([5], **Remark 1.9**) Every b -generalized skew derivation \mathcal{F} with associated term (b, α, δ) can be extended to \mathcal{Q} and assumes of the form $\mathcal{F}(x) = ax + b^\delta(x)$, where $a, b \in \mathcal{Q}$.

We begin our discussion with the following key result, which have been proved in [9]. Nevertheless, for the sake of completeness, here, we would like to provide the proof:

Proposition 2.1 Let \mathcal{A} be a unital prime Banach algebra and Θ_1, Θ_2 be open subsets of \mathcal{A} , $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ and $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{A}$ be two continuous linear maps. If, for each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x, y) > 1$ such that either $\mathcal{F}((xy)^m) - \mathcal{G}(x^m y^m) \in \mathcal{M}$ or $\mathcal{F}((xy)^m) - \mathcal{G}(y^m x^m) \in \mathcal{M}$, then the following facts hold simultaneously:

1. $\mathcal{F}(x) - \mathcal{G}(x) \in \mathcal{M}$ for all $x \in \mathcal{A}$;
2. $\mathcal{F}([x, y]) \in \mathcal{M}$ and $\mathcal{G}([x, y]) \in \mathcal{M}$ for all $x, y \in \mathcal{A}$;
3. there exists a fixed integer $m > 1$ such that for all $x, y \in \mathcal{A}$, both $\mathcal{F}((xy)^m - x^m y^m) \in \mathcal{M}$ and $\mathcal{F}((xy)^m - y^m x^m) \in \mathcal{M}$.

In the proof of above proposition, we adapt the arguments from the proof of [[12], Theorem 1]. However, we omit the details of arguments for brevity.

Proof Fix any $x \in \Theta_1$. For each $n > 1$ we define the set $U_n = \{y \in \mathcal{A} \mid \mathcal{F}((xy)^n) - \mathcal{G}(x^n y^n) \notin \mathcal{M} \text{ and } \mathcal{F}((xy)^n) - \mathcal{G}(y^n x^n) \notin \mathcal{M}\}$. It is easy to prove that U_n is open. Thus, by the Baire category theorem, if every U_n is dense then their intersection is also dense, which contradicts the existence of Θ_2 . Hence, there exists a positive integer r such that U_r is not dense. Therefore, there exists a nonempty open set Θ_3 in the complement of U_r such that for all $y \in \Theta_3$ either $\mathcal{F}((xy)^r) - \mathcal{G}(x^r y^r) \in \mathcal{M}$ or $\mathcal{F}((xy)^r) - \mathcal{G}(y^r x^r) \in \mathcal{M}$. Let $e_0 \in \Theta_3$ and $z \in \mathcal{A}$. Then $e_0 + tz \in \Theta_3$ for all sufficiently small real t . Thus, for each such t , we have

$$\mathcal{F}((x(e_0 + tz))^r) - \mathcal{G}(x^r(e_0 + tz)^r) \in \mathcal{M} \tag{2.1}$$

or

$$\mathcal{F}((x(e_0 + tz))^r) - \mathcal{G}((e_0 + tz)^r n^r) \in \mathcal{M}. \tag{2.2}$$

Thus at least one of (2.1) and (2.2) is valid for infinitely many t . Suppose (2.1) holds for these t . Then the expression $\mathcal{F}((x(e_0 + tz))^r) - \mathcal{G}(n^r(e_0 + tz)^r)$ can be written as

$$\begin{aligned} & \mathcal{F}(\Upsilon_{r,0}(x, e_0, z)) - \mathcal{G}(x^r \Upsilon'_{r,0}(e_0, z)) \\ & + \mathcal{F}(\Upsilon_{r-1,1}(x, e_0, z))t - \mathcal{G}(x^r \Upsilon'_{r-1,1}(e_0, z))t \\ & + \dots \\ & + \mathcal{F}(\Upsilon_{1,r-1}(x, e_0, z))t^{r-1} - \mathcal{G}(n^r \Upsilon'_{1,r-1}(e_0, z))t^{r-1} \\ & + \mathcal{F}(\Upsilon_{0,r}(x, e_0, z))t^r - \mathcal{G}(n^r \Upsilon'_{0,r}(e_0, z))t^r, \end{aligned}$$

where $\Upsilon_{i,j}(x, e_0, z)$ denotes the sum of all terms in which xe_0 appears exactly i times and xz appears exactly j times in the expansion of $(x(e_0 + tz))^r$ where i and j are nonnegative integers such that $i + j = r$. Similarly, $\Upsilon'_{i,j}(e_0, z)$ is sum of all terms in which e_0 appears exactly i times and z appears exactly j times in the expansion of $(e_0 + tz)^r$, where i and j are nonnegative integers such that $i + j = r$. The above expression is a polynomial in t and the coefficient of t^r in this polynomial is $\mathcal{F}((xz)^r) - \mathcal{G}(x^r z^r)$. Therefore in view of Fact 1, we have $\mathcal{F}((xz)^r) - \mathcal{G}(x^r z^r) \in \mathcal{M}$. On the other hand, if (2.2) holds for infinitely many t , then $\mathcal{F}((xz)^r) - \mathcal{G}(z^r x^r) \in \mathcal{M}$. Thus, given $x \in \Theta_1$ and for all $z \in \mathcal{A}$, there is a positive integer $r > 1$ depending on x such that either $\mathcal{F}((xz)^r) - \mathcal{G}(x^r z^r) \in \mathcal{M}$ or $\mathcal{F}((xz)^r) - \mathcal{G}(z^r x^r) \in \mathcal{M}$. Next, fix $y \in \mathcal{A}$ and for each positive integer k , set $V_k = \{e \in \mathcal{A} \mid \mathcal{F}((ey)^k) - \mathcal{G}(e^k y^k) \notin \mathcal{M} \text{ and } \mathcal{F}((ey)^k) - \mathcal{G}(y^k e^k) \notin \mathcal{M}\}$. Each V_k is open (X). If each V_k is dense, then by the Baire category theorem, their intersection is also dense; which contradicts the existence of the open set Θ_1 . Thus, there is an integer $m = m(y) > 1$ and a nonempty open subset Θ_4 in the complement of V_m . If $x_0 \in \Theta_4$ and $u \in \mathcal{A}$, then $x_0 + tu \in \Theta_4$ for all sufficiently small real t . Hence for positive integer $m > 1$ either

$$\mathcal{F}\left(\left((x_0 + tu)y\right)^m\right) - \mathcal{G}\left(\left(x_0 + tu\right)^m y^m\right) \in \mathcal{M}$$

or

$$\mathcal{F}\left(\left((x_0 + tu)y\right)^m\right) - \mathcal{G}\left(y^m \left(x_0 + tu\right)^m\right) \in \mathcal{M}$$

for each $u \in \mathcal{A}$ and $x_0 \in \Theta_4$. Arguing as above we see that, for any $y \in \mathcal{A}$ there exists $m = m(y) > 1$ such that, for any $u \in \mathcal{A}$, either

$$\mathcal{F}((uy)^m) - \mathcal{G}(u^m y^m) \in \mathcal{M} \tag{2.3}$$

or

$$\mathcal{F}((uy)^m) - \mathcal{G}(y^m u^m) \in \mathcal{M}. \tag{2.4}$$

Let S_k , $k > 1$ be the set of $y \in \mathcal{A}$ such that for each $z \in \mathcal{A}$ either $\mathcal{F}((zy)^k) - \mathcal{G}(z^k y^k) \in \mathcal{M}$ or $\mathcal{F}((zy)^k) - \mathcal{G}(y^k z^k) \in \mathcal{M}$, then the union of S_k will be \mathcal{A} . It can be easily proved that each S_k is closed. Hence again by Baire category theorem some S_n , $n > 1$ must have a nonempty open subset Θ_5 . Let $y_0 \in \Theta_5$, for all sufficiently small real t and each $v, z \in \mathcal{A}$ either

$$\mathcal{F}\left(\left(z(y_0 + tv)\right)^n\right) - \mathcal{G}\left(z^n \left(y_0 + tv\right)^n\right) \in \mathcal{M}$$

or

$$\mathcal{F}\left(\left(z(y_0 + tv)\right)^n\right) - \mathcal{G}\left(\left(y_0 + tv\right)^n z^n\right) \in \mathcal{M}.$$

By the same above argument, we have for each $v, z \in \mathcal{A}$ either

$$\mathcal{F}((zv)^n) - \mathcal{G}(z^n v^n) \in \mathcal{M} \tag{2.5}$$

or

$$\mathcal{F}((zv)^n) - \mathcal{G}(v^n z^n) \in \mathcal{M}. \tag{2.6}$$

Let e be the unity of \mathcal{A} . Hence, for all real t and for any $x, y \in \mathcal{A}$, either

$$\mathcal{F}\left(\left((e+tx)y\right)^n\right) - \mathcal{G}\left(\left(e+tx\right)^n y^n\right) \in \mathcal{M}$$

or

$$\mathcal{F}\left(\left((e+tx)y\right)^n\right) - \mathcal{G}\left(y^n(e+tx)^n\right) \in \mathcal{M}.$$

By the computation of the coefficient of t in the expansion of the above equations and using Fact 1, it follows that, for all $x, y \in \mathcal{A}$, either

$$\mathcal{F}\left(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}\right) - n\mathcal{G}(xy^n) \in \mathcal{M} \tag{2.7}$$

or

$$\mathcal{F}\left(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}\right) - n\mathcal{G}(y^n x) \in \mathcal{M}. \tag{2.8}$$

Now, taking $\mathcal{F}([(y(e+tx))^n])$ in place of $\mathcal{G}([(e+tx)y^n])$, we have that, for all $x, y \in \mathcal{A}$, either

$$\mathcal{F}\left(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}\right) - n\mathcal{G}(xy^n) \in \mathcal{M} \tag{2.9}$$

or

$$\mathcal{F}\left(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}\right) - n\mathcal{G}(y^n x) \in \mathcal{M}. \tag{2.10}$$

Then, at least one of pairs of equations $\{(2.7), (2.9)\}$, $\{(2.7), (2.10)\}$, $\{(2.8), (2.9)\}$ and $\{(2.8), (2.10)\}$ must hold.

Firstly, we notice that, for $y = e$ in any one of the equations (2.7) to (2.10), one has that $n\mathcal{F}(x) - n\mathcal{G}(x) \in \mathcal{M}$, that is $\mathcal{F}(x) - \mathcal{G}(x) \in \mathcal{M}$, for any $x \in \mathcal{A}$.

On the other hand, by combining the equations in the pairs $\{(2.7), (2.9)\}$, $\{(2.7), (2.10)\}$, $\{(2.8), (2.9)\}$ and $\{(2.8), (2.10)\}$, we have that, for any $x, y \in \mathcal{A}$, one of the following relation holds:

$$\mathcal{F}([x, y^n]) \in \mathcal{M} \tag{2.11}$$

$$\mathcal{F}([x, y^n]) + n\mathcal{G}([x, y^n]) \in \mathcal{M} \tag{2.12}$$

$$\mathcal{F}([x, y^n]) - n\mathcal{G}([x, y^n]) \in \mathcal{M}. \tag{2.13}$$

Replacing y by $e + ty$ in (2.11, 2.12 and 2.13), and using same above arguments, it follows that, for any $x, y \in \mathcal{A}$, one of the following holds:

$$\mathcal{F}([x, y]) \in \mathcal{M} \tag{2.14}$$

$$\mathcal{F}([x, y]) + n\mathcal{G}([x, y]) \in \mathcal{M} \tag{2.15}$$

$$\mathcal{F}([x, y]) - n\mathcal{G}([x, y]) \in \mathcal{M}. \tag{2.16}$$

In any case, since $n > 1$ and $\mathcal{F}(x) - \mathcal{G}(x) \in \mathcal{M}$, for all $x \in \mathcal{A}$, it follows that both $\mathcal{F}([x, y]) \in \mathcal{M}$ and $\mathcal{G}([x, y]) \in \mathcal{M}$, for any $x, y \in \mathcal{A}$. Since $\mathcal{F}(xy) - \mathcal{G}(yx) \in \mathcal{M}$, for any $x, y \in \mathcal{A}$, the relations (2.5) and (2.6) are equivalent. Moreover, since (2.5) and (2.6) hold simultaneously and $\mathcal{F}(x) - \mathcal{G}(x) \in \mathcal{M}$, for all $x \in \mathcal{A}$, then there exists a fixed integer $n > 1$ such that,

$$\mathcal{F}\left((xy)^n - y^n x^n\right) \in \mathcal{M} \text{ for all } x \in \mathcal{A},$$

and also

$$\mathcal{F}\left((xy)^n - x^n y^n\right) \in \mathcal{M} \text{ for all } x \in \mathcal{A},$$

as required. This completes the proof. □

Proof of Theorem 1.3. Since the centre $\mathcal{Z}(\mathcal{A})$ of \mathcal{A} , is also a closed linear subspace of \mathcal{A} , so we can replace $\mathcal{Z}(\mathcal{A})$ by \mathcal{M} . Thus, by Proposition 2.1, we have the following relations:

$$\mathcal{F}(x) - \mathcal{G}(x) \in \mathcal{Z}(\mathcal{A}), \text{ for all } x \in \mathcal{A}, \tag{2.17}$$

$$\mathcal{F}([x, y]) \in \mathcal{Z}(\mathcal{A}) \text{ for all } x, y \in \mathcal{A}, \tag{2.18}$$

$$\mathcal{G}([x, y]) \in \mathcal{Z}(\mathcal{A}), \text{ for all } x, y \in \mathcal{A}. \tag{2.19}$$

Moreover there exists a fixed integer $m > 1$, such that, for any $x, y \in \mathcal{A}$

$$\mathcal{F}((xy)^m - y^m x^m) \in \mathcal{Z}(\mathcal{A}). \tag{2.20}$$

Suppose, for that sake of a contradiction $[x, y] \notin \mathcal{Z}(\mathcal{A})$, for some $x, y \in \mathcal{A}$. In particular, since \mathcal{A} is prime, by [7, Theorem 1.1.8] there is a commutator which does not commute with $[x, y]$. Consequently, $\Lambda = [\mathcal{A}, \mathcal{A}]$ is a noncommutative Lie ideal of \mathcal{A} . Since $\mathcal{F}([x, y]) \in \mathcal{Z}(\mathcal{A})$, for any $x, y \in \mathcal{A}$, it follows that $[\mathcal{F}(z), z] = 0$, for any $z \in \Lambda$. Thus Theorem 1.5 in [5] for $a = e$ applies. If the first case holds, that is, if $\mathcal{F}(x) = \lambda x$ for each $x \in \mathcal{A}$, where $\lambda \in \mathcal{C}$ is fixed, then $\mathcal{F} \neq 0$ and the primeness of \mathcal{A} yields $[x, y] \in \mathcal{Z}(\mathcal{A})$, a contradiction. Hence, the second case must hold for \mathcal{F} , that is \mathcal{A} satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree four and there exist $b \in \mathcal{Q}_r$ and $\lambda \in \mathcal{C}$ such that $\mathcal{F}(x) = bx + xb + \lambda x$, for any $x \in \mathcal{A}$. If $b \in \mathcal{C}$, then this case reduces to the first one. Hence, we may assume that $b \notin \mathcal{C}$. From (2.20) it follows that for any $\alpha, \beta, \gamma \in \mathcal{A}$,

$$\left[\alpha, (b + \lambda) \left((\beta\gamma)^m - \gamma^m \beta^m \right) + \left((\beta\gamma)^m - \gamma^m \beta^m \right) b \right] = 0. \tag{2.21}$$

Consequently, \mathcal{A} satisfies a generalized polynomial identity. Thus, by [3] \mathcal{Q}_r also satisfies the same identity. In case \mathcal{C} is infinite, we obtain that $\mathcal{Q}_r \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ satisfies (2.21), where $\overline{\mathcal{C}}$ is the algebraic closure of \mathcal{C} . Since

both \mathcal{Q}_r and $\mathcal{Q}_r \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ are centrally closed (Theorems 2.5 and 3.5 in [6]), we may replace \mathcal{Q}_r by either \mathcal{Q}_r or $\mathcal{Q}_r \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ depending whether \mathcal{C} is finite or infinite. Thus, we may assume that \mathcal{Q}_r is centrally closed over \mathcal{C} which is either finite or algebraically closed. By Martindale's theorem [11], \mathcal{Q}_r is a primitive ring having a nonzero socle with \mathcal{C} as the associated division ring. In light of Jacobson's theorem ([8], p. 75) \mathcal{A} is isomorphic to a dense ring of linear transformations on some vector space \mathcal{V} over \mathcal{C} .

Since $b \notin \mathcal{C}$, there exists $v \in \mathcal{V}$ such that v, bv are linearly \mathcal{C} -independent. By the Jacobson density theorem there exist $\alpha_0, \beta_0, \gamma_0 \in \mathcal{Q}_r$ such that

$$\alpha_0 v = 0, \quad \alpha_0(bv) = v, \quad \beta_0 v = bv, \quad \beta_0(bv) = 0, \quad \gamma_0 v = 0, \quad \gamma_0(bv) = v.$$

However, then

$$\left[\gamma_0, (b + \lambda) \left((\alpha_0 \beta_0)^n - \beta_0^n \alpha_0^n \right) + \left((\alpha_0 \beta_0)^n - \beta_0^n \alpha_0^n \right) b \right] v = v \neq 0.$$

From this contradiction, it therefore follows that $[x, y] \in \mathcal{Z}(\mathcal{A})$ for all $x, y \in \mathcal{A}$. Thus \mathcal{A} is commutative by [7, Theorem 1.1.8], a contradiction. This complete the proof.

Proof of Theorem 1.4. By using the same techniques as in Proposition 2.1, we deduce that, for any $x \in \mathcal{A}$, either $\mathcal{F}(x) - \mathcal{G}(x) \in \mathcal{Z}(\mathcal{A})$ or $\mathcal{F}(x) + \mathcal{G}(x) \in \mathcal{Z}(\mathcal{A})$. Moreover, there is a positive integer $n > 1$ such that, for any $x, y \in \mathcal{A}$, either

$$\mathcal{F}((xy)^n) + \mathcal{G}(x^n y^n) \in \mathcal{Z}(\mathcal{A}) \tag{2.22}$$

or

$$\mathcal{F}((xy)^n) - \mathcal{G}(y^n x^n) \in \mathcal{Z}(\mathcal{A}). \tag{2.23}$$

Suppose there exists $x \in \mathcal{A}$ such that $\mathcal{F}(x) - \mathcal{G}(x) \in \mathcal{Z}(\mathcal{A})$ and $\mathcal{F}(x) + \mathcal{G}(x) \notin \mathcal{Z}(\mathcal{A})$. Let $y \in \mathcal{A}$ such that $\mathcal{F}(x + y) - \mathcal{G}(x + y) \in \mathcal{Z}(\mathcal{A})$. Then, also $\mathcal{F}(y) - \mathcal{G}(y) \in \mathcal{Z}(\mathcal{A})$ holds. On the other hand, if $\mathcal{F}(x + y) + \mathcal{G}(x + y) \in \mathcal{Z}(\mathcal{A})$, and since $\mathcal{F}(x) + \mathcal{G}(x) \notin \mathcal{Z}(\mathcal{A})$, then also $\mathcal{F}(y) + \mathcal{G}(y) \notin \mathcal{Z}(\mathcal{A})$. Therefore $\mathcal{F}(y) - \mathcal{G}(y) \in \mathcal{Z}(\mathcal{A})$, for any $y \in \mathcal{A}$.

By using a similar argument, one may prove that if there exists $x \in \mathcal{A}$, such that $\mathcal{F}(x) + \mathcal{G}(x) \in \mathcal{Z}(\mathcal{A})$ and $\mathcal{F}(x) - \mathcal{G}(x) \notin \mathcal{Z}(\mathcal{A})$, then $\mathcal{F}(y) + \mathcal{G}(y) \in \mathcal{Z}(\mathcal{A})$, for any $y \in \mathcal{A}$.

In other words, either

$$\mathcal{F}(y) - \mathcal{G}(y) \in \mathcal{Z}(\mathcal{A}), \text{ for all } y \in \mathcal{A} \tag{2.24}$$

or

$$\mathcal{F}(y) + \mathcal{G}(y) \in \mathcal{Z}(\mathcal{A}), \text{ for all } y \in \mathcal{A}. \tag{2.25}$$

For infinitely many real t , by relations (2.22)–(2.23), and for any $x, y \in \mathcal{A}$ we have that either

$$\mathcal{F} \left(((e + tx)y)^n \right) + \mathcal{G} \left((e + tx)^n y^n \right) \in \mathcal{Z}(\mathcal{A})$$

or

$$\mathcal{F} \left(((e + tx)y)^n \right) - \mathcal{G} \left(y^n (e + tx)^n \right) \in \mathcal{Z}(\mathcal{A}).$$

Hence, taking coefficient of t in the expansion of above equations and using Fact 1, it follows that for any $x, y \in \mathcal{A}$, either

$$\mathcal{F}(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}) + n\mathcal{G}(xy^n) \in \mathcal{L}(\mathcal{A}) \quad (2.26)$$

or

$$\mathcal{F}(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}) - n\mathcal{G}(y^n x) \in \mathcal{L}(\mathcal{A}). \quad (2.27)$$

Now, taking $\mathcal{F}((y(e' + tx))^n)$ in place of $\mathcal{G}(((e' + tx)y)^n)$, we see that for any $x, y \in \mathcal{A}$, either

$$\mathcal{F}(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}) + n\mathcal{G}(xy^n) \in \mathcal{L}(\mathcal{A}) \quad (2.28)$$

or

$$\mathcal{F}(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}) - n\mathcal{G}(y^n x) \in \mathcal{L}(\mathcal{A}). \quad (2.29)$$

Then, at least one of pairs of equations $\{(2.26), (2.28)\}$, $\{(2.26), (2.29)\}$, $\{(2.27), (2.28)\}$ and $\{(2.27), (2.29)\}$ must hold. By comparing the equations in these expressions, it follows that, for any $x, y \in \mathcal{A}$, one of the following holds:

$$\mathcal{F}([x, y^n]) \in \mathcal{L}(\mathcal{A}) \quad (2.30)$$

$$\mathcal{F}([x, y^n]) + \mathcal{G}(nxy^n + ny^n x) \in \mathcal{L}(\mathcal{A}) \quad (2.31)$$

$$\mathcal{F}([x, y^n]) - \mathcal{G}(nxy^n + ny^n x) \in \mathcal{L}(\mathcal{A}). \quad (2.32)$$

By proceeding as in Proposition 2.1, and as a consequence of relations (2.30, 2.31 and 2.32) one has that, for any $x, y \in \mathcal{A}$, one of the following holds:

$$\mathcal{F}([x, y]) \in \mathcal{L}(\mathcal{A}) \quad (2.33)$$

$$\mathcal{F}([x, y]) + \mathcal{G}(nxy + nyx) \in \mathcal{L}(\mathcal{A}) \quad (2.34)$$

$$\mathcal{F}([x, y]) - \mathcal{G}(nxy + nyx) \in \mathcal{L}(\mathcal{A}). \quad (2.35)$$

By the fact that (2.24) or (2.25) holds true, we deduce that for any $x, y \in \mathcal{A}$, one of the following holds:

$$\mathcal{F}([x, y]) \in \mathcal{L}(\mathcal{A}) \quad (2.36)$$

$$\mathcal{F}([x, y] + nxy + nyx) \in \mathcal{L}(\mathcal{A}) \quad (2.37)$$

$$\mathcal{G}([x, y] - nxy - nyx) \in \mathcal{L}(\mathcal{A}). \quad (2.38)$$

Our aim is to prove that $\mathcal{F}([x, y]) \in \mathcal{L}(\mathcal{A})$, for any $x, y \in \mathcal{A}$. To do this, we assume there exist $x_0, y_0 \in \mathcal{A}$ such that $\mathcal{F}([x_0, y_0]) \notin \mathcal{L}(\mathcal{A})$. In this case, since $\mathcal{F}([x_0, y_0 + e']) = \mathcal{F}([x_0, y_0]) \notin \mathcal{L}(\mathcal{A})$ and by the above relations, we have that either

$$\mathcal{F}([x_0, y_0 + e'] + nx_0(y_0 + e') + n(y_0 + e')x_0) \in \mathcal{L}(\mathcal{A})$$

or

$$\mathcal{F}([x_0, y_0 + e'] - nx_0(y_0 + e') - n(y_0 + e')x_0) \in \mathcal{L}(\mathcal{A})$$

that is, either

$$\mathcal{F}([x_0, y_0] + nx_0y_0 + ny_0x_0 + 2nx_0) \in \mathcal{L}(\mathcal{A}) \quad (2.39)$$

or

$$\mathcal{F}([x_0, y_0] - nx_0y_0 - ny_0x_0 - 2nx_0) \in \mathcal{L}(\mathcal{A}). \quad (2.40)$$

Moreover, since $\mathcal{F}([tx_0, e + ty_0]) \notin \mathcal{L}(\mathcal{A})$, for any real element t , then, by relations (2.39) and (2.40), it follows that either

$$\mathcal{F}\left([tx_0, e + ty_0] + n(tx_0)(e + ty_0) + n(e + ty_0)(tx_0) + 2n(tx_0)\right) \in \mathcal{L}(\mathcal{A}) \quad (2.41)$$

or

$$\mathcal{F}\left([tx_0, e + ty_0] - n(tx_0)(e + ty_0) - n(e + ty_0)(tx_0) - 2n(tx_0)\right) \in \mathcal{L}(\mathcal{A}). \quad (2.42)$$

By the computation of the coefficient of t and by Fact 1, we get in any case $\mathcal{F}(x_0) \in \mathcal{L}(\mathcal{A})$.

On the other hand, let $x_1 \in \mathcal{A}$ such that $\mathcal{F}([x_1, y_0]) \in \mathcal{L}(\mathcal{A})$. Since $\mathcal{F}([x_1 + x_0, y_0]) \notin \mathcal{L}(\mathcal{A})$, then $\mathcal{F}(x_1 + x_0) \in \mathcal{L}(\mathcal{A})$ by the above argument, that is $\mathcal{F}(x_1) \in \mathcal{L}(\mathcal{A})$.

Hence, it is shown that either $\mathcal{F}([\mathcal{A}, \mathcal{A}]) \subseteq \mathcal{L}(\mathcal{A})$ or $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A})$. In any case, we may assume that $\mathcal{F}([x, y]) \in \mathcal{L}(\mathcal{A})$, for any $x, y \in \mathcal{A}$. Moreover, since $\mathcal{F}([x^n, y^n]) \in \mathcal{L}(\mathcal{A})$, by relations (2.22), (2.23), (2.24) and (2.25), it follows, for any $x, y \in \mathcal{A}$, that either

$$\mathcal{F}((xy)^n) + \mathcal{F}(y^n x^n) \in \mathcal{L}(\mathcal{A}), \quad (2.43)$$

or

$$\mathcal{F}((xy)^n) - \mathcal{F}(y^n x^n) \in \mathcal{L}(\mathcal{A}). \quad (2.44)$$

Starting from (2.43) and (2.44), here we follow the same argument used in the proof of Theorem 1.3, so we omit some details for brevity. Suppose, by contradiction, \mathcal{A} is not commutative, that is $[x, y] \notin \mathcal{Z}(\mathcal{A})$, for some $x, y \in \mathcal{A}$. Therefore, there is a commutator which does not commute with $[x, y]$, so that $\Lambda = [\mathcal{A}, \mathcal{A}]$ is a noncommutative Lie ideal of \mathcal{A} . Since $\mathcal{F}([x, y]) \in \mathcal{Z}(\mathcal{A})$, for any $x, y \in \mathcal{A}$, it follows that $[\mathcal{F}(z), z] = 0$, for any $z \in \Lambda$. Thus by Theorem 1.5 in [5] (for $a = e$) either $\mathcal{F}(x) = \mu x$, where $\mu \in \mathcal{Z}(\mathcal{A})$ or \mathcal{A} satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree four and there exist $b \in \mathcal{Q}_r$ and $\lambda \in \mathcal{C}$ such that $\mathcal{F}(x) = bx + xb + \lambda x$ for all $x \in \mathcal{A}$.

If the first case occurs, since $\mathcal{F} \neq 0$, so the primeness of \mathcal{A} yields $[x, y] \in \mathcal{Z}(\mathcal{A})$, a contradiction. Hence, the second case must hold and we may assume that $b \notin \mathcal{C}$. By relations (2.43) and (2.44), it follows for any $\alpha, \beta, \gamma \in \mathcal{A}$ that either

$$\left[\alpha, (b + \lambda) \left((\beta\gamma)^n + \gamma^n \beta^n \right) + \left((\beta\gamma)^n + \gamma^n \beta^n \right) b \right] = 0, \quad (2.45)$$

or

$$\left[\alpha, (b + \lambda) \left((\beta\gamma)^n - \gamma^n \beta^n \right) + \left((\beta\gamma)^n - \gamma^n \beta^n \right) b \right] = 0. \quad (2.46)$$

In any case, \mathcal{A} satisfies a generalized polynomial identity. As in the proof of Theorem 1.3, we may assume that \mathcal{Q}_r is centrally closed over \mathcal{C} which is either finite or algebraically closed. Moreover, \mathcal{Q}_r is a primitive ring having a nonzero socle with \mathcal{C} as the associated division ring. In light of Jacobson's theorem ([8], p. 75) \mathcal{A} is isomorphic to a dense ring of linear transformations on some vector space \mathcal{V} over \mathcal{C} .

Since $b \notin \mathcal{C}$, there exists $v \in \mathcal{V}$ such that v, bv are linearly \mathcal{C} -independent. By the Jacobson density theorem there exist $\alpha, \beta, \gamma \in \mathcal{Q}_r$ such that

$$\alpha v = 0, \quad \alpha(bv) = v, \quad \beta v = bv, \quad \beta(bv) = 0, \quad \gamma v = 0, \quad \gamma(bv) = v.$$

However, both

$$\left[\gamma, (b + \lambda) \left((\alpha\beta)^n + \beta^n \alpha^n \right) + \left((\alpha\beta)^n + \beta^n \alpha^n \right) b \right] v = v \neq 0$$

and

$$\left[\gamma, (b + \lambda) \left((\alpha\beta)^n - \beta^n \alpha^n \right) + \left((\alpha\beta)^n - \beta^n \alpha^n \right) b \right] v = v \neq 0.$$

These last two relations contradict the fact that one of (2.45) and (2.46) must hold. From this contradiction it follows that $[x, y] \in \mathcal{Z}(\mathcal{A})$ for all $x, y \in \mathcal{A}$, this leads to a contradiction again. Thus \mathcal{A} satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree four.

We conclude our paper with following open questions:

Open Questions. Let \mathcal{A} be a semisimple Banach algebra (unital or not) with centre $\mathcal{Z}(\mathcal{A})$, Θ_1, Θ_2 be open subsets of \mathcal{A} . Suppose $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ and $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{A}$ two continuous linear b -generalized skew derivations of \mathcal{A} . What can be said about the structure of \mathcal{A} and the form of \mathcal{F} and \mathcal{G} in the following cases:

1. For each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x, y) > 1$ such that $\mathcal{F}((xy)^m) - \mathcal{G}(x^m y^m) \in \mathcal{Z}(\mathcal{A})$ or $\mathcal{F}((xy)^m) - \mathcal{G}(y^m x^m) \in \mathcal{Z}(\mathcal{A})$.
2. For each $x \in \Theta_1, y \in \Theta_2$, there exists an integer $m = m(x, y) > 1$ such that $\mathcal{F}((xy)^m) + \mathcal{G}(x^m y^m) \in \mathcal{Z}(\mathcal{A})$ or $\mathcal{F}((xy)^m) - \mathcal{G}(y^m x^m) \in \mathcal{Z}(\mathcal{A})$.

References

- [1] Bonsall FF , Duncan J. Complete Normed Algebras. Berlin, Germany: Springer-Verlag, 1973.
- [2] Chang JC. On the identity $h(x) = af(x) + g(x)b$. Taiwanese Journal of Mathematics 2003; 7: 103-113. doi: 10.11650/twjmath/1500407520
- [3] Chuang CL. GPIs having coefficients in Utumi quotient rings. Proceedings of American Mathematical Society 1988; 103: 723-728. doi: 10.1090/S0002-9939-1988-0947646-4
- [4] De Filippis V. Annihilators and power values of generalized skew derivations on Lie ideals. Canadian Mathematical Bulletin 2016; 59: 258-270. doi: 10.4153/CMB-2015-077-x
- [5] De Filippis V, Wei F. b -generalized skew derivations on Lie ideals. Mediterrenian Journal of Mathematics 2018; 15: 65. doi: 10.1007/s0009-018-1103-2
- [6] Erickson TS, Martindale III WS, Osborn JM. Prime nonassociative algebras. Pacific Journal of Mathematics 1975; 60: 49-63. doi: 10.2140/pjm.1975.60.49
- [7] Herstein IN. Rings With Involution (Chicago Lectures in Mathematics). Chicago, IL, USA: University of Chicago, 1976.
- [8] Jacobson N. Structure of Rings. Providence, RI, USA: American Mathematical Society, 1964.
- [9] Khan AK, Ali S, Alhazmi H, De Filippis V. On skew derivations and generalized skew derivations in Banach algebras. Quaestiones Mathematicae 2020; 43: 1259-1272. doi: 10.2989/16073606.2019.1607604
- [10] Koşan MT, Lee TK. b -Generalized derivations of semiprime rings having nilpotent values. Journal of Australian Mathematical Society 2014; 96: 326-337. doi: 10.1017/S1446788713000670
- [11] Martindale III WS. Prime rings satisfying a generalized polynomial identity. Journal of Algebra 1969; 12: 576-584. doi: 10.1016/0021-8693(69)90029-5
- [12] Yood B. On commutativity of unital Banach algebras. Bulletin of London Mathematical Society 1991; 23: 278-280. doi: 10.1112/blms/23.3.278