Ideals in semigroups of partial transformations with invariant set

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Abstract: This paper explores the ideals and their structural properties in two generalizations of the partial transformation semigroup. Furthermore, principal, maximal, and minimal ideals within these semigroups are elucidated.

Key words: Partial transformation semigroups, Ideals, Principal ideals, Minimal ideals, Maximal ideals, Mathematics.

1. Introduction and Preliminaries

Let $S$ be a semigroup, and let $S^1$ denote a semigroup obtained from $S$ by adding an identity element if $S$ lacks one. If $S$ already contains an identity element, then $S^1$ is equivalent to $S$. For a nonempty subset $I$ of $S$, the term ideal is assigned to $I$ if both $SI$ and $IS$ are subsets of $I$. If $a \in S$, the smallest ideal of $S$ containing $a$ is identified as $S^1aS^1$ and is referred to as the principal ideal generated by $a$. Moreover, an ideal $I$ is considered minimal if there is no ideal $J$ such that $I \subseteq J$. Conversely, an ideal $I$ is deemed maximal if there is no ideal $J$ such that $I \subset J \subset S$.

Consider a nonempty set $X$, and let $T(X)$ represent the full transformation semigroup on $X$ under the composition of functions. Within semigroup theory, the semigroup $T(X)$ holds paramount significance as it serves as a foundational framework, allowing any semigroup to be viewed as an isomorphic subsemigroup. A comprehensive exploration of $T(X)$ has revealed numerous fundamental properties, and substantial research efforts have been dedicated to investigating various specific subsemigroups within the structure.

Henceforth, the cardinality of any set $A$ will be denoted by $|A|$. In 1952, Malcev [15] demonstrated that the ideals in $T(X)$ precisely take the form

$$T_r = \{ \alpha \in T(X) : |X\alpha| < r \},$$

where $2 \leq r \leq |X|'$, and $|X|'$ represents the minimum cardinality greater than $|X|$. It is evident that the ideals in $T(X)$ form a chain under set inclusion. Over the years, the concept of full transformation semigroups has

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of functions. It is noteworthy that the three previously mentioned transformation semigroups are strictly defined as follows:

$$\overline{T}(X, Y) = \{ \alpha \in T(X) : Y \alpha \subseteq Y \}$$ and $$\text{Fix}(X, Y) = \{ \alpha \in T(X) : y \alpha = y \text{ for all } y \in Y \}.$$  

Since $$\overline{T}(X, X) = T(X)$$ and $$\text{Fix}(X, \emptyset) = T(X),$$ both are considered generalizations of $$T(X).$$ Specifically, all three aforementioned semigroups contain $$\text{id}_X,$$ the identity map on $$X,$$ as an identity element. Furthermore, it holds that $$\text{Fix}(X, Y) \subseteq \overline{T}(X, Y) \subseteq T(X),$$ with the inclusion being strictly observed in general.

The exploration of $$\overline{T}(X, Y)$$ was initiated by Magill [14] in 1966, while Honyam and Sanwong [12] delved into $$\text{Fix}(X, Y)$$ in 2013. Extensive examination of the algebraic properties of these semigroups has been undertaken. For $$\overline{T}(X, Y),$$ please refer to [3, 6, 10, 11, 16, 21, 23, 24]. For $$\text{Fix}(X, Y),$$ consult [1, 2, 4, 12, 17, 18]. Additionally, Honyam and Sanwong determined the ideals of both $$\overline{T}(X, Y)$$ and $$\text{Fix}(X, Y)$$ in [10] and [12], respectively. For the semigroup $$\overline{T}(X, Y),$$ its ideals precisely consist of sets in the form

$$K(Z) = \{ \alpha \in \overline{T}(X, Y) : |X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta|, \text{ and } |X\alpha \setminus Y| \leq |X\beta \setminus Y| \text{ for some } \beta \in Z \},$$

where $$\emptyset \neq Z \subseteq \overline{T}(X, Y).$$ Concerning the semigroup $$\text{Fix}(X, Y),$$ its ideals are exactly the sets

$$\text{Fix}_r = \{ \alpha \in \text{Fix}(X, Y) : |X\alpha \setminus Y| < r \},$$

where $$1 \leq r \leq |X \setminus Y|'.$$ The ideals in $$\text{Fix}(X, Y)$$ form a chain under set inclusion, whereas the ideals in $$\overline{T}(X, Y)$$ do not.

Consider $$P(X),$$ the semigroup comprising all partial transformations on $$X$$ under the composition of functions. It is noteworthy that the three previously mentioned transformation semigroups are strictly encompassed within $$P(X).$$ The concept of construction semigroups $$\overline{T}(X, Y)$$ and $$\text{Fix}(X, Y)$$ can be employed to formulate generalizations of $$P(X)$$ as follows:

$$\overline{PT}(X, Y) = \{ \alpha \in P(X) : (\text{dom } \alpha \cap Y) \alpha \subseteq Y \},$$

where $$\emptyset \neq Y \subseteq X$$ and $$\text{dom } \alpha$$ denotes the domain of $$\alpha.$$ Furthermore, for $$Y \subseteq X,$$ let

$$\text{PF} \text{ix}(X, Y) = \{ \alpha \in P(X) : y \alpha = y \text{ for all } y \in \text{dom } \alpha \cap Y \}.$$  

Since $$\overline{PT}(X, X) = P(X)$$ and $$\text{PF} \text{ix}(X, \emptyset) = P(X),$$ both semigroups are regarded as extensions of $$P(X).$$ However, they find application in distinct scenarios and complement each other. Various algebraic properties of $$\overline{PT}(X, Y)$$ and $$\text{PF} \text{ix}(X, Y)$$ have been explored; for example, refer to [5, 7, 19, 20, 25, 26].

In this article, we systematically identify all ideals and their respective properties within $$\overline{PT}(X, Y)$$ and $$\text{PF} \text{ix}(X, Y).$$ Additionally, we conduct an examination of principal, minimal, and maximal ideals in these semigroups, illustrating that the ideals do not generally form a chain under set inclusion.

In the context of this paper, we adhere to the convention of right-to-left function application. Specifically, in the composition $$\alpha \beta,$$ the transformation $$\alpha$$ is applied first. For any $$\alpha \in P(X),$$ we denote the domain and image of $$\alpha$$ as $$\text{dom } \alpha$$ and $$\text{im } \alpha,$$ respectively. For notions and notations that are not explicitly defined herein, the reader is referred to [8, 9, 13].
2. Main Results

Consider any cardinal number $p$ and define $p'$ to be the minimum cardinal $q$ such that $q > p$, i.e., $p' = \min\{q : q > p\}$. It is crucial to emphasize that the existence of $p'$ is guaranteed due to the well-ordered nature of cardinals. When $p$ is finite, $p' = p + 1$, representing its successor. The ideals of $P(X)$, as presented in [22], constitute the only sets of the form

$$P_r = \{ \alpha \in P(X) : |\im\alpha| < r \},$$

where $2 \leq r \leq |X'|$. Clearly, the ideals of $P(X)$ form a chain under set inclusion.

To characterize the ideals of $\mathcal{PT}(X,Y)$, unless otherwise stated, we let $|X| = a$, $|Y| = b$, and $|X\setminus Y| = c$. Furthermore, for each triplet of cardinals $r$, $s$, and $t$ satisfying $1 \leq r \leq a'$, $1 \leq s \leq b'$, and $1 \leq t \leq c'$, we define the subset $\mathcal{PT}(r,s,t)$ of $\mathcal{PT}(X,Y)$ as follows:

$$\mathcal{PT}(r,s,t) = \{ \alpha \in \mathcal{PT}(X,Y) : |\im\alpha| < r, |Y\alpha| < s, \text{ and } |\im\alpha\setminus Y| < t \}.$$

Evidently, $\mathcal{PT}(r,s,t)$ can be empty, and $\mathcal{PT}(a',b',c') = \mathcal{PT}(X,Y)$. In cases where $\mathcal{PT}(r,s,t)$ is not empty, we obtain the following:

**Theorem 2.1** Let $\mathcal{PT}(r,s,t) \neq \emptyset$. Then the set $\mathcal{PT}(r,s,t)$ is an ideal of $\mathcal{PT}(X,Y)$.

**Proof** Let $\alpha \in \mathcal{PT}(r,s,t)$ and $\lambda, \mu \in \mathcal{PT}(X,Y)$. Then $|\im\alpha| < r, |Y\alpha| < s$ and $|\im\alpha\setminus Y| < t$. By simple set-theoretical arguments, we can conclude that $|\im\lambda\alpha\mu| \leq |\im\alpha| < r$, $|Y\lambda\alpha\mu| \leq |Y\alpha| < s$, and $|\im\lambda\alpha\setminus Y| \leq |\im\alpha\setminus Y| < t$. Thus, $\lambda\alpha\mu \in \mathcal{PT}(r,s,t)$, and consequently, $\mathcal{PT}(r,s,t)$ forms an ideal of $\mathcal{PT}(X,Y)$.

Observe that if $r \leq u, s \leq v$, and $t \leq w$, then we have $\mathcal{PT}(r,s,t) \subseteq \mathcal{PT}(u,v,w)$. However, the following example demonstrates that there exists an ideal in $\mathcal{PT}(X,Y)$ that does not conform to the form of $\mathcal{PT}(r,s,t)$.

This also illustrates that the ideals in $\mathcal{PT}(X,Y)$ do not form a chain under set inclusion.

**Example 2.2** Considering $X = \{1,2,3,4\}$ and $Y = \{1,2\}$, we have $|X| = 4$, $|Y| = 2$, and $|X\setminus Y| = 2$. Both $\mathcal{PT}(3,3,1)$ and $\mathcal{PT}(4,2,2)$ are ideals of $\mathcal{PT}(X,Y)$, and therefore, the union of $\mathcal{PT}(3,3,1)$ and $\mathcal{PT}(4,2,2)$ is also an ideal of $\mathcal{PT}(X,Y)$. To demonstrate that $\mathcal{PT}(3,3,1) \cup \mathcal{PT}(4,2,2)$ does not constitute a member of the form $\mathcal{PT}(r,s,t)$, we suppose, to the contrary, that $\mathcal{PT}(3,3,1) \cup \mathcal{PT}(4,2,2) = \mathcal{PT}(r,s,t)$ for some $1 \leq r \leq 5, 1 \leq s \leq 3$, and $1 \leq t \leq 3$. If $r < 4$ or $t < 2$, then there is

$$\alpha = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 4 \end{pmatrix} \in \mathcal{PT}(4,2,2) \setminus \mathcal{PT}(r,s,t),$$

and if $s < 3$, then there is

$$\beta = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \in \mathcal{PT}(3,3,1) \setminus \mathcal{PT}(r,s,t).$$

Both cases contradict the supposition. Hence, $r \geq 4$, $s = 3$ and $t \geq 2$. However, there exists

$$\gamma = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix} \in \mathcal{PT}(r,3,t),$$

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but $\gamma \notin \mathcal{PT}(3, 3, 1) \cup \mathcal{PT}(4, 2, 2)$, so $\mathcal{PT}(3, 3, 1) \cup \mathcal{PT}(4, 2, 2) \neq \mathcal{PT}(r, 3, t)$ for all $r \geq 4$, and $t \geq 2$. Since $\alpha \in \mathcal{PT}(4, 2, 2) \setminus \mathcal{PT}(3, 3, 1)$ and $\beta \in \mathcal{PT}(3, 3, 1) \setminus \mathcal{PT}(4, 2, 2)$, we obtain that the ideals of $\mathcal{PT}(X, Y)$ do not form a chain.

In order to determine all ideals of $\mathcal{PT}(X, Y)$, we refer to the result from [19] as follows:

**Lemma 2.3** [19] Let $\alpha, \beta \in \mathcal{PT}(X, Y)$. Then $\alpha = \lambda \beta \mu$ for some $\lambda, \mu \in \mathcal{PT}(X, Y)$ if and only if $|\alpha| \leq |\beta|$, $|\alpha| \leq |\beta|$, and $|\alpha \setminus \beta| \leq |\beta \setminus \beta|$. Moreover, we define the set $\mathcal{PT}[Z]$, for $\emptyset \neq Z \subseteq \mathcal{PT}(X, Y)$, as:

$$\mathcal{PT}[Z] = \{ \alpha \in \mathcal{PT}(X, Y) : |\alpha| \leq |\beta|, |\alpha| \leq |\beta|, |\alpha \setminus \beta| \leq |\beta \setminus \beta| \text{ for some } \beta \in Z \}.$$ 

It is evident that $Z \subseteq \mathcal{PT}[Z]$, and furthermore, if $Z_1 \subseteq Z_2$, then $\mathcal{PT}[Z_1] \subseteq \mathcal{PT}[Z_2]$.

**Theorem 2.4** The ideals of $\mathcal{PT}(X, Y)$ are precisely those sets of the form $\mathcal{PT}[Z]$, where $Z$ is a nonempty subset of $\mathcal{PT}(X, Y)$.

**Proof** To prove that $\mathcal{PT}[Z]$ is an ideal of $\mathcal{PT}(X, Y)$, let $\alpha \in \mathcal{PT}[Z]$ and $\lambda, \mu \in \mathcal{PT}(X, Y)$. Then $|\alpha| \leq |\beta|$, $|\alpha| \leq |\beta|$, and $|\alpha \setminus \beta| \leq |\beta \setminus \beta|$. By employing a comparable proof as demonstrated in Theorem 2.1, we obtain $|\lambda \alpha \mu| \leq |\alpha|$, $|\lambda \alpha \mu| \leq |\alpha|$, and $|\lambda \alpha \mu| \leq |\alpha|$. Thus $|\lambda \alpha \mu| \leq |\alpha|$, $|\alpha \setminus \beta| \leq |\beta \setminus \beta|$ and $|\lambda \alpha \mu| \leq |\beta \setminus \beta|$. Hence, $\lambda \alpha \mu$ is an ideal of $\mathcal{PT}(X, Y)$.

Now, let $I$ be an ideal of $\mathcal{PT}(X, Y)$. To prove that $I = \mathcal{PT}[I]$, we begin by considering $\alpha \in \mathcal{PT}[I]$. Then $|\alpha| \leq |\beta|$, $|\alpha| \leq |\beta|$, and $|\alpha \setminus \beta| \leq |\beta \setminus \beta|$. By Lemma 2.3, we have $\alpha = \lambda \beta \mu$ for some $\lambda, \mu \in \mathcal{PT}(X, Y)$. Since $\beta \in I$ and $I$ is an ideal of $\mathcal{PT}(X, Y)$, it follows that $\alpha = \lambda \beta \mu \in I$. Hence, $\mathcal{PT}[I] \subseteq I$. Since $I$ is already included in $\mathcal{PT}[I]$, we conclude that $I = \mathcal{PT}[I]$, as required.

Note that for an ideal $I$ of $\mathcal{PT}(X, Y)$, as indicated in the proof of Theorem 2.4, we have $\mathcal{PT}[I] = I$. Additionally, it is possible for the difference sets $Z$ to yield the same ideal in $\mathcal{PT}(X, Y)$. To distinguish subsets of $\mathcal{PT}(X, Y)$ that form distinct ideals, we define a subset $J_{r,s,t}$ of $\mathcal{PT}(X, Y)$, where $0 \leq r \leq a, 0 \leq s \leq b$, and $0 \leq t \leq c$, as follows:

$$J_{r,s,t} = \{ \alpha \in \mathcal{PT}(X, Y) : |\alpha| = r, |Y \alpha| = s, \text{ and } |\alpha \setminus \beta| = t \}.$$ 

Observe that if $r, s$, and $t$ satisfy any of the conditions $s + t > r, r - s - t > b - s$, or $r - s - t > c - t$, then $J_{r,s,t} = \emptyset$. On the other hand, if $s + t \leq r, r - s - t \leq b - s$, and $r - s - t \leq c - t$, then we define $J_{r,s,t} \in J_{r,s,t}$ by choosing $S \subseteq Y$ and $T \subseteq X \setminus Y$ with $|S| = s$ and $|T| = t$. Next, we let $R \subseteq (X \setminus Y) \setminus T$ and $R' \subseteq X \setminus S$ with $|R| = r - s - t = |R'|$. Now, fixing a bijection $\sigma : R \rightarrow R'$, we define $J_{r,s,t} = \sigma \cup \text{id}_S \cup \text{id}_T$, where $\text{id}_S$ and $\text{id}_T$ are the identity maps on $S$ and $T$, respectively.

Let $Z$ be a collection of all $J_{r,s,t}$, where $J_{r,s,t} \neq \emptyset$. It is evident that $|Z \cup J_{r,s,t}| = 1$. A nonempty subset $Z$ of $Z$ is called $pt$-pure if for any distinct two elements $\alpha_{n_1,n_2,n_3}$ and $\alpha_{m_1,m_2,m_3}$ in $Z$, there exist $i, j \in \{1, 2, 3\}$ such that $n_i > m_i$ and $m_j > n_j$. 
Theorem 2.5 Let $X$ be a finite set. The ideals of $\mathcal{PT}(X,Y)$ are precisely those sets of the form $\mathcal{PT}[Z]$, where $Z$ is a pt-pure subset of $Z$. In particular, distinct pt-pure subsets of $Z$ result in distinct ideals.

Proof Let $I$ be any ideal of $\mathcal{PT}(X,Y)$. Let $r = \max\{|\alpha| : \alpha \in I\}$, $s = \max\{|\gamma \alpha| : \alpha \in I\}$, and $t = \max\{|\alpha \gamma Y : \alpha \in I\}$. Choose $\alpha \in J_{r,s,t} \cap I$, where $s_r = \max\{u : J_{r,u} \cap I \neq \emptyset\}$ and $t_r = \max\{v : J_{r,v,u} \cap I \neq \emptyset\}$. Similarly, we choose $\beta \in J_{r,s,t} \cap I$ and $\gamma \in J_{r,s,t} \cap I$. By Lemma 2.3, we obtain that $\alpha_{r,s,t}$, $\alpha_{r,s,t}$, and $\alpha_{r,s,t}$ belong to $I$. Let $Z_I = \{\alpha_{r,s,t}, \alpha_{r,s,t}, \alpha_{r,s,t} \}$. Note that the elements in $Z_I$ may not differ at all and $Z_I$ is a pt-pure subset. It is clear that $I \subseteq \mathcal{PT}[Z_I]$. Then $|\alpha | \leq u, |Y \beta| \leq v$, and $|\alpha \gamma Y| \leq w$ for some $u, v, w$ with $\alpha_{u,v,w} \in Z_I$. According to Lemma 2.3, we have $\delta = \lambda \alpha_{u,v,w} \in I$, thus implying $I = \mathcal{PT}[Z_I]$.

Next, we consider any pure subsets $Z_1$ and $Z_2$ of $Z$ with $Z_1 \neq Z_2$. Without loss of generality, if one is strictly contained in the other, we assume that $Z_1 \subset Z_2$. Then there exists $\alpha_{r,s,t} \in Z_2 \setminus Z_1$. Since $Z_2$ is a pt-pure subset of $Z$, for each $\alpha_{u,v,w} \in Z_1$, $u > r$ or $v > s$ or $w > t$. Hence, $\alpha_{r,s,t} \in \mathcal{PT}[Z_2] \setminus \mathcal{PT}[Z_1]$. For the case $Z_1 \subseteq Z_2$ and $Z_2 \subset Z_1$, we have $Z_1 \setminus Z_2 \neq \emptyset$ and $Z_2 \setminus Z_1 \neq \emptyset$. Let $r_1 = \max\{u : \alpha_{u,v,w} \in Z_1 \setminus Z_2\}$ and $r_2 = \max\{u : \alpha_{u,v,w} \in Z_2 \setminus Z_1\}$. Then there exist $\alpha_{r_1,s_1,t_1} \in Z_1 \setminus Z_2$ and $\alpha_{r_2,s_2,t_2} \in Z_2 \setminus Z_1$.

Case 1: $r_1 > r_2$. If $\alpha_{r_1,s_1,t_1} \in \mathcal{PT}[Z_2]$, then there exists $\alpha_{u,v,w} \in Z_2$ such that $r_2 < r_1 \leq u$, $s_2 < v$, and $t_2 < w_2$. The maximum value of $r_2$ implies that $\alpha_{u,v,w} \in Z_1$, which contradicts the fact that $Z_1$ is a pure subset. Hence, $\alpha_{r_1,s_1,t_1} \in \mathcal{PT}[Z_1] \setminus \mathcal{PT}[Z_2]$.

Case 2: $r_2 > r_1$. Using the same argument as in Case 1, we can conclude that $\alpha_{r_2,s_2,t_2} \in \mathcal{PT}[Z_2] \setminus \mathcal{PT}[Z_1]$.

Case 3: $r_1 = r_2$. Let $v_1 = \max\{v : \alpha_{r_1,v,w} \in Z_1 \setminus Z_2\}$ and $v_2 = \max\{v : \alpha_{r_2,v,w} \in Z_2 \setminus Z_1\}$. If $v_1 \neq v_2$, applying the same previous argument, we conclude that $\mathcal{PT}[Z_1] \neq \mathcal{PT}[Z_2]$. In the case where $v_1 = v_2$, we let $w_1 = \max w : \alpha_{r_1,v_1,w} \in Z_1 \setminus Z_2$ and $w_2 = \max w : \alpha_{r_2,v_2,w} \in Z_2 \setminus Z_1$. Consequently, we have $w_1 \neq w_2$ and also establish $\mathcal{PT}[Z_1] \neq \mathcal{PT}[Z_2]$.

To simplify notation, in the case of $Z$ being a finite set such that $Z = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we use the notation $\mathcal{PT}[\alpha_1, \alpha_2, \ldots, \alpha_n]$ instead of $\mathcal{PT}[\{\alpha_1, \alpha_2, \ldots, \alpha_n\}]$. It is clear that $\mathcal{PT}[Z] = \bigcup_{\gamma \in Z} \mathcal{PT}[\gamma]$.

For $\alpha, \beta \in \mathcal{PT}(X,Y)$, $\mathcal{PT}[\alpha] \subseteq \mathcal{PT}[\beta]$ if and only if $|\alpha \gamma Y| \leq |\beta \gamma Y|$. Consequently, $\mathcal{PT}[\alpha] = \mathcal{PT}[\beta]$ if and only if $|\alpha \gamma Y| = |\beta \gamma Y|$. Additionally, if $\alpha, \beta \in Z$, $\mathcal{PT}[\alpha]$ and $\mathcal{PT}[\beta]$ are distinct.

Proposition 2.6 The principal ideals of $\mathcal{PT}(X,Y)$ are precisely those sets of the form $\mathcal{PT}[\alpha_{r,s,t}]$. 

Proof Let $\alpha_{r,s,t} \in Z$. Our objective is to demonstrate that $\mathcal{PT}[\alpha_{r,s,t}] = \mathcal{PT}(X,Y)\alpha_{r,s,t}$, $\mathcal{PT}(X,Y)$. We begin by considering $\beta \in \mathcal{PT}[\alpha_{r,s,t}]$. This implies that $|\alpha \gamma Y| \leq r, |\beta \gamma Y| \leq s$, and $|\alpha \gamma Y| \leq t$. According to Lemma 2.3, we can express $\beta$ as $\lambda \alpha_{r,s,t} \mu$ for some $\lambda, \mu \in \mathcal{PT}(X,Y)$. Consequently, we have established that $\beta$ belongs to $\mathcal{PT}(X,Y)\alpha_{r,s,t}$, $\mathcal{PT}(X,Y)$. On the other hand, consider $\gamma$ in $\mathcal{PT}(X,Y)\alpha_{r,s,t}$, $\mathcal{PT}(X,Y)$. This implies that $\gamma = \theta \alpha_{r,s,t} \eta$ for some $\theta, \eta \in \mathcal{PT}(X,Y)$. Since $\alpha_{r,s,t} \in \mathcal{PT}[\alpha_{r,s,t}]$ and $\mathcal{PT}[\alpha_{r,s,t}]$ is an ideal, we can conclude that $\gamma$ is an element of $\mathcal{PT}[\alpha_{r,s,t}]$. Therefore, $\mathcal{PT}[\alpha_{r,s,t}] = \mathcal{PT}(X,Y)\alpha_{r,s,t}$, $\mathcal{PT}(X,Y)$ is a principal ideal within $\mathcal{PT}(X,Y)$.
Let $I$ be any principal ideal of $\mathcal{PT}(X, Y)$. Then $I = \mathcal{PT}(X, Y)\alpha\mathcal{PT}(X, Y)$ for some $\alpha \in \mathcal{PT}(X, Y)$. Let $|\text{im} \alpha| = r, |Y \alpha| = s,$ and $|\text{im} \alpha \setminus Y| = t$. By Lemma 2.3, $\alpha = \lambda_{r,s,t} \mu$ and $\alpha_{r,s,t} = X'\alpha' \mu'$ for some $\lambda, X', \mu, \mu' \in \mathcal{PT}(X, Y)$. Hence, $I = \mathcal{PT}(X, Y)\alpha\mathcal{PT}(X, Y) \subseteq \mathcal{PT}(X, Y)\alpha_{r,s,t} \mathcal{PT}(X, Y) \subseteq \mathcal{PT}(X, Y)\alpha \mathcal{PT}(X, Y) = I$. Therefore, $I = \mathcal{PT}(X, Y)\alpha_{r,s,t} \mathcal{PT}(X, Y) = \mathcal{PT}[\alpha_{r,s,t}]$. □

Next, we will discuss the minimal and maximal ideals of $\mathcal{PT}(X, Y)$. It is clear that $J_{0,0,0} = \{\emptyset\} = \mathcal{PT}[\alpha_{0,0,0}]$ is the minimum ideal of $\mathcal{PT}(X, Y)$.

As $\{\emptyset\}$ represents the minimum ideal within $\mathcal{PT}(X, Y)$, we can define a minimal ideal in $\mathcal{PT}(X, Y)$ as an ideal $I$ such that $\{\emptyset\} \subseteq I$ and $I$ satisfies the condition: if there exists an ideal $J$ such that $\{\emptyset\} \subseteq J \subseteq I$, then either $J = \{\emptyset\}$ or $J = I$. The following theorem elaborates on the details of the minimal ideal in $\mathcal{PT}(X, Y)$.

**Theorem 2.7** $\{\emptyset\} \cup J_{1,0,0}$ is the unique minimal ideal of $\mathcal{PT}(X, Y)$.

**Proof** It is routine to verify that $\{\emptyset\} \cup J_{1,0,0} = \mathcal{PT}(2, 1, 1)$ is an ideal of $\mathcal{PT}(X, Y)$. To prove the minimality, we let $J$ be an ideal of $\mathcal{PT}(X, Y)$ such that $\{\emptyset\} \subseteq J \subseteq \{\emptyset\} \cup J_{1,0,0}$. Then there exists $\alpha \in J_{1,0,0}$, but $\alpha \notin J$. To demonstrate that $J = \{\emptyset\}$, we assume the contrary. In this case, there exists $\emptyset \neq \beta \in J$. Since both $\alpha$ and $\beta$ belong to $J_{1,0,0}$, by Lemma 2.3, there exist $\lambda, \mu \in \mathcal{PT}(X, Y)$ such that $\alpha = \lambda \beta \mu$. Since $\beta \in J$ and $J$ is an ideal, we obtain $\alpha = \lambda \beta \mu \in J$, which leads to a contradiction. Consequently, $\{\emptyset\} \cup J_{1,0,0}$ qualifies as a minimal ideal within $\mathcal{PT}(X, Y)$. For the uniqueness, we let $M$ be a minimal ideal of $\mathcal{PT}(X, Y)$. As $M$ is an ideal of $\mathcal{PT}(X, Y)$, it can be expressed as $M = \mathcal{PT}[Z]$ for some a nonempty subset $Z$ of $\mathcal{PT}(X, Y)$. Since $\{\emptyset\} \subseteq M$, there must exist $\alpha \in M$ such that $|\text{im} \alpha| \geq 1$. Since $\alpha \in M = \mathcal{PT}[Z]$, we have $|\text{im} \alpha| \leq |\text{im} \beta|, |Y \alpha| \leq |Y \beta|, \text{ and } |\text{im} \alpha \setminus Y| \leq |\text{im} \beta \setminus Y|$ for some $\beta \in Z$. Now, let $\gamma \in J_{1,0,0}$. Then $|\text{im} \gamma| = 1 \leq |\text{im} \alpha| \leq |\text{im} \beta|, |Y \gamma| = 0 \leq |Y \alpha| \leq |Y \beta|, \text{ and } |\text{im} \gamma \setminus Y| = 0 \leq |\text{im} \alpha \setminus Y| \leq |\text{im} \beta \setminus Y|$.

This implies that $\gamma \in \mathcal{PT}[Z] = M$. Consequently, we have shown that $\{\emptyset\} \cup J_{1,0,0} \subseteq M$, and therefore, $M = \{\emptyset\} \cup J_{1,0,0}$ by the minimality of $M$. □

Now, we will introduce the concept of a maximal ideal in $\mathcal{PT}(X, Y)$. An ideal $I$ in $\mathcal{PT}(X, Y)$ is categorized as a maximal ideal if, for any ideal $M$ such that $I \subseteq M \subseteq \mathcal{PT}(X, Y)$, it holds that either $M = I$ or $M = \mathcal{PT}(X, Y)$.

**Theorem 2.8** $\mathcal{PT}(X, Y) \setminus J_{a,b,c}$ is the unique maximal ideal of $\mathcal{PT}(X, Y)$.

**Proof** It is clear that $\mathcal{PT}(X, Y) \setminus J_{a,b,c} = \mathcal{PT}[\mathcal{PT}(X, Y) \setminus J_{a,b,c}]$ is an ideal of $\mathcal{PT}(X, Y)$. To show that $\mathcal{PT}(X, Y) \setminus J_{a,b,c}$ is a maximal ideal of $\mathcal{PT}(X, Y)$, we let $M$ be an ideal of $\mathcal{PT}(X, Y)$ such that $\mathcal{PT}(X, Y) \setminus J_{a,b,c} \subseteq M \subseteq \mathcal{PT}(X, Y)$. This implies that there exists $\alpha \in M$, but $\alpha \notin \mathcal{PT}(X, Y) \setminus J_{a,b,c}$. As a result, we have $|\text{im} \alpha| = a, |Y \alpha| = b,$ and $|\text{im} \alpha \setminus Y| = c$. Now, let $\beta \in J_{a,b,c}$. Since $\alpha, \beta \in J_{a,b,c}$, there exist $\lambda$ and $\mu$ in $\mathcal{PT}(X, Y)$ such that $\beta = \lambda \alpha \mu$. Consequently, $\beta = \lambda \alpha \mu \in M$ since $\alpha \in M$ and $M$ is an ideal. Thus, $M = \mathcal{PT}(X, Y)$. For the uniqueness, we let $M'$ be a maximal ideal of $\mathcal{PT}(X, Y)$. Then $M \cup M'$ is an ideal and $id_X \notin M \cup M'$, whence $M \cup M' \subseteq \mathcal{PT}(X, Y)$. Since $M \subseteq M \cup M'$ and $M$ is a maximal ideal, we have $M \cup M' = M$. Similarly, we can conclude that $M \cup M' = M'$. Thus, $M = M \cup M' = M'$. □
If $Y \neq X$, then $PT[\alpha_{1,0,1}]$ and $PT[\alpha_{1,1,0}]$ neither contains the other. This means that if $Y \neq \emptyset$, then the ideals does not form a chain.

We conclude the study of ideals on $PT(X, Y)$ by elucidating the set $J_{r,s,t}$ and the poset of ideals in $PT(X, Y)$ for the sets $X = \{1, 2, 3\}$ and $Y = \{1, 2\}$. To enhance clarity, an element $\alpha$ in $PT(X, Y)$ satisfying $1\alpha = x$, $2\alpha = y$, and $3\alpha = z$ is denoted as $(x, y, z)$. Specifically, the vacant positions in the 3-tuple signify their exclusion from the domain of those elements. The subsets $J_{r,s,t}$ with $\alpha_{r,s,t}$ in red and the Hasse diagram of ideals in $PT(X, Y)$ are presented in Table 1 and Figure 1, respectively.

<table>
<thead>
<tr>
<th>$J_{3,2,1}$</th>
<th>$(1, 2, 3)$</th>
<th>$(2, 1, 3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{2,1,0}$</td>
<td>$(1, 1, 2)$</td>
<td>$(2, 2, 1)$</td>
</tr>
<tr>
<td>$(1, 2)$</td>
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<td>$(2, 2, 3)$</td>
</tr>
<tr>
<td>$(1, , 3)$</td>
<td>$(1, , 2)$</td>
<td>$(1, , 2)$</td>
</tr>
<tr>
<td>$J_{2,2,0}$</td>
<td>$(1, 1, 2)$</td>
<td>$(2, 1, 1)$</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>$(2, 1)$</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td>$J_{1,0,0}$</td>
<td>$(1, , 1)$</td>
<td>$(2, , 1)$</td>
</tr>
<tr>
<td>$(1, , 2)$</td>
<td>$(1, , 2)$</td>
<td>$(1, , 2)$</td>
</tr>
<tr>
<td>$J_{1,0,1}$</td>
<td>$(1, , 3)$</td>
<td>$(2, , 3)$</td>
</tr>
<tr>
<td>$(1, , 3)$</td>
<td>$(1, , 3)$</td>
<td>$(1, , 3)$</td>
</tr>
<tr>
<td>$J_{1,1,0}$</td>
<td>$(1, , 1)$</td>
<td>$(2, , 2)$</td>
</tr>
<tr>
<td>$(1, , 1)$</td>
<td>$(1, , 1)$</td>
<td>$(1, , 1)$</td>
</tr>
<tr>
<td>$J_{0,0,0}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Table 1. The subsets $J_{r,s,t}$ of $PT(\{1, 2, 3\}, \{1, 2\})$

Figure 1. The Hasse diagram of ideals in $PT(\{1, 2, 3\}, \{1, 2\})$
Our next propose is to explore the ideals of \(PFix(X, Y)\) in the case where \(Y\) is a proper subset of \(X\). Recall that the ideals of \(\overline{PT}(X, Y)\) are of the form \(\overline{PT}[Z]\) where \(\emptyset \neq Z \subseteq \overline{PT}(X, Y)\). Since \(PFix(X, Y)\) is a subsemigroup of \(\overline{PT}(X, Y)\), we easily obtain the following:

**Lemma 2.9** \(\overline{PT}[Z] \cap PFix(X, Y)\) is an ideal of \(PFix(X, Y)\).

The following example demonstrates that there exists an ideal in \(PFix(X, Y)\) that does not conform to the form of \(\overline{PT}[Z] \cap PFix(X, Y)\).

**Example 2.10** Let \(X = \{1, 2, 3\}\) and \(Y = \{1, 2\}\). Consider the ideal
\[
I = \{\emptyset, ( , 1), ( , 2), (1, ), (1, 1)\}
\]
in \(PFix(X, Y)\). If \(Z\) takes the form \(\{( , 1), ( , 2)\}\), then the corresponding \(\overline{PT}[Z]\) is \(\{\emptyset, ( , 1), ( , 2)\}\); if \(Z = \{( , 3)\}\), then \(\overline{PT}[Z]\) is \(\{\emptyset, ( , 1), ( , 2), ( , 3)\}\); and if \(Z = \{( , 1), ( , 3)\}\), then \(\overline{PT}[Z]\) is \(\{\emptyset, ( , 1), ( , 2), ( , 3)\}\). In all mentioned cases, it is implied that \(\overline{PT}[Z] \cap PFix(X, Y) \neq I\). Furthermore, for any \(Z \subseteq \overline{PT}(X, Y)\) not falling within the previously mentioned scenarios, \(\overline{PT}[Z]\) consistently contains \(( , 2)\), which results in \(\overline{PT}[Z] \cap PFix(X, Y) \neq I\).

Consequently, we assert that \(I \neq \overline{PT}[Z] \cap PFix(X, Y)\) for all \(Z \subseteq \overline{PT}(X, Y)\).

To identify all ideals of \(PFix(X, Y)\), we refer to the result from [26] as follows:

**Lemma 2.11** \([26]\) Let \(\alpha, \beta \in PFix(X, Y)\). Then \(\alpha = \lambda \beta \mu\) for some \(\lambda, \mu \in PFix(X, Y)\) if and only if \(\text{dom } \alpha \cap Y \subseteq \text{dom } \beta \cap Y, |\text{im } \alpha| \leq |\text{im } \beta| \text{ and } |\text{im } \alpha \setminus (\text{im } \beta \cap Y)| \leq |\text{im } \beta \setminus Y|\).

Moreover, we define a subset \(PF[Z]\), where \(\emptyset \neq Z \subseteq PFix(X, Y)\), as the set \(PF[Z] = \{\alpha \in PFix(X, Y) : \text{dom } \alpha \cap Y \subseteq \text{dom } \beta \cap Y, |\text{im } \alpha| \leq |\text{im } \beta|, |\text{im } \alpha \setminus (\text{im } \beta \cap Y)| \leq |\text{im } \beta \setminus Y| \text{ for some } \beta \in Z\}\). Clearly, \(Z \subseteq PF[Z]\), and if \(Z_1 \subseteq Z_2\), then \(PF[Z_1] \subseteq PF[Z_2]\).

Following the argument presented in the proof of Theorem 2.4, we establish the following theorem:

**Theorem 2.12** The ideals of \(PFix(X, Y)\) are precisely those sets of the form \(PF[Z]\), where \(Z\) is a nonempty subset of \(PFix(X, Y)\).

If \(Z\) is a finite set such that \(Z = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}\), similar to the notation used in \(\overline{PT}[Z]\), we use the notation \(PF[\alpha_1, \alpha_2, \ldots, \alpha_n]\) instead of \(PF[\{\alpha_1, \alpha_2, \ldots, \alpha_n\}]\). It is clear that \(PF[Z] = \bigcup_{\gamma \in Z} PF[\gamma]\).

For \(\alpha\) and \(\beta\) in \(PFix(X, Y)\). Then, \(PF[\alpha] \subseteq PF[\beta]\) if and only if \(\text{dom } \alpha \cap Y \subseteq \text{dom } \beta \cap Y, |\text{im } \alpha| \leq |\text{im } \beta|, \text{ and } |\text{im } \alpha \setminus (\text{im } \beta \cap Y)| \leq |\text{im } \beta \setminus Y|\). Consequently, \(PF[\alpha] = PF[\beta]\) if and only if \(\text{dom } \alpha = \text{dom } \beta, |\text{im } \alpha| = |\text{im } \beta|, \text{ dom } \alpha \cap Y = \text{dom } \beta \cap Y, |\text{im } \alpha \setminus (\text{im } \beta \cap Y)| \leq |\text{im } \beta \setminus Y| \text{ and } |\text{im } \beta \setminus (\text{im } \alpha \cap Y)| \leq |\text{im } \alpha \setminus Y|\).

Note that, according to Lemma 2.11, we directly obtain that \(\alpha \in PF[\beta]\) if and only if \(\alpha = \lambda \beta \mu\) for some \(\lambda, \mu \in PFix(X, Y)\). By applying the same argument used in the proof of Proposition 2.6, we arrive at the following theorem:
**Proposition 2.13** The principal ideals of $PFix(X,Y)$ are precisely those sets of the form $PF[\alpha]$, where $\alpha \in PFix(X,Y)$.

Next, we will examine the minimal and maximal ideals of $PFix(X,Y)$. Henceforth, let $|X \setminus Y| = c$, we will then proceed to define

$$J(A, B, t) = \{\alpha \in PFix(X,Y) : \text{dom } \alpha \cap Y = A, \text{im } \alpha \cap Y = B \text{ and } |\text{im } \alpha \setminus Y| = t\},$$

where $A, B \subseteq Y$, and $0 \leq t \leq c$. It is clear that $J(\emptyset, \emptyset, 0) = \{\emptyset\} = PF[\emptyset]$ is the minimum ideal of $PFix(X,Y)$.

**Lemma 2.14** Let $y \in Y$. Then $\{\emptyset\} \cup J(\emptyset, \{y\}, 0)$ is a minimal ideal of $PFix(X,Y)$.

**Proof** Let $y \in Y$, $x \in X \setminus Y$, and $\gamma = \begin{pmatrix} x \\ y \end{pmatrix}$. It is clear that $\{\emptyset\} \cup J(\emptyset, \{y\}, 0) = PF[\gamma]$ is an ideal of $PFix(X,Y)$. To prove the minimality, we let $J$ be an ideal of $PFix(X,Y)$ such that $\{\emptyset\} \subseteq J \subseteq \{\emptyset\} \cup J(\emptyset, \{y\}, 0)$. Then, there exists $\alpha \in J(\emptyset, \{y\}, 0)$, but $\alpha \notin J$. To demonstrate that $J = \{\emptyset\}$, we assume the contrary. In this case, there exists $\emptyset \neq \beta \in J$. Since both $\alpha$ and $\beta$ belong to $J(\emptyset, \{y\}, 0)$, it follows that $\text{dom } \alpha \cap Y = \text{dom } \beta \cap Y$, $|\text{im } \alpha| = 1 = |\text{im } \beta|$, and $|\text{im } \alpha \setminus (\text{im } \beta \cap Y)| = 0 = |\text{im } \beta \setminus Y|$. By Lemma 2.11, there exist $\lambda, \mu \in PFix(X,Y)$ such that $\alpha = \lambda \beta \mu$. Since $\beta \in J$ and $J$ is an ideal, we obtain $\alpha \in J$, which leads to a contradiction. Consequently, $\{\emptyset\} \cup J(\emptyset, \{y\}, 0)$ qualifies as a minimal ideal within $PFix(X,Y)$.

**Lemma 2.15** Let $\emptyset \neq Z \subseteq PFix(X,Y)$. If $\text{im } \alpha \cap Y = \emptyset$ for all $\alpha \in Z$, then $PF[Z]$ is not a minimal ideal of $PFix(X,Y)$.

**Proof** Assume that the given condition holds. The assertion is clear in the case where $Z = \{\emptyset\}$. Therefore, we consider the case where $\emptyset \neq \alpha \in Z$. Let $x \in X \setminus Y$ and consider $\gamma = \begin{pmatrix} x \\ \alpha \end{pmatrix}$. We can see that $\text{dom } \gamma \cap Y = \emptyset \subseteq \text{dom } \alpha \cap Y$, $|\text{im } \gamma| = 1 \leq |\text{im } \alpha|$, and $|\text{im } \gamma \setminus (\text{im } \alpha \cap Y)| = |\text{im } \gamma| \leq |\text{im } \alpha| = |\text{im } \alpha \setminus Y|$. This follows that $\gamma \in PF[\alpha] \subseteq PF[Z]$. To show $J(\emptyset, \{y\}, 0) \subseteq PF[Z]$, we let $\beta \in J(\emptyset, \{y\}, 0)$. Then $\text{dom } \beta \cap Y = \emptyset = \text{dom } \gamma \cap Y$, $|\text{im } \beta| = 1 = |\text{im } \gamma|$, and $|\text{im } \beta \setminus (\text{im } \gamma \cap Y)| = |\text{im } \beta| = |\text{im } \gamma| = |\text{im } \gamma \setminus Y|$. This implies, by Lemma 2.11, that $\beta = \lambda \gamma \mu$ for some $\lambda, \mu \in PFix(X,Y)$. Since $PF[Z]$ is an ideal and $\gamma \in PF[Z]$, we get $\beta \in PF[Z]$, which implies $J(\emptyset, \{y\}, 0) \subseteq PF[Z]$. Hence, $\{\emptyset\} \cup J(\emptyset, \{y\}, 0) \subseteq PF[Z]$ since $\gamma \notin J(\emptyset, \{y\}, 0)$. Therefore, $PF[Z]$ is not minimal.

**Theorem 2.16** The minimal ideals of $PFix(X,Y)$ are precisely those sets of the form $\{\emptyset\} \cup J(\emptyset, \{y\}, 0)$, where $y \in Y$.

**Proof** Let $I$ be any minimal ideal of $PFix(X,Y)$. According to Theorem 2.12, $I = PF[Z]$ for some nonempty set $Z \subseteq PFix(X,Y)$. Since $I$ is minimal, as indicated by Lemma 2.15, there exists $\alpha \in Z$ such that $\text{im } \alpha \cap Y \neq \emptyset$. Choose $y \in \text{im } \alpha \cap Y$. To demonstrate that $J(\emptyset, \{y\}, 0) \subseteq I$, let $\beta \in J(\emptyset, \{y\}, 0)$. Then $\text{dom } \beta \cap Y = \emptyset \subseteq \text{dom } \alpha \cap Y$, $|\text{im } \beta| = 1 \leq |\text{im } \alpha|$, and $|\text{im } \beta \setminus (\text{im } \alpha \cap Y)| = 0 \leq |\text{im } \alpha \setminus Y|$. Consequently, $\beta \in PF[Z] = I$, implying $\{\emptyset\} \cup J(\emptyset, \{y\}, 0) \subseteq I$. Since $I$ is minimal, we conclude that $I = \{\emptyset\} \cup J(\emptyset, \{y\}, 0)$, as required.
Theorem 2.17 \(PFix(X,Y) \setminus J(Y,Y,c)\) is the unique maximal ideal of \(PFix(X,Y)\).

Proof It is routine to verify that \(PFix(X,Y) \setminus J(Y,Y,c) = PF[PFix(X,Y) \setminus J(Y,Y,c)]\) is an ideal of \(PFix(X,Y)\). To show that \(PFix(X,Y) \setminus J(Y,Y,c)\) is a maximal ideal of \(PFix(X,Y)\), we let \(M\) be an ideal of \(PFix(X,Y)\) such that \(PFix(X,Y) \setminus J(Y,Y,c) \subseteq M \subseteq PFix(X,Y)\). This implies that there exists an \(\alpha \in M\), but \(\alpha \notin PFix(X,Y) \setminus J(Y,Y,c)\). As a result, we have \(\text{dom} \alpha \cap Y = Y\), \(\text{im} \alpha \cap Y = Y\) and \(|\text{im} \alpha \setminus Y| = c\).

Now, let \(\beta \in J(Y,Y,c)\). Since \(\alpha, \beta \in J(Y,Y,c)\), there exist \(\lambda\) and \(\mu\) in \(PFix(X,Y)\) such that \(\beta = \lambda \alpha \mu\). Consequently, \(\beta = \lambda \alpha \mu \in M\) since \(\alpha \in M\) and \(M\) is an ideal. Thus, \(M = PFix(X,Y)\). The uniqueness can be proved similar to Theorem 2.8. □

Theorem 2.18 The ideals of \(PFix(X,Y)\) form a chain under the set inclusion if and only if \(Y = \emptyset\).

Proof Assume that \(Y \neq \emptyset\). Then there exist an element \(y\) in \(Y\) and an element \(x\) from \(X \setminus Y\). Define \(\alpha\) and \(\beta\) in \(PFix(X,Y)\) by

\[
\alpha = \left(\begin{array}{c} x \\ x \end{array}\right) \text{ and } \beta = \left(\begin{array}{c} y \\ y \end{array}\right).
\]

Since \(|\text{im} \alpha \setminus (\text{im} \beta \cap Y)| = 1 \neq 0 = |\text{im} \beta \setminus Y|\), it follows that \(\alpha \in PF[\alpha] \setminus PF[\beta]\). Also, since \(\text{dom} \beta \cap Y = \emptyset = \text{im} \alpha \cap Y\), implies \(\beta \in PF[\beta] \setminus PF[\alpha]\). This implies that neither contains the other. Hence, the ideals of \(PFix(X,Y)\) do not form a chain. The converse is trivial, since \(PFix(X,Y) = P(X)\) when \(Y = \emptyset\). □

Note that in the case where \(X\) is a finite set, we have \(\mathcal{D} = \mathcal{J}\); this implies, by Theorem 3.5 in [26], that \(PF[\alpha] = PF[\beta]\) for all \(\alpha, \beta \in J(A,B,t)\). Also, for \(\alpha \in J(A,B,t)\) and \(\beta \in J(U,V,w)\) such that \(A \subseteq U\), \(B \subseteq V\) and \(t \leq w\), then \(PF[\alpha] \subseteq PF[\beta]\). However, the converse of this statement does not hold.

This section concludes by explicating the set \(J(A,B,t)\) and the poset of ideals within \(PFix(X,Y)\) concerning the sets \(X = \{1,2,3\}\) and \(Y = \{1,2\}\), as depicted in Table 2 and Figure 2, respectively. The utilization of blue color in these depictions signifies the representation of ideals in the form \(\mathcal{P}|Z| \cap PFix(X,Y)\).

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<th>(J(Y,Y,1))</th>
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<tr>
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<td>(J(0),\emptyset,0)</td>
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Table 2. The subsets \(J(A,B,t)\) of \(PFix(\{1,2,3\},\{1,2\})\).
Figure 2. The Hasse diagram of ideals in $PF_{1x}(\{1,2,3\}, \{1,2\})$

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References


