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Ideals in semigroups of partial transformations with invariant set

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Abstract: This paper explores the ideals and their structural properties in two generalizations of the partial transformation semigroup. Furthermore, principal, maximal, and minimal ideals within these semigroups are elucidated.

Key words: Partial transformation semigroups, Ideals, Principal ideals, Minimal ideals, Maximal ideals, Mathematics.

1. Introduction and Preliminaries

Let S be a semigroup, and let S^1 denote a semigroup obtained from S by adding an identity element if S lacks one. If S already contains an identity element, then S^1 is equivalent to S. For a nonempty subset I of S, the term *ideal* is assigned to I if both SI and IS are subsets of I. If $a \in S$, the smallest ideal of S containing a is identified as S^1aS^1 and is referred to as the *principal ideal generated by* a. Moreover, an ideal I is considered *minimal* if there is no ideal J such that $J \subsetneq I$. Conversely, an ideal I is deemed *maximal* if there is no ideal J such that $I \subsetneq J \subsetneq S$.

Consider a nonempty set X, and let $T(X)$ represent the full transformation semigroup on X under the composition of functions. Within semigroup theory, the semigroup $T(X)$ holds paramount significance as it serves as a foundational framework, allowing any semigroup to be viewed as an isomorphic subsemigroup. A comprehensive exploration of $T(X)$ has revealed numerous fundamental properties, and substantial research efforts have been dedicated to investigating various specific subsemigroups within the structure.

Henceforth, the cardinality of any set A will be denoted by $|A|$. In 1952, Malcev [15] demonstrated that the ideals in $T(X)$ precisely take the form

$$T_r = \{\alpha \in T(X) : |X\alpha| < r\},$$

where $2 \leq r \leq |X|'$, and $|X|'$ represents the minimum cardinality greater than $|X|$. It is evident that the ideals in $T(X)$ form a chain under set inclusion. Over the years, the concept of full transformation semigroups has

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experienced significant growth, incorporating and building upon earlier discoveries. A well-recognized extension of $T(X)$ is represented by the semigroups $\overline{T}(X,Y)$ and $Fix(X,Y)$, where Y is a subset of X. These are defined as follows:

$$\overline{T}(X,Y)=\{\alpha\in T(X): Y\alpha\subseteq Y\} \textit{ and } Fix(X,Y)=\{\alpha\in T(X): y\alpha=y \textit{ for all } y\in Y\}.$$

Since $\overline{T}(X,X)=T(X)$ and $Fix(X,\emptyset)=T(X)$, both are considered generalizations of $T(X)$. Specifically, all three aforementioned semigroups contain id_X, the identity map on X, as an identity element. Furthermore, it holds that $Fix(X,Y)\subseteq\overline{T}(X,Y)\subseteq T(X)$, with the inclusion being strictly observed in general.

The exploration of $\overline{T}(X,Y)$ was initiated by Magill [14] in 1966, while Honyam and Sanwong [12] delved into $Fix(X,Y)$ in 2013. Extensive examination of the algebraic properties of these semigroups has been undertaken. For $\overline{T}(X,Y)$, please refer to [3, 6, 10, 11, 16, 21, 23, 24]. For $Fix(X,Y)$, consult [1, 2, 4, 12, 17, 18]. Additionally, Honyam and Sanwong determined the ideals of both $\overline{T}(X,Y)$ and $Fix(X,Y)$ in [10] and [12], respectively. For the semigroup $\overline{T}(X,Y)$, its ideals precisely consist of sets in the form

$$K(Z)=\{\alpha\in\overline{T}(X,Y): |X\alpha|\leq|X\beta|, |Y\alpha|\leq|Y\beta|, \text{ and } |X\alpha\setminus Y|\leq|X\beta\setminus Y| \text{ for some } \beta\in Z\},$$

where $\emptyset\neq Z\subseteq\overline{T}(X,Y)$. Concerning the semigroup $Fix(X,Y)$, its ideals are exactly the sets

$$Fix_r=\{\alpha\in Fix(X,Y): |X\alpha\setminus Y|<r\},$$

where $1\leq r\leq|X\setminus Y|'$. The ideals in $Fix(X,Y)$ form a chain under set inclusion, whereas the ideals in $\overline{T}(X,Y)$ do not.

Consider $P(X)$, the semigroup comprising all partial transformations on X under the composition of functions. It is noteworthy that the three previously mentioned transformation semigroups are strictly encompassed within $P(X)$. The concept of construction semigroups $\overline{T}(X,Y)$ and $Fix(X,Y)$ can be employed to formulate generalizations of $P(X)$ as follows:

$$\overline{PT}(X,Y)=\{\alpha\in P(X): (\operatorname{dom}\alpha\cap Y)\alpha\subseteq Y\},$$

where $\emptyset\neq Y\subseteq X$ and $\operatorname{dom}\alpha$ denotes the domain of α. Furthermore, for $Y\subsetneq X$, let

$$PFix(X,Y)=\{\alpha\in P(X): y\alpha=y \textit{ for all } y\in\operatorname{dom}\alpha\cap Y\}.$$

Since $\overline{PT}(X,X)=P(X)$ and $PFix(X,\emptyset)=P(X)$, both semigroups are regarded as extensions of $P(X)$. However, they find application in distinct scenarios and complement each other. Various algebraic properties of $\overline{PT}(X,Y)$ and $PFix(X,Y)$ have been explored; for example, refer to [5, 7, 19, 20, 25, 26].

In this article, we systematically identify all ideals and their respective properties within $\overline{PT}(X,Y)$ and $PFix(X,Y)$. Additionally, we conduct an examination of principal, minimal, and maximal ideals in these semigroups, illustrating that the ideals do not generally form a chain under set inclusion.

In the context of this paper, we adhere to the convention of right-to-left function application. Specifically, in the composition $\alpha\beta$, the transformation α is applied first. For any $\alpha\in P(X)$, we denote the domain and image of α as $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$, respectively. For notions and notations that are not explicitly defined herein, the reader is referred to [8, 9, 13].

2. Main Results

Consider any cardinal number p and define p' to be the minimum cardinal q such that $q > p$, i.e., $p' = \min\{q : q > p\}$. It is crucial to emphasize that the existence of p' is guaranteed due to the well-ordered nature of cardinals. When p is finite, $p' = p + 1$, representing its successor. The ideals of $P(X)$, as presented in [22], constitute the only sets of the form

$$P_r = \{\alpha \in P(X) : |\operatorname{im}\alpha| < r\},$$

where $2 \leq r \leq |X|'$. Clearly, the ideals of $P(X)$ form a chain under set inclusion.

To characterize the ideals of $\overline{PT}(X,Y)$, unless otherwise stated, we let $|X| = a$, $|Y| = b$, and $|X \backslash Y| = c$. Furthermore, for each triplet of cardinals r, s, and t satisfying $1 \leq r \leq a'$, $1 \leq s \leq b'$, and $1 \leq t \leq c'$, we define the subset $\overline{PT}(r,s,t)$ of $\overline{PT}(X,Y)$ as follows:

$$\overline{PT}(r,s,t) = \{\alpha \in \overline{PT}(X,Y) : |\operatorname{im}\alpha| < r, |Y\alpha| < s, \text{ and } |\operatorname{im}\alpha \backslash Y| < t\}.$$

Evidently, $\overline{PT}(r,s,t)$ can be empty, and $\overline{PT}(a',b',c') = \overline{PT}(X,Y)$. In cases where $\overline{PT}(r,s,t)$ is not empty, we obtain the following:

Theorem 2.1 *Let $\overline{PT}(r,s,t) \neq \emptyset$. Then the set $\overline{PT}(r,s,t)$ is an ideal of $\overline{PT}(X,Y)$.*

Proof Let $\alpha \in \overline{PT}(r,s,t)$ and $\lambda, \mu \in \overline{PT}(X,Y)$. Then $|\operatorname{im}\alpha| < r, |Y\alpha| < s$ and $|\operatorname{im}\alpha \backslash Y| < t$. By simple set-theoretical arguments, we can conclude that $|\operatorname{im}\lambda\alpha\mu| \leq |\operatorname{im}\alpha| < r$, $|Y\lambda\alpha\mu| \leq |Y\alpha| < s$, and $|\operatorname{im}\lambda\alpha\mu \backslash Y| \leq |\operatorname{im}\alpha \backslash Y| < t$. Thus, $\lambda\alpha\mu \in \overline{PT}(r,s,t)$, and consequently, $\overline{PT}(r,s,t)$ forms an ideal of $\overline{PT}(X,Y)$. □

Observe that if $r \leq u, s \leq v$, and $t \leq w$, then we have $\overline{PT}(r,s,t) \subseteq \overline{PT}(u,v,w)$. However, the following example demonstrates that there exists an ideal in $\overline{PT}(X,Y)$ that does not conform to the form of $\overline{PT}(r,s,t)$. This also illustrates that the ideals in $\overline{PT}(X,Y)$ do not form a chain under set inclusion.

Example 2.2 *Considering $X = \{1,2,3,4\}$ and $Y = \{1,2\}$, we have $|X| = 4$, $|Y| = 2$, and $|X \backslash Y| = 2$. Both $\overline{PT}(3,3,1)$ and $\overline{PT}(4,2,2)$ are ideals of $\overline{PT}(X,Y)$, and therefore, the union of $\overline{PT}(3,3,1)$ and $\overline{PT}(4,2,2)$ is also an ideal of $\overline{PT}(X,Y)$. To demonstrate that $\overline{PT}(3,3,1) \cup \overline{PT}(4,2,2)$ does not constitute a member of the form $\overline{PT}(r,s,t)$, we suppose, to the contrary, that $\overline{PT}(3,3,1) \cup \overline{PT}(4,2,2) = \overline{PT}(r,s,t)$ for some $1 \leq r \leq 5, 1 \leq s \leq 3$, and $1 \leq t \leq 3$. If $r < 4$ or $t < 2$, then there is*

$$\alpha = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix} \in \overline{PT}(4,2,2) \backslash \overline{PT}(r,s,t),$$

and if $s < 3$, then there is

$$\beta = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \in \overline{PT}(3,3,1) \backslash \overline{PT}(r,s,t).$$

Both cases contradict with the supposition. Hence, $r \geq 4$, $s = 3$ and $t \geq 2$. However, there exists

$$\gamma = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix} \in \overline{PT}(r,3,t),$$

but $\gamma \notin \overline{PT}(3,3,1) \cup \overline{PT}(4,2,2)$, *so* $\overline{PT}(3,3,1) \cup \overline{PT}(4,2,2) \neq \overline{PT}(r,3,t)$ *for all* $r \geq 4$, *and* $t \geq 2$. *Since* $\alpha \in \overline{PT}(4,2,2) \backslash \overline{PT}(3,3,1)$ *and* $\beta \in \overline{PT}(3,3,1) \backslash \overline{PT}(4,2,2)$, *we obtain that the ideals of* $\overline{PT}(X,Y)$ *do not form a chain.*

In order to determine all ideals of $\overline{PT}(X,Y)$, we refer to the result from [19] as follows:

Lemma 2.3 *[19] Let* $\alpha, \beta \in \overline{PT}(X,Y)$. *Then* $\alpha = \lambda\beta\mu$ *for some* $\lambda, \mu \in \overline{PT}(X,Y)$ *if and only if* $|\operatorname{im}\alpha| \leq |\operatorname{im}\beta|$, $|Y\alpha| \leq |Y\beta|$ *and* $|\operatorname{im}\alpha \backslash Y| \leq |\operatorname{im}\beta \backslash Y|$.

Moreover, we define the set $\overline{PT}[Z]$, for $\emptyset \neq Z \subseteq \overline{PT}(X,Y)$, as:

$$\overline{PT}[Z] = \{\alpha \in \overline{PT}(X,Y) : |\operatorname{im}\alpha| \leq |\operatorname{im}\beta|, |Y\alpha| \leq |Y\beta|, |\operatorname{im}\alpha \backslash Y| \leq |\operatorname{im}\beta \backslash Y| \text{ for some } \beta \in Z\}.$$

It is evident that $Z \subseteq \overline{PT}[Z]$, and furthermore, if $Z_1 \subseteq Z_2$, then $\overline{PT}[Z_1] \subseteq \overline{PT}[Z_2]$.

Theorem 2.4 *The ideals of* $\overline{PT}(X,Y)$ *are precisely those sets of the form* $\overline{PT}[Z]$, *where* Z *is a nonempty subset of* $\overline{PT}(X,Y)$.

Proof To prove that $\overline{PT}[Z]$ is an ideal of $\overline{PT}(X,Y)$, let $\alpha \in \overline{PT}[Z]$ and $\lambda, \mu \in \overline{PT}(X,Y)$. Then $|\operatorname{im}\alpha| \leq |\operatorname{im}\beta|, |Y\alpha| \leq |Y\beta|$ and $|\operatorname{im}\alpha \backslash Y| \leq |\operatorname{im}\beta \backslash Y|$ for some $\beta \in Z$. By employing a comparable proof as demonstrated in Theorem 2.1, we obtain $|\operatorname{im}\lambda\alpha\mu| \leq |\operatorname{im}\alpha|, |Y\lambda\alpha\mu| \leq |Y\alpha|$ and $|\operatorname{im}\lambda\alpha\mu \backslash Y| \leq |\operatorname{im}\alpha \backslash Y|$. Thus $|\operatorname{im}\lambda\alpha\mu| \leq |\operatorname{im}\beta|, |Y\lambda\alpha\mu| \leq |Y\beta|$ and $|\operatorname{im}\lambda\alpha\mu \backslash Y| \leq |\operatorname{im}\beta \backslash Y|$. Hence, $\lambda\alpha\mu \in \overline{PT}[Z]$, implying that $\overline{PT}[Z]$ is an ideal of $\overline{PT}(X,Y)$.

Now, let I be an ideal of $\overline{PT}(X,Y)$. To prove that $I = \overline{PT}[I]$, we begin by considering $\alpha \in \overline{PT}[I]$. Then $|\operatorname{im}\alpha| \leq |\operatorname{im}\beta|$, $|Y\alpha| \leq |Y\beta|$, and $|\operatorname{im}\alpha \backslash Y| \leq |\operatorname{im}\beta \backslash Y|$ for some $\beta \in I$. By Lemma 2.3, we have $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in \overline{PT}(X,Y)$. Since $\beta \in I$ and I is an ideal of $\overline{PT}(X,Y)$, it follows that $\alpha = \lambda\beta\mu \in I$. Hence, $\overline{PT}[I] \subseteq I$. Since I is already included in $\overline{PT}[I]$, we conclude that $I = \overline{PT}[I]$, as required. □

Note that for an ideal I of $\overline{PT}(X,Y)$, as indicated in the proof of Theorem 2.4, we have $\overline{PT}[I] = I$. Additionally, it is possible for the difference sets Z to yield the same ideal in $\overline{PT}(X,Y)$. To distinguish subsets of $\overline{PT}(X,Y)$ that form distinct ideals, we define a subset $J_{r,s,t}$ of $\overline{PT}(X,Y)$, where $0 \leq r \leq a, 0 \leq s \leq b$, and $0 \leq t \leq c$, as follows:

$$J_{r,s,t} = \{\alpha \in \overline{PT}(X,Y) : |\operatorname{im}\alpha| = r, |Y\alpha| = s \text{ and } |\operatorname{im}\alpha \backslash Y| = t\}.$$

Observe that if r, s, and t satisfy any of the conditions $s + t > r$, $r - s - t > b - s$, or $r - s - t > c - t$, then $J_{r,s,t} = \emptyset$. On the other hand, if $s + t \leq r$, $r - s - t \leq b - s$, and $r - s - t \leq c - t$, then we define $\alpha_{r,s,t} \in J_{r,s,t}$ by choosing $S \subseteq Y$ and $T \subseteq X \setminus Y$ with $|S| = s$ and $|T| = t$. Next, we let $R \subseteq (X \setminus Y) \setminus T$ and $R' \subseteq Y \setminus S$ with $|R| = r - s - t = |R'|$. Now, fixing a bijection $\sigma : R \to R'$, we define $\alpha_{r,s,t} = \sigma \cup \mathrm{id}_S \cup \mathrm{id}_T$, where id_S and id_T are the identity maps on S and T, respectively.

Let $\mathcal{Z}$ be a collection of all $\alpha_{r,s,t}$, where $J_{r,s,t} \neq \emptyset$. It is evident that $|\mathcal{Z} \cap J_{r,s,t}| = 1$. A nonempty subset Z of $\mathcal{Z}$ is called *pt-pure* if for any distinct two elements α_{n_1,n_2,n_3} and α_{m_1,m_2,m_3} in Z, there exist $i, j \in \{1,2,3\}$ such that $n_i > m_i$ and $m_j > n_j$.

Theorem 2.5 *Let X be a finite set. The ideals of $\overline{PT}(X,Y)$ are precisely those sets of the form $\overline{PT}[Z]$, where Z is a pt-pure subset of $\mathcal{Z}$. In particular, distinct pt-pure subsets of $\mathcal{Z}$ result in distinct ideals.*

Proof Let I be any ideal of $\overline{PT}(X,Y)$. Let $r = \max\{|\operatorname{im}\alpha| : \alpha \in I\}$, $s = \max\{|Y\alpha| : \alpha \in I\}$, and $t = \max\{|\operatorname{im}\alpha \setminus Y| : \alpha \in I\}$. Choose $\alpha \in J_{r,s_r,t_r} \cap I$, where $s_r = \max\{u : J_{r,u,v} \cap I \neq \emptyset\}$ and $t_r = \max\{v : J_{r,u,v} \cap I \neq \emptyset\}$. Similarly, we choose $\beta \in J_{r_s,s,t_s} \cap I$ and $\gamma \in J_{r_t,s_t,t} \cap I$. By Lemma 2.3, we obtain that $\alpha_{r,s_r,t_r}, \alpha_{r_s,s,t_s}$, and $\alpha_{r_t,s_t,t}$ belong to I. Let $Z_I = \{\alpha_{r,s_r,t_r}, \alpha_{r_s,s,t_s}, \alpha_{r_t,s_t,t}\}$. Note that the elements in Z_I may not differ at all and Z_I is a pt-pure subset. It is clear that $I \subseteq \overline{PT}[Z_I]$. Let $\delta \in \overline{PT}[Z_I]$. Then $|\operatorname{im}\delta| \leq u, |Y\delta| \leq v$, and $|\operatorname{im}\delta \setminus Y| \leq w$ for some u, v, w with $\alpha_{u,v,w} \in Z_I$. According to Lemma 2.3, we have $\delta = \lambda\alpha_{u,v,w}\mu \in I$, thus implying $I = \overline{PT}[Z_I]$.

Next, we consider any pure subsets Z_1 and Z_2 of $\mathcal{Z}$ with $Z_1 \neq Z_2$. Without loss of generality, if one is strictly contained in the other, we assume that $Z_1 \subsetneq Z_2$. Then there exists $\alpha_{r,s,t} \in Z_2 \setminus Z_1$. Since Z_2 is a pt-pure subset of $\mathcal{Z}$, for each $\alpha_{u,v,w} \in Z_1$, $u > r$ or $v > s$ or $w > t$. Hence, $\alpha_{r,s,t} \in \overline{PT}[Z_2] \setminus \overline{PT}[Z_1]$. For the case $Z_1 \nsubseteq Z_2$ and $Z_2 \nsubseteq Z_1$, we have $Z_1 \setminus Z_2 \neq \emptyset$ and $Z_2 \setminus Z_1 \neq \emptyset$. Let $r_1 = \max\{u : \alpha_{u,v,w} \in Z_1 \setminus Z_2\}$ and $r_2 = \max\{u : \alpha_{u,v,w} \in Z_2 \setminus Z_1\}$. Then there exist $\alpha_{r_1,s_1,t_1} \in Z_1 \setminus Z_2$ and $\alpha_{r_2,s_2,t_2} \in Z_2 \setminus Z_1$.

Case 1: $r_1 > r_2$. If $\alpha_{r_1,s_1,t_1} \in \overline{PT}[Z_2]$, then there exists $\alpha_{u,v,w} \in Z_2$ such that $r_2 < r_1 \leq u$, $s_2 \leq v$, and $t_2 \leq w_2$. The maximum value of r_2 implies that $\alpha_{u,v,w} \in Z_1$, which contradicts the fact that Z_1 is a pure subset. Hence, $\alpha_{r_1,s_1,t_1} \in \overline{PT}[Z_1] \setminus \overline{PT}[Z_2]$.

Case 2: $r_2 > r_1$. Using the same argument as in Case 1, we can conclude that $\alpha_{r_2,s_2,t_2} \in \overline{PT}[Z_2] \setminus \overline{PT}[Z_1]$.

Case 3: $r_1 = r_2$. Let $v_1 = \max\{v : \alpha_{r_1,v,w} \in Z_1 \setminus Z_2\}$ and $v_2 = \max\{v : \alpha_{r_2,v,w} \in Z_2 \setminus Z_1\}$. If $v_1 \neq v_2$, applying the same previous argument, we conclude that $\overline{PT}[Z_1] \neq \overline{PT}[Z_2]$. In the case where $v_1 = v_2$, we let $w_1 = \max w : \alpha_{r_1,v_1,w} \in Z_1 \setminus Z_2$ and $w_2 = \max w : \alpha_{r_2,v_2,w} \in Z_2 \setminus Z_1$. Consequently, we have $w_1 \neq w_2$ and also establish $\overline{PT}[Z_1] \neq \overline{PT}[Z_2]$. □

To simplify notation, in the case of Z being a finite set such that $Z = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we use the notation $\overline{PT}[\alpha_1, \alpha_2, \ldots, \alpha_n]$ instead of $\overline{PT}[\{\alpha_1, \alpha_2, \ldots, \alpha_n\}]$. It is clear that $\overline{PT}[Z] = \bigcup_{\gamma \in Z} \overline{PT}[\gamma]$.

For $\alpha, \beta \in \overline{PT}(X,Y)$, $\overline{PT}[\alpha] \subseteq \overline{PT}[\beta]$ if and only if $|\operatorname{im}\alpha| \leq |\operatorname{im}\beta|, |Y\alpha| \leq |Y\beta|$, and $|\operatorname{im}\alpha \setminus Y| \leq |\operatorname{im}\beta \setminus Y|$. Consequently, $\overline{PT}[\alpha] = \overline{PT}[\beta]$ if and only if $|\operatorname{im}\alpha| = |\operatorname{im}\beta|, |Y\alpha| = |Y\beta|$, and $|\operatorname{im}\alpha \setminus Y| = |\operatorname{im}\beta \setminus Y|$. Additionally, if $\alpha, \beta \in \mathcal{Z}$, $\overline{PT}[\alpha]$ and $\overline{PT}[\beta]$ are distinct.

Proposition 2.6 *The principal ideals of $\overline{PT}(X,Y)$ are precisely those sets of the form $\overline{PT}[\alpha_{r,s,t}]$.*

Proof Let $\alpha_{r,s,t} \in \mathcal{Z}$. Our objective is to demonstrate that $\overline{PT}[\alpha_{r,s,t}] = \overline{PT}(X,Y)\alpha_{r,s,t}\overline{PT}(X,Y)$. We begin by considering $\beta \in \overline{PT}[\alpha_{r,s,t}]$. This implies that $|\operatorname{im}\beta| \leq r, |Y\beta| \leq s$, and $|\operatorname{im}\beta \setminus Y| \leq t$. According to Lemma 2.3, we can express β as $\lambda\alpha_{r,s,t}\mu$ for some $\lambda, \mu \in \overline{PT}(X,Y)$. Consequently, we have established that β belongs to $\overline{PT}(X,Y)\alpha_{r,s,t}\overline{PT}(X,Y)$. On the other hand, consider γ in $\overline{PT}(X,Y)\alpha_{r,s,t}\overline{PT}(X,Y)$. This implies that $\gamma = \theta\alpha_{r,s,t}\eta$ for some $\theta, \eta \in \overline{PT}(X,Y)$. Since $\alpha_{r,s,t} \in \overline{PT}[\alpha_{r,s,t}]$ and $\overline{PT}[\alpha_{r,s,t}]$ is an ideal, we can conclude that γ is an element of $\overline{PT}[\alpha_{r,s,t}]$. Therefore, $\overline{PT}[\alpha_{r,s,t}] = \overline{PT}(X,Y)\alpha_{r,s,t}\overline{PT}(X,Y)$ is a principal ideal within $\overline{PT}(X,Y)$.

Let I be any principal ideal of $\overline{PT}(X,Y)$. Then $I=\overline{PT}(X,Y)\alpha\overline{PT}(X,Y)$ for some $\alpha\in\overline{PT}(X,Y)$. Let $|\operatorname{im}\alpha|=r, |Y\alpha|=s$, and $|\operatorname{im}\alpha\setminus Y|=t$. By Lemma 2.3, $\alpha=\lambda\alpha_{r,s,t}\mu$ and $\alpha_{r,s,t}=\lambda'\alpha\mu'$ for some $\lambda,\lambda',\mu,\mu'\in\overline{PT}(X,Y)$. Hence, $I=\overline{PT}(X,Y)\alpha\overline{PT}(X,Y)\subseteq\overline{PT}(X,Y)\alpha_{r,s,t}\overline{PT}(X,Y)\subseteq\overline{PT}(X,Y)\alpha\overline{PT}(X,Y)=I$. Therefore, $I=\overline{PT}(X,Y)\alpha_{r,s,t}\overline{PT}(X,Y)=\overline{PT}[\alpha_{r,s,t}]$. □

Next, we will discuss the minimal and maximal ideals of $\overline{PT}(X,Y)$. It is clear that $J_{0,0,0}=\{\emptyset\}=\overline{PT}[\alpha_{0,0,0}]$ is the minimum ideal of $\overline{PT}(X,Y)$.

As $\{\emptyset\}$ represents the minimum ideal within $\overline{PT}(X,Y)$, we can define a *minimal ideal* in $\overline{PT}(X,Y)$ as an ideal I such that $\{\emptyset\}\subsetneq I$ and I satisfies the condition: if there exists an ideal J such that $\{\emptyset\}\subseteq J\subseteq I$, then either $J=\{\emptyset\}$ or $J=I$. The following theorem elaborates on the details of the minimal ideal in $\overline{PT}(X,Y)$.

Theorem 2.7 *$\{\emptyset\}\cup J_{1,0,0}$ is the unique minimal ideal of $\overline{PT}(X,Y)$.*

Proof It is routine to verify that $\{\emptyset\}\cup J_{1,0,0}=\overline{PT}(2,1,1)$ is an ideal of $\overline{PT}(X,Y)$. To prove the minimality, we let J be an ideal of $\overline{PT}(X,Y)$ such that $\{\emptyset\}\subseteq J\subsetneq\{\emptyset\}\cup J_{1,0,0}$. Then there exists $\alpha\in J_{1,0,0}$, but $\alpha\notin J$. To demonstrate that $J=\{\emptyset\}$, we assume the contrary. In this case, there exists $\emptyset\neq\beta\in J$. Since both α and β belong to $J_{1,0,0}$, by Lemma 2.3, there exist $\lambda,\mu\in\overline{PT}(X,Y)$ such that $\alpha=\lambda\beta\mu$. Since $\beta\in J$ and J is an ideal, we obtain $\alpha=\lambda\beta\mu\in J$, which leads to a contradiction. Consequently, $\{\emptyset\}\cup J_{1,0,0}$ qualifies as a minimal ideal within $\overline{PT}(X,Y)$. For the uniqueness, we let M be a minimal ideal of $\overline{PT}(X,Y)$. As M is an ideal of $\overline{PT}(X,Y)$, it can be expressed as $M=\overline{PT}[Z]$ for some a nonempty subset Z of $\overline{PT}(X,Y)$. Since $\{\emptyset\}\subsetneq M$, there must exist $\alpha\in M$ such that $|\operatorname{im}\alpha|\geq 1$. Since $\alpha\in M=\overline{PT}[Z]$, we have $|\operatorname{im}\alpha|\leq|\operatorname{im}\beta|, |Y\alpha|\leq|Y\beta|$, and $|\operatorname{im}\alpha\backslash Y|\leq|\operatorname{im}\beta\backslash Y|$ for some $\beta\in Z$. Now, let $\gamma\in J_{1,0,0}$. Then $|\operatorname{im}\gamma|=1\leq|\operatorname{im}\alpha|\leq|\operatorname{im}\beta|$, $|Y\gamma|=0\leq|Y\alpha|\leq|Y\beta|$, and $|\operatorname{im}\gamma\backslash Y|=0\leq|\operatorname{im}\alpha\backslash Y|\leq|\operatorname{im}\beta\backslash Y|$. This implies that $\gamma\in\overline{PT}[Z]=M$. Consequently, we have shown that $\{\emptyset\}\cup J_{1,0,0}\subseteq M$, and therefore, $M=\{\emptyset\}\cup J_{1,0,0}$ by the minimality of M. □

Now, we will introduce the concept of a maximal ideal in $\overline{PT}(X,Y)$. An ideal I in $\overline{PT}(X,Y)$ is categorized as a maximal ideal if, for any ideal M such that $I\subseteq M\subseteq\overline{PT}(X,Y)$, it holds that either $M=I$ or $M=\overline{PT}(X,Y)$.

Theorem 2.8 *$\overline{PT}(X,Y)\backslash J_{a,b,c}$ is the unique maximal ideal of $\overline{PT}(X,Y)$.*

Proof It is clear that $\overline{PT}(X,Y)\backslash J_{a,b,c}=\overline{PT}[\overline{PT}(X,Y)\backslash J_{a,b,c}]$ is an ideal of $\overline{PT}(X,Y)$. To show that $\overline{PT}(X,Y)\backslash J_{a,b,c}$ is a maximal ideal of $\overline{PT}(X,Y)$, we let M be an ideal of $\overline{PT}(X,Y)$ such that $\overline{PT}(X,Y)\backslash J_{a,b,c}\subsetneq M\subseteq\overline{PT}(X,Y)$. This implies that there exists $\alpha\in M$, but $\alpha\notin\overline{PT}(X,Y)\backslash J_{a,b,c}$. As a result, we have $|\operatorname{im}\alpha|=a, |Y\alpha|=b$, and $|\operatorname{im}\alpha\backslash Y|=c$. Now, let $\beta\in J_{a,b,c}$. Since $\alpha,\beta\in J_{a,b,c}$, there exist λ and μ in $\overline{PT}(X,Y)$ such that $\beta=\lambda\alpha\mu$. Consequently, $\beta=\lambda\alpha\mu\in M$ since $\alpha\in M$ and M is an ideal. Thus, $M=\overline{PT}(X,Y)$. For the uniqueness, we let M' be a maximal ideal of $\overline{PT}(X,Y)$. Then $M\cup M'$ is an ideal and $id_X\notin M\cup M'$, whence $M\cup M'\subseteq\overline{PT}(X,Y)$. Since $M\subseteq M\cup M'$ and M is a maximal ideal, we have $M\cup M'=M$. Similarly, we can conclude that $M\cup M'=M'$. Thus, $M=M\cup M'=M'$ □

If $Y \neq X$, then $\overline{PT}[\alpha_{1,0,1}]$ and $\overline{PT}[\alpha_{1,1,0}]$ neither contains the other. This means that if $Y \neq \emptyset$, then the ideals does not form a chain.

we conclude the study of ideals on $\overline{PT}(X,Y)$ by elucidating the set $J_{r,s,t}$ and the poset of ideals in $\overline{PT}(X,Y)$ for the sets $X = \{1,2,3\}$ and $Y = \{1,2\}$. To enhance clarity, an element α in $\overline{PT}(X,Y)$ satisfying $1\alpha = x$, $2\alpha = y$, and $3\alpha = z$ is denoted as (x,y,z). Specifically, the vacant positions in the 3-tuple signify their exclusion from the domain of those elements. The subsets $J_{r,s,t}$ with $\alpha_{r,s,t}$ in red and the Hasse diagram of ideals in $\overline{PT}(X,Y)$ are presented in Table 1 and Figure 1, respectively.

$J_{3,2,1}$ $(1,2,3)$ $(2,1,3)$		
$J_{2,1,0}$ $(1,1,2)$ $(2,2,1)$ $(1, \ ,2)$ $(2, \ ,1)$ $(\ ,1,2)$ $(\ ,2,1)$	$J_{2,1,1}$ $(1,1,3)$ $(2,2,3)$ $(1, \ ,3)$ $(2, \ ,3)$ $(\ ,1,3)$ $(\ ,2,3)$	$J_{2,2,0}$ $(1,2,1)$ $(2,1,2)$ $(1,2,2)$ $(2,1,1)$ $(1,2, \ )$ $(2,1, \ )$
$J_{1,0,0}$ $(\ , \ ,1)$ $(\ , \ ,2)$	$J_{1,0,1}$ $(\ , \ ,3)$	$J_{1,1,0}$ $(1, \ , \ )$ $(2, \ , \ )$ $(\ ,1, \ )$ $(\ ,2, \ )$ $(1,1, \ )$ $(2,2, \ )$ $(1, \ ,1)$ $(2, \ ,2)$ $(\ ,1,1)$ $(\ ,2,2)$ $(1,1,1)$ $(2,2,2)$
$J_{0,0,0}$ $\emptyset$		

Table 1. The subsets $J_{r,s,t}$ of $\overline{PT}(\{1,2,3\},\{1,2\})$

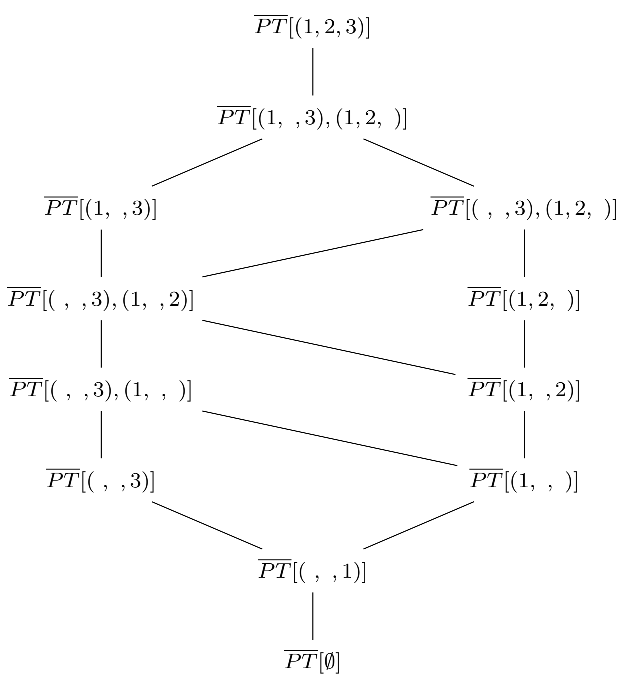

Figure 1. The Hasse diagram of ideals in $\overline{PT}(\{1,2,3\},\{1,2\})$

Our next propose is to explore the ideals of $PFix(X,Y)$ in the case where Y is a proper subset of X. Recall that the ideals of $\overline{PT}(X,Y)$ are of the form $\overline{PT}[Z]$ where $\emptyset \neq Z \subseteq \overline{PT}(X,Y)$. Since $PFix(X,Y)$ is a subsemigroup of $\overline{PT}(X,Y)$, we easily obtain the following:

Lemma 2.9 *$\overline{PT}[Z] \cap PFix(X,Y)$ is an ideal of $PFix(X,Y)$.*

The following example demonstrates that there exists an ideal in $PFix(X,Y)$ that does not conform to the form of $\overline{PT}[Z] \cap PFix(X,Y)$.

Example 2.10 *Let $X = \{1,2,3\}$ and $Y = \{1,2\}$. Consider the ideal*

$$I = \{\emptyset, (\ ,\ ,1), (\ ,\ ,2), (1,\ ,\ ), (1,\ ,1)\}$$

in $PFix(X,Y)$. If Z takes the form $\{(\ ,\ ,1)\}$, $\{(\ ,\ ,2)\}$, or $\{(\ ,\ ,1),(\ ,\ ,2)\}$, then the corresponding $\overline{PT}[Z]$ is $\{\emptyset,(\ ,\ ,1),(\ ,\ ,2)\}$; if $Z = \{(\ ,\ ,3)\}$, then $\overline{PT}[Z] = \{\emptyset,(\ ,\ ,1),(\ ,\ ,2),(\ ,\ ,3)\}$; and if $Z = \{(\ ,\ ,1),(\ ,\ ,3)\}$, then $\overline{PT}[Z] = \{\emptyset,(\ ,\ ,1),(\ ,\ ,2),(\ ,\ ,3)\}$. In all mentioned cases, it is implied that $\overline{PT}[Z] \cap PFix(X,Y) = \overline{PT}[Z] \neq I$. Furthermore, for any $Z \subseteq \overline{PT}(X,Y)$ not falling within the previously mentioned scenarios, $\overline{PT}[Z]$ consistently contains $(\ ,2,\ )$, which results in $\overline{PT}[Z] \cap PFix(X,Y) \neq I$. Consequently, we assert that $I \neq \overline{PT}[Z] \cap PFix(X,Y)$ for all $Z \subseteq \overline{PT}(X,Y)$.

To identify all ideals of $PFix(X,Y)$, we refer to the result from [26] as follows:

Lemma 2.11 *[26] Let $\alpha,\beta \in PFix(X,Y)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda,\mu \in PFix(X,Y)$ if and only if* $\operatorname{dom}\alpha \cap Y \subseteq \operatorname{dom}\beta \cap Y, |\operatorname{im}\alpha| \leq |\operatorname{im}\beta|$ *and* $|\operatorname{im}\alpha \backslash (\operatorname{im}\beta \cap Y)| \leq |\operatorname{im}\beta \backslash Y|$.

Moreover, we define a subset $PF[Z]$, where $\emptyset \neq Z \subseteq PFix(X,Y)$, as the set $PF[Z] = \{\alpha \in PFix(X,Y) : \operatorname{dom}\alpha \cap Y \subseteq \operatorname{dom}\beta \cap Y, |\operatorname{im}\alpha| \leq |\operatorname{im}\beta|, |\operatorname{im}\alpha \backslash (\operatorname{im}\beta \cap Y)| \leq |\operatorname{im}\beta \backslash Y| \text{ for some } \beta \in Z\}$. Clearly, $Z \subseteq PF[Z]$, and if $Z_1 \subseteq Z_2$, then $PF[Z_1] \subseteq PF[Z_2]$.

Following the argument presented in the proof of Theorem 2.4, we establish the following theorem:

Theorem 2.12 *The ideals of $PFix(X,Y)$ are precisely those sets of the form $PF[Z]$, where Z is a nonempty subset of $PFix(X,Y)$.*

If Z is a finite set such that $Z = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, similar to the notation used in $\overline{PT}[Z]$, we use the notation $PF[\alpha_1, \alpha_2, \ldots, \alpha_n]$ instead of $PF[\{\alpha_1, \alpha_2, \ldots, \alpha_n\}]$. It is clear that $PF[Z] = \bigcup_{\gamma \in Z} PF[\gamma]$.

For α and β in $PFix(X,Y)$. Then, $PF[\alpha] \subseteq PF[\beta]$ if and only if $\operatorname{dom}\alpha \cap Y \subseteq \operatorname{dom}\beta \cap Y$, $|\operatorname{im}\alpha| \leq |\operatorname{im}\beta|$, and $|\operatorname{im}\alpha \backslash (\operatorname{im}\beta \cap Y)| \leq |\operatorname{im}\beta \backslash Y|$. Consequently, $PF[\alpha] = PF[\beta]$ if and only if $|\operatorname{im}\alpha| = |\operatorname{im}\beta|$, $\operatorname{dom}\alpha \cap Y = \operatorname{dom}\beta \cap Y$, $|\operatorname{im}\alpha \backslash (\operatorname{im}\beta \cap Y)| \leq |\operatorname{im}\beta \backslash Y|$ and $|\operatorname{im}\beta \backslash (\operatorname{im}\alpha \cap Y)| \leq |\operatorname{im}\alpha \backslash Y|$.

Note that, according to Lemma 2.11, we directly obtain that $\alpha \in PF[\beta]$ if and only if $\alpha = \lambda\beta\mu$ for some $\lambda,\mu \in PFix(X,Y)$. By applying the same argument used in the proof of Proposition 2.6, we arrive at the following theorem:

Proposition 2.13 *The principal ideals of* $PFix(X,Y)$ *are precisely those sets of the form* $PF[\alpha]$*, where* $\alpha \in PFix(X,Y)$.

Next, we will examine the minimal and maximal ideals of $PFix(X,Y)$. Henceforth, let $|X\backslash Y| = c$, we will then proceed to define

$$J(A,B,t) = \{\alpha \in PFix(X,Y) : \operatorname{dom}\alpha \cap Y = A, \operatorname{im}\alpha \cap Y = B \text{ and } |\operatorname{im}\alpha\backslash Y| = t\},$$

where $A, B \subseteq Y$, and $0 \leq t \leq c$. It is clear that $J(\emptyset,\emptyset,0) = \{\emptyset\} = PF[\emptyset]$ is the minimum ideal of $PFix(X,Y)$.

Lemma 2.14 *Let* $y \in Y$*. Then* $\{\emptyset\} \cup J(\emptyset,\{y\},0)$ *is a minimal ideal of* $PFix(X,Y)$.

Proof Let $y \in Y$, $x \in X \setminus Y$, and $\gamma = \begin{pmatrix} x \\ y \end{pmatrix}$. It is clear that $\{\emptyset\} \cup J(\emptyset,\{y\},0) = PF[\gamma]$ is an ideal of $PFix(X,Y)$. To prove the minimality, we let J be an ideal of $PFix(X,Y)$ such that $\{\emptyset\} \subseteq J \subsetneq \{\emptyset\} \cup J(\emptyset,\{y\},0)$. Then, there exists $\alpha \in J(\emptyset,\{y\},0)$, but $\alpha \notin J$. To demonstrate that $J = \{\emptyset\}$, we assume the contrary. In this case, there exists $\emptyset \neq \beta \in J$. Since both α and β belong to $J(\emptyset,\{y\},0)$, it follows that $\operatorname{dom}\alpha \cap Y = \operatorname{dom}\beta \cap Y$, $|\operatorname{im}\alpha| = 1 = |\operatorname{im}\beta|$, and $|\operatorname{im}\alpha \setminus (\operatorname{im}\beta \cap Y)| = 0 = |\operatorname{im}\beta \setminus Y|$. By Lemma 2.11, there exist $\lambda, \mu \in PFix(X,Y)$ such that $\alpha = \lambda\beta\mu$. Since $\beta \in J$ and J is an ideal, we obtain $\alpha \in J$, which leads to a contradiction. Consequently, $\{\emptyset\} \cup J(\emptyset,\{y\},0)$ qualifies as a minimal ideal within $PFix(X,Y)$. □

Lemma 2.15 *Let* $\emptyset \neq Z \subseteq PFix(X,Y)$*. If* $\operatorname{im}\alpha \cap Y = \emptyset$ *for all* $\alpha \in Z$*, then* $PF[Z]$ *is not a minimal ideal of* $PFix(X,Y)$.

Proof Assume that the given condition holds. The assertion is clear in the case where $Z = \{\emptyset\}$. Therefore, we consider the case where $\emptyset \neq \alpha \in Z$. Let $x \in X \setminus Y$ and consider $\gamma = \begin{pmatrix} x \\ x \end{pmatrix}$. We can see that $\operatorname{dom}\gamma \cap Y = \emptyset \subseteq \operatorname{dom}\alpha \cap Y$, $|\operatorname{im}\gamma| = 1 \leq |\operatorname{im}\alpha|$, and $|\operatorname{im}\gamma \setminus (\operatorname{im}\alpha \cap Y)| = |\operatorname{im}\gamma| \leq |\operatorname{im}\alpha| = |\operatorname{im}\alpha \setminus Y|$. This follows that $\gamma \in PF[\alpha] \subseteq PF[Z]$. To show $J(\emptyset,\{y\},0) \subseteq PF[Z]$, we let $\beta \in J(\emptyset,\{y\},0)$. Then $\operatorname{dom}\beta \cap Y = \emptyset = \operatorname{dom}\gamma \cap Y$, $|\operatorname{im}\beta| = 1 = |\operatorname{im}\gamma|$, and $|\operatorname{im}\beta \setminus (\operatorname{im}\gamma \cap Y)| = |\operatorname{im}\beta| = |\operatorname{im}\gamma| = |\operatorname{im}\gamma \setminus Y|$. This implies, by Lemma 2.11, that $\beta = \lambda\gamma\mu$ for some $\lambda, \mu \in PFix(X,Y)$. Since $PF[Z]$ is an ideal and $\gamma \in PF[Z]$, we get $\beta \in PF[Z]$, which implies $J(\emptyset,\{y\},0) \subseteq PF[Z]$. Hence, $\{\emptyset\} \cup J(\emptyset,\{y\},0) \subsetneq PF[Z]$ since $\gamma \notin J(\emptyset,\{y\},0)$. Therefore, $PF[Z]$ is not minimal. □

Theorem 2.16 *The minimal ideals of* $PFix(X,Y)$ *are precisely those sets of the form* $\{\emptyset\} \cup J(\emptyset,\{y\},0)$*, where* $y \in Y$.

Proof Let I be any minimal ideal of $PFix(X,Y)$. According to Theorem 2.12, $I = PF[Z]$ for some nonempty set $Z \subseteq PFix(X,Y)$. Since I is minimal, as indicated by Lemma 2.15, there exists $\alpha \in Z$ such that $\operatorname{im}\alpha \cap Y \neq \emptyset$. Choose $y \in \operatorname{im}\alpha \cap Y$. To demonstrate that $J(\emptyset,\{y\},0) \subseteq I$, let $\beta \in J(\emptyset,\{y\},0)$. Then $\operatorname{dom}\beta \cap Y = \emptyset \subseteq \operatorname{dom}\alpha \cap Y$, $|\operatorname{im}\beta| = 1 \leq |\operatorname{im}\alpha|$, and $|\operatorname{im}\beta \setminus (\operatorname{im}\alpha \cap Y)| = 0 \leq |\operatorname{im}\alpha \setminus Y|$. Consequently, $\beta \in PF[Z] = I$, implying $\{\emptyset\} \cup J(\emptyset,\{y\},0) \subseteq I$. Since I is minimal, we conclude that $I = \{\emptyset\} \cup J(\emptyset,\{y\},0)$, as required. □

Theorem 2.17 *$PFix(X,Y)\backslash J(Y,Y,c)$ is the unique maximal ideal of $PFix(X,Y)$.*

Proof It is routine to verify that $PFix(X,Y)\backslash J(Y,Y,c) = PF[PFix(X,Y)\backslash J(Y,Y,c)]$ is an ideal of $PFix(X,Y)$. To show that $PFix(X,Y)\backslash J(Y,Y,c)$ is a maximal ideal of $PFix(X,Y)$, we let M be an ideal of $PFix(X,Y)$ such that $PFix(X,Y)\backslash J(Y,Y,c) \subsetneq M \subseteq PFix(X,Y)$. This implies that there exists an $\alpha \in M$, but $\alpha \notin PFix(X,Y)\backslash J(Y,Y,c)$. As a result, we have $\operatorname{dom}\alpha \cap Y = Y$, $\operatorname{im}\alpha \cap Y = Y$ and $|\operatorname{im}\alpha\backslash Y| = c$. Now, let $\beta \in J(Y,Y,c)$. Since $\alpha, \beta \in J(Y,Y,c)$, there exist λ and μ in $PFix(X,Y)$ such that $\beta = \lambda\alpha\mu$. Consequently, $\beta = \lambda\alpha\mu \in M$ since $\alpha \in M$ and M is an ideal. Thus, $M = PFix(X,Y)$. The uniqueness can be proved similar to Theorem 2.8. □

Theorem 2.18 *The ideals of $PFix(X,Y)$ form a chain under the set inclusion if and only if $Y = \emptyset$.*

Proof Assume that $Y \neq \emptyset$. Then there exist an element y in Y and an element x from $X\backslash Y$. Define α and β in $PFix(X,Y)$ by

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} y \\ y \end{pmatrix}.$$

Since $|\operatorname{im}\alpha \setminus (\operatorname{im}\beta \cap Y)| = 1 \nleq 0 = |\operatorname{im}\beta \setminus Y|$, it follows that $\alpha \in PF[\alpha] \setminus PF[\beta]$. Also, since $\operatorname{dom}\beta \cap Y = \{y\} \nsubseteq \emptyset = \operatorname{im}\alpha \cap Y$, implies $\beta \in PF[\beta] \setminus PF[\alpha]$. This implies that neither contains the other. Hence, the ideals of $PFix(X,Y)$ do not form a chain. The converse is trivial, since $PFix(X,Y) = P(X)$ when $Y = \emptyset$. □

Note that in the case where X is a finite set, we have $\mathcal{D} = \mathcal{J}$; this implies, by Theorem 3.5 in [26], that $PF[\alpha] = PF[\beta]$ for all $\alpha, \beta \in J(A,B,t)$. Also, for $\alpha \in J(A,B,t)$ and $\beta \in J(U,V,w)$ such that $A \subseteq U$, $B \subseteq V$ and $t \leq w$, then $PF[\alpha] \subseteq PF[\beta]$. However, the converse of this statement dose not hold.

This section concludes by explicating the set $J(A,B,t)$ and the poset of ideals within $PFix(X,Y)$ concerning the sets $X = \{1,2,3\}$ and $Y = \{1,2\}$, as depicted in Table 2 and Figure 2, respectively. The utilization of blue color in these depictions signifies the representation of ideals in the form $\overline{PT}[Z] \cap PFix(X,Y)$.

$J(Y,Y,1)$: $(1,2,3)$				
$J(\{1\},Y,0)$: $(1, \ ,2)$	$J(\{2\},Y,0)$: $(\ ,2,1)$	$J(\{1\},\{1\},1)$: $(1, \ ,3)$	$J(\{2\},\{2\},1)$: $(\ ,2,3)$	$J(Y,Y,0)$: $(1,2, \ )$ $(1,2,1)$ $(1,2,2)$
$J(\{1\},\{1\},0)$: $(1, \ , \ )$ $(1, \ ,1)$	$J(\{2\},\{2\},0)$: $(\ ,2, \ )$ $(\ ,2,2)$	$J(\emptyset,\{1\},0)$: $(\ , \ ,1)$	$J(\emptyset,\{2\},0)$: $(\ , \ ,2)$	$J(\emptyset,\emptyset,1)$: $(\ , \ ,3)$
$J(\emptyset,\emptyset,0)$: $\emptyset$				

Table 2. The subsets $J(A,B,t)$ of $PFix(\{1,2,3\},\{1,2\})$

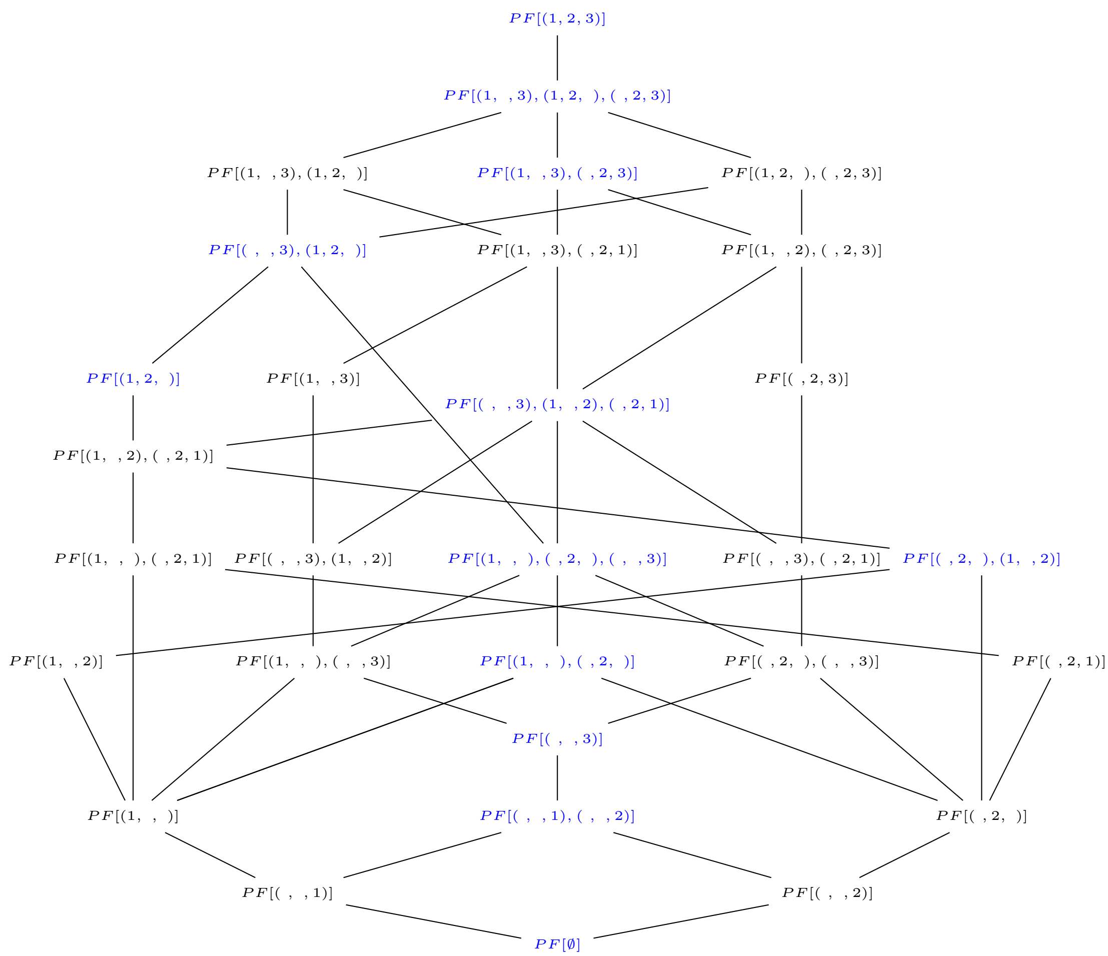

Figure 2. The Hasse diagram of ideals in $PFix(\{1,2,3\},\{1,2\})$

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