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Gradient almost Yamabe solitons immersed into a Riemannian warped product manifold

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Abstract: The purpose of this article is to study the geometry of gradient almost Yamabe solitons immersed into warped product manifolds $I \times_f M^n$ whose potential is given by the height function from the immersion. First, we present some geometric rigidity on compact solitons due to a curvature condition on the warped product manifold. In the sequel, we investigate conditions for the existence of totally geodesic, totally umbilical, and minimal solitons. Furthermore, in the scope of constant angle immersions, a classification of rotational gradient almost Yamabe solitons immersed into $\mathbb{R} \times_f \mathbb{R}^n$ is also made.

Key words: Yamabe solitons, almost Yamabe solitons, totally geodesic hypersurface, warped product, totally umbilical hypersurface, rotational gradient Yamabe solitons.

1. Introduction

The concept of almost Yamabe soliton, introduced in the celebrated work [3], corresponds to a natural generalization of Yamabe solitons [17, 18] and Yamabe metrics [26]. We recall that a Riemannian manifold (Σ^n, g) is an *almost Yamabe soliton* if it admits a vector field $X \in \mathfrak{X}(\Sigma)$ and a smooth function $\lambda : \Sigma \to \mathbb{R}$ satisfying the equation

$$\frac{1}{2}\mathfrak{L}_X g = (scal_g - \lambda)g, \tag{1.1}$$

where $\mathfrak{L}_X g$ and $scal_g$ stand, respectively, for the Lie derivative of g in the direction of X and the scalar curvature of g. The quadruple $(\Sigma^n, g, X, \lambda)$ is classified into three types according to the sign of λ: expanding if $\lambda < 0$, steady if $\lambda = 0$, and shrinking if $\lambda > 0$. If λ occurs as a constant, the soliton is usually referred to as a *Yamabe soliton*. It may happen that $X = \nabla h$ is the gradient vector field of a smooth function $h : M \to \mathbb{R}$, called *potential*, in which case the soliton $(\Sigma^n, g, h, \lambda)$ is referred to as a *gradient almost Yamabe soliton*. Equation

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(1.1) then becomes

$$\nabla^2 h = (scal_g - \lambda)g, \tag{1.2}$$

where $\nabla^2 h$ is the Hessian of h. We pointed out that, almost Yamabe solitons, gradient or not, are regarded as *trivial* if their defining equation vanishes identically.

The geometry of almost Yamabe solitons isometrically immersed in space forms has recently received a great deal of attention [2, 7, 16, 21, 22]. Chen and Deshmukh [7], study Yamabe solitons whose soliton vector field is the tangent component from the position vector on Euclidean space and, as a result, they established some rigidity results. Under a concurrent vector field assumption on the soliton vector field, Seko and Maeta [22] showed that any almost Yamabe soliton has a gradient almost Yamabe soliton structure. Furthermore, for almost Yamabe solitons on ambient spaces furnished with a concurrent vector field, the authors give a classification of such solitons. On the other hand, Aquino et al. [2] present the study of gradient almost Ricci solitons immersed into the space of constant sectional curvature $M^{n+1} \subset \mathbb{R}^{n+2}$ with potential given by the height function from the soliton associated to a fixed direction on $\mathbb{R}^{n+2}$.

The above works bring to light that important geometric results are obtained if we choose an appropriate soliton vector field. In this sense, a vector field that has already proven itself to be a rich source for producing examples of solitons fields is the one generated by the gradient of the height function from the immersion. Examples of solitons in which the height function is taken as the potential function are given in [2–5, 10, 23].

From the previous works, gradient almost Yamabe solitons in which the height function is chosen as the potential, might be interesting for further investigation. Moreover, to extend the above works to a larger class of ambient spaces, it appears convenient to consider the immersions in a sufficiently large family of manifolds, including the spaces of constant sectional curvature. A natural metric, which includes the spaces of constant sectional curvature in its range, is described by warped product metrics [20]. Warped product manifolds have already proven themselves to be a profitable ambient space to obtain a wide range of distinct geometrical proprieties for immersions (cf. [1, 6, 8, 12, 13]). In this context, as in [1], we can extend the concept of height function using the projection onto the base of the warped product (see Section 2).

The purpose of this manuscript is to study the geometry of gradient almost Yamabe solitons $(\Sigma^n, g, h, \lambda)$ immersed into warped product manifold $I \times_f M^n$ whose potential h is given by the height function from the immersion. In this setting, we derive a necessary and sufficient condition for the immersion to be a gradient almost Yamabe soliton. We use this result to investigate conditions for the existence of totally geodesic, totally umbilical, minimal and trivial solitons. Furthermore, when the ambient space is taken as $\mathbb{R} \times_f \mathbb{R}^n$, we provide the classification of rotational gradient almost Yamabe solitons with a constant angle.

This manuscript is organized in the following way: In Section 2, we recall some basic facts and notations that will appear throughout the paper. Afterward, in Section 3, we exhibit some examples of immersions satisfying the gradient almost Yamabe soliton equation (1.2), and we establish our first main results concerning the geometry of these geometric objects. Finally, in Section 4, we provide the classification of rotationally symmetric gradient almost Yamabe solitons.

2. Preliminaries

Let (M^n, g_M) be a connected, n-dimensional oriented Riemannian manifold, $I \subset \mathbb{R}$ an interval and $f : I \to (0, \infty)$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, consider the projections π_I and π_M onto the spaces I and M^n, respectively. A particular class of Riemannian manifold is the one obtained by furnishing $\overline{M}^{n+1}$ with the metric

$$\langle\ ,\ \rangle = \pi_I^*(dt^2) + f^2(\pi_I)\pi_M^*(g_M),$$

such a space is called a warped product manifold with base I, fiber M^n and warping function f. In this setting, for a fixed $t_0 \in \mathbb{R}$, we say that $\Sigma_{t_0}^n := \{t_0\} \times M^n$ is a slice of $\overline{M}^{n+1}$.

Let $\overline{\nabla}$ and ∇ the Levi-Civita connection in $I \times_f M^n$ and Σ^n, respectively. Then, the Gauss-Weingarten formulas for an isometric immersion $\psi : \Sigma^n \to I \times_f M^n$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N, \qquad AX = -\overline{\nabla}_X N, \tag{2.1}$$

for any $X \in \mathfrak{X}(\Sigma^n)$, where $A : T\Sigma^n \to T\Sigma^n$ denotes the Weingarten operator of Σ^n with respect to its Gauss map N. We consider two particular functions naturally attached to Σ^n, namely, the height function $h := (\pi_I)|_\Sigma$ and the angle function $\theta = \langle N, \partial_t\rangle$, where ∂_t is the standard unit vector field tangent to I. By a straightforward computation, we obtain that the gradient of π_I on $I \times_f M^n$ is given by

$$\overline{\nabla}\pi_I = \langle\overline{\nabla}\pi_I, \partial_t\rangle\partial_t = \partial_t,$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = \partial_t^\top = \partial_t - \theta N, \tag{2.2}$$

where $(\ \cdot\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along Σ^n. In particular, we get

$$|\nabla h|^2 = 1 - \theta^2,$$

where $|\cdot|$ denotes the norm of a vector field on Σ^n.

Let $\overline{R}$ and R be the curvature tensors of $I \times_f M^n$ and Σ^n, respectively. Therefore, for any X, Y, $Z \in \mathfrak{X}(\Sigma^n)$ we have the following Gauss equation:

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top + \langle AX, Z\rangle AY - \langle AY, Z\rangle AX. \tag{2.3}$$

Denote by Ric the Ricci tensor of Σ^n and consider a local orthonormal frame $\{E_i\}_{i=1}^n$ of $\mathfrak{X}(\Sigma^n)$, as well as $X \in \mathfrak{X}(\Sigma^n)$. Then, it follows from the Gauss equation (2.3) that

$$Ric(X,X) = \sum_{i=1}^n \langle\overline{R}(X,E_i)X, E_i\rangle + nH\langle AX, X\rangle - \langle AX, AX\rangle. \tag{2.4}$$

where H is the mean curvature of Σ^n.

Taking into account the properties of the Riemannian tensor $\overline{R}$ of a warped product (see, for instance, Proposition 7.42 in [20]), we deduce

$$\begin{aligned}\overline{R}(X,Y)Z = R^M(X^*,Y^*)Z^* &- [(\log f)'(h)]^2 \left[\langle X, Z\rangle Y - \langle Y, Z\rangle X\right] \\ &+ (\log f)''(h)\langle Z, \partial_t\rangle \left[\langle Y, \partial_t\rangle X - \langle X, \partial_t\rangle Y\right] \\ &- (\log f)''(h) \left[\langle Y, \partial_t\rangle\langle X, Z\rangle - \langle X, \partial_t\rangle\langle Y, Z\rangle\right]\partial_t,\end{aligned}$$

where R^M is the curvature tensor of the fiber and $X^* = X - \langle X, \partial_t\rangle\partial_t$, $E_i^* = E_i - \langle E_i, \partial_t\rangle\partial_t$ are, respectively, the projections of the tangent vector fields X and E_i onto M^n. Then we arrive at

$$\begin{aligned}\sum_{i=1}^{n}\langle\overline{R}(X,E_i)X,E_i\rangle =& f(h)^{-2}\sum_{i=1}^{n}K^M(X^*,E_i^*)\Big[|X|^2-\langle X,\nabla h\rangle^2-|X|^2\langle\nabla h,E_i\rangle^2\\ &-\langle X,E_i\rangle^2+2\langle X,\nabla h\rangle\langle X,E_i\rangle\langle\nabla h,E_i\rangle\Big]+[(\log f)'(h)]^2\Big(|\nabla h|^2\\ &-(n-1)\Big)|X|^2-(n-2)(\log f)''(h)\langle X,\nabla h\rangle^2-\frac{f''}{f}|\nabla h|^2|X|^2,\end{aligned} \tag{2.5}$$

where K^M is the sectional curvature of M^n. Combining equations (2.4) and (2.5), we deduce the following expression for the scalar curvature of Σ^n:

$$\begin{aligned} scal_g =& f(h)^{-2}\sum_{i,j=1}^{n}K^M(E_j^*,E_i^*)\Big[1-\langle E_j,\nabla h\rangle^2-\langle\nabla h,E_i\rangle^2-\langle E_j,E_i\rangle^2\\ &+2\langle E_j,\nabla h\rangle\langle E_j,E_i\rangle\langle\nabla h,E_i\rangle\Big]+n[(\log f)'(h)]^2\left[|\nabla h|^2-(n-1)\right]\\ &-(n-2)(\log f)''(h)|\nabla h|^2-n\frac{f''}{f}|\nabla h|^2+n^2H^2-|A|^2.\end{aligned} \tag{2.6}$$

From [20], we know that $\overline{M}^{n+1}$ has constant sectional curvature c if, and only if, M^n has constant sectional curvature k and the warping function f satisfy the following ODE:

$$\frac{(f')^2-k}{f^2}=-c=\frac{f''}{f}. \tag{2.7}$$

The next example provides warped product manifolds with constant sectional curvature, i.e., satisfying (2.7).

Example 2.1 *We remark that a Riemannian manifold of constant sectional curvature $c\in\{-1,0,1\}$ can be expressed as a warped product manifold $I\times_f M^n$, namely*

a) $\mathbb{R}^{n+1}\setminus\{0\}=(0,+\infty)\times_f\mathbb{S}^n$ *with* $f(t)=t$,

b) $\mathbb{R}^{n+1}=\mathbb{R}\times_f\mathbb{R}^n$ *with* $f(t)=1$,

c) $\mathbb{S}^{n+1}\setminus\{\pm p\}=(0,\pi)\times_f\mathbb{S}^n$ *with* $f(t)=\sin t$,

d) $\mathbb{H}^{n+1}=\mathbb{R}\times_f\mathbb{R}^n$ *with* $f(t)=e^t$,

e) $\mathbb{H}^{n+1}\setminus\{p\}=(0,+\infty)\times_f\mathbb{S}^n$ *with* $f(t)=\sinh t$,

Proceeding, in order to establish our main results, we will need the following key lemma, which provides a necessary and sufficient condition for a hypersurface to be a gradient almost Yamabe soliton with a height function as the potential.

Lemma 2.2 *Let $\psi : \Sigma^n \to I \times_f M^n$ be an isometric immersion. Then (Σ^n, g) is a gradient almost Yamabe soliton with potential $h = (\pi_I)|_\Sigma$ if, and only if,*

$$(scal_g - \lambda)g(X,Y) = (\log f)'(h)\,[g(X,Y) - dh \otimes dh(X,Y)] + \theta g(AX,Y) \tag{2.8}$$

for all $X\,,\ Y \in \mathfrak{X}(\Sigma^n)$.

Proof Taking into account the properties of the Levi-Civita connection of a warped product (see, for instance, Proposition 7.35 in [20]), it easily follows that

$$\overline{\nabla}_X \partial_t = \frac{f'}{f}(X - \langle X, \partial_t\rangle \partial_t), \qquad \forall X \in \mathfrak{X}(\Sigma^n).$$

Thus, from equations (2.1) and (2.2), we deduce the following expression for the Hessian of h

$$\nabla^2 h(X) = \nabla_X \nabla h = \frac{f'(h)}{f(h)}\,(X - \langle X, \nabla h\rangle \nabla h) + \langle N, \partial_t\rangle AX.$$

Therefore,

$$\nabla^2 h(X,Y) = g(\nabla_X \nabla h, Y) = \frac{f'(h)}{f(h)}\,[g(X,Y) - dh \otimes dh(X,Y)] + \theta g(AX,Y). \tag{2.9}$$

The result follows by the fundamental equation (1.2). □

We finalize this section by quoting the generalized Hopf's maximum principle due to S.T. Yau. In the following, $L^1(\Sigma^n)$ stands for the space of the Lebesgue integrable functions on Σ^n.

Lemma 2.3 ([25]) *Let (Σ^n, g) be a complete, noncompact Riemannian manifold. If $h : \Sigma^n \to \mathbb{R}$ is a smooth subharmonic function such that $|\nabla h| \in L^1(\Sigma^n)$, then h must be actually harmonic.*

3. Examples and main results

Before presenting the main results, we will exhibit some examples of immersions satisfying the gradient almost Yamabe soliton equation (1.2).

Example 3.1 *Let $(\mathbb{S}^n, g_1)$ the standard sphere immersed into Euclidean space $(\mathbb{R}^{n+1}, g_0)$. According to [3], if we take the height function from the sphere given by*

$$h : \mathbb{S}^n \to \mathbb{R}, \quad x \mapsto g_1(x, \eta_1),$$

where $\eta_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ and $x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$ is the position vector, then $(\mathbb{S}^n, g_1)$ is a gradient almost Yamabe soliton with the height function as the potential and the soliton function given by $\lambda = \frac{1}{n}(\Delta h - scal_{g_1})$.

Example 3.2 *Let $\mathbb{P}^n_\epsilon := \{(x_1, x_2, x_3, \ldots, x_\epsilon, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_\epsilon = 0\}$ the hyperplane isometrically imersed into Euclidean space $(\mathbb{R}^{n+1}, g_0)$. Hence, taking the height function from the hyperplane given by*

$$h : \mathbb{P}^n_\epsilon \to \mathbb{R}, \quad x \mapsto g_0(x, \eta_1),$$

where $\eta_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ and $x \in \mathbb{P}^n$, we deduce that $(\mathbb{P}^n, g_0)$ is a steady gradient almost Yamabe soliton with the height function as the potential function.

Examples 3.1 and 3.2 are totally umbilical hypersurfaces and provide particular instances of the classification of Corollary 3.9.

Example 3.3 *Consider the hyperbolic space $\mathbb{R} \times_{e^t} \mathbb{R}^n$ furnished with the warped product structure. It is well known that the horospheres of the hyperbolic space are totally umbilical hypersurfaces isometric to $\mathbb{R}^n$ and correspond to slices $\{t_0\} \times \mathbb{R}^n$, $t_0 \in \mathbb{R}$. Hence, taking the inclusion $i : \{t_0\} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, we deduce that the height function satisfies $h(x) = t_0$, and then the standard Euclidean space $\{t_0\} \times \mathbb{R}^n$ is a trivial gradient almost Yamabe soliton with potential $h(x) = t_0$.*

The above example allows us to conclude, in a broad sense, that for each fixed number $t_0 \in I$, the inclusion $i : \{t_0\} \times M^n \to I \times M^n$ produces a constant height function $h(x) = t_0$. Hence, $\{t_0\} \times M^n$ is a trivial gradient almost Yamabe soliton with potential $h(x) = t_0$. This observation allows us to produce infinitely many examples of gradient almost Yamabe solitons. More precisely, we have the following example.

Example 3.4 *Every manifold $\Sigma^n \subset M^n$, isometrically included into the warped product manifold $I \times_f M^n$ is a trivial gradient almost Yamabe soliton with potential $h = (\pi_I)|_\Sigma = const.$ and scalar curvature $scal_g = \lambda$.*

The next example deals with a rotationally symmetric gradient almost Yamabe soliton with a constant angle (see Figure 1).

Example 3.5 *Let $\psi : \Sigma^2 = (0, \infty) \times (0, 2\pi) \to \mathbb{R} \times_{e^t} \mathbb{R}^2$ be an isometric immersion given by:*

$$\psi(u, v) = (u\sqrt{1-\theta^2}, -\frac{\theta}{\sqrt{1-\theta^2}} e^{-u\sqrt{1-\theta^2}} \cos v, -\frac{\theta}{\sqrt{1-\theta^2}} e^{-u\sqrt{1-\theta^2}} \sin v), \quad \theta \in (0,1),$$

then, Σ^2 is a gradient almost Yamabe soliton with potential $h(u,v) = u\sqrt{1-\theta^2}$ and soliton function $\lambda = scal_g$ (see Section 4).

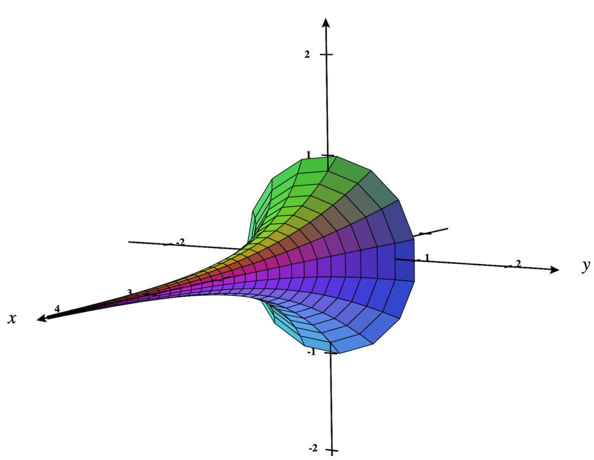

Figure 1. Rotational soliton immersed into hyperbolic space with $\theta = \frac{\sqrt{2}}{2}$.

Initially, we focus our attention on compact gradient almost Yamabe soliton immersions $\psi : \Sigma^n \to I \times_f M^n$. It has been known that every compact gradient Yamabe soliton is of constant scalar curvature, hence trivial since h is harmonic (cf. [11, 19, 24]). For gradient almost Yamabe solitons, the previous result was generalized by Barbosa and Ribeiro [3], where the authors proved that any compact gradient almost Yamabe soliton (M^n, g, h, λ) satisfying

$$\int_M g(\nabla\lambda, \nabla h)dv_g \geq 0, \tag{3.1}$$

is trivial. Our first result drops the condition (3.1) in favor of a hypothesis about the geometry of $I \times_f M^n$ and produces the following result.

Theorem 3.6 *Let $(\Sigma^n, g, h = (\pi_I)|_\Sigma, \lambda)$ be a compact gradient almost Yamabe soliton immersed into $I \times_f M^n$. If the mean curvature of Σ^n satisfies $|H| \leq (\log f)'(h)$, then Σ^n is trivial.*

Proof Taking the trace in (2.9), we deduce

$$\Delta h = \frac{f'(h)}{f(h)}\left(n - |\nabla h|^2\right) + n\theta H.$$

Hence,

$$\Delta h + \langle \nabla \log f(h), \nabla h\rangle = n\left(\frac{f'(h)}{f(h)} + \theta H\right) \geq n\left(\frac{f'(h)}{f(h)} - |\theta||H|\right) \geq n\left(\frac{f'(h)}{f(h)} - |H|\right) \geq 0. \tag{3.2}$$

It follows from the maximum principle, see page 35 of [15], that h is constant. Therefore, Σ^n is a trivial gradient almost Yamabe soliton. □

We pointed out that each leaf $\Sigma_t = \{t\} \times M^n$ of the foliation $t \to \Sigma_t$ of $\overline{M}^{n+1}$ is a totally umbilical hypersurface with constant mean curvature

$$H = (\log f)'(h),$$

with respect to $-\partial t$. From Example 3.4, each Σ_t is a trivial almost Yamabe soliton. This implies that the inequality in Theorem 3.6 is optimal.

In the particular case in which the ambient space is a space form, we obtain the following corollary.

Corollary 3.7 *Let $(\Sigma^n, g, h = (\pi_I)|_\Sigma, \lambda)$ be a compact gradient almost Yamabe soliton immersed into $I \times_f M^n$. Then, the following statements hold:*

(a) *If $I \times_f M^n$ is the Euclidean sphere $(0, \pi) \times_{\sin t} \mathbb{S}^n$ and the mean curvature of Σ^n satisfies:*

$$|H| \leq \cot(h),$$

then Σ^n is trivial.

(b) *If $I \times_f M^n$ is the hyperbolic space $(0, +\infty) \times_{\sinh t} \mathbb{S}^n$ and the mean curvature of Σ^n satisfies:*

$$|H| \leq \coth(h),$$

then Σ^n is trivial.

(c) *If $I \times_f M^n$ is the Euclidean space $(0,+\infty)\times_t \mathbb{S}^n$ and the mean curvature of Σ^n satisfies:*

$$|H| \leq h^{-1},$$

then Σ^n is trivial.

Recently, B. Chen and Deshmukh [7] study Yamabe solitons immersed on Euclidean space $\mathbb{R}^n$ whose soliton vector field is the tangent component from the position vector field and proved that Yamabe solitons hypersurfaces are totally umbilical. Seko and Maeta [22] improved on this result by showing that it may be extended to the class of almost Yamabe solitons. In our context, we give the following result for hypersurfaces immersed into Riemann product manifolds.

Theorem 3.8 *Let $(\Sigma^n, g, h = (\pi_I)|_\Sigma, \lambda)$ be a gradient almost Yamabe soliton immersed into a Riemannian product manifold $I \times M^n$. If the angle function θ does not change sign, then Σ^n is a totally umbilical hypersurface.*

Proof First, let us consider a local orthonormal frame $\{E_i\}_{i=1}^n$ of $\mathfrak{X}(\Sigma^n)$ associated with the Weingarten operator, i.e., $A(E_i) = \lambda_i E_i$, where $\{\lambda_i\}_{i=1}^n$ are the principal curvatures of Σ^n. Since the warping function f is constant, we deduce from Lemma 2.2 that

$$g(AE_i, E_j) = \lambda_i g(E_i, E_j) = \theta^{-1}(scal_g - \lambda) g(E_i, E_j), \qquad i, j = 1, \ldots, n,$$

which implies that

$$\lambda_i = \theta^{-1}(scal_g - \lambda), \qquad i = 1, \ldots, n.$$

Therefore, Σ^n is totally umbilical with mean curvature $H = \theta^{-1}(scal_g - \lambda)$. □

In the particular case in which $I \times M^n = \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$, we obtain the following classification.

Corollary 3.9 *Let $(\Sigma^n, g, h = (\pi_I)|_\Sigma, \lambda)$ be a gradient almost Yamabe soliton immersed into the Euclidean space $\mathbb{R}^{n+1}$. If θ does not change sign, then Σ^n is contained in a hypersphere or a hyperplane.*

In order to investigate minimal gradient almost Yamabe solitons, we prove the following.

Theorem 3.10 *Let $(\Sigma^n, g, h = (\pi_I)|_\Sigma, \lambda)$ be a minimal gradient almost Yamabe soliton immersed into $I \times_f M^n$ with $f'(h) \geq 0$, then the scalar curvature of Σ^n satisfies $scal_g \geq \lambda$. Moreover, if h reaches the maximum, then $scal_g \equiv \lambda$, $f'(h) = 0$ and Σ^n is a slice of $I \times M^n$.*

Proof Since $(\Sigma^n, g, h, \lambda)$ is a minimal gradient almost Yamabe soliton, we deduce from the trace of (2.8) in Lemma 2.2 that

$$(scal_g - \lambda)n = \frac{f'(h)}{f(h)}\left(n - |\nabla h|^2\right).$$

From $|\nabla h|^2 = 1 - \theta^2$ and by the hypothesis $f'(h) \geq 0$, we derive

$$(scal_g - \lambda)n = \frac{f'(h)}{f(h)}\left(n - |\nabla h|^2\right) = \frac{f'(h)}{f(h)}\left(n - 1 + \theta^2\right) \geq 0, \tag{3.3}$$

which proves that $scal_g \geq \lambda$. On the other hand, it follows from equation (1.2) that

$$\Delta h = n(scal_g - \lambda) \geq 0. \tag{3.4}$$

Now, assume that h attains its maximum h_0 in the point $x_0 \in \Sigma^n$ and define

$$\Omega_0 := \{x \in \Sigma^n \ ; \ h(x) = h_0\}.$$

since $x_0 \in \Omega_0$, it must be closed and non-empty. Let now $y \in \Omega_0$, then applying the maximum principle (see [15] p. 35) to (3.4) we obtain that, $h(x) = h_0$ in a neighborhood of y so that Ω_0 is open. The connectedness of Σ^n yields $\Omega_0 = \Sigma$. Hence h is constant, which implies that Σ^n is a slice, $scal_g = \lambda$ and $f'(h) = 0$. □

As a consequence of Theorem 3.10, we derive a condition for the nonexistence of minimal immersion of gradient almost Yamabe solitons into the hyperbolic space $\mathbb{R} \times_{e^t} \mathbb{R}^n$ or the Euclidean space $(0, \infty) \times_t \mathbb{S}^n$. More precisely, we derive the following corollary.

Corollary 3.11 *Let $(\Sigma^n, g, h = (\pi_I)|_\Sigma, \lambda)$ be a gradient almost Yamabe soliton immersed into $I \times_f M^n$. Then the following statements hold.*

(a) *If $I \times_f M^n = \mathbb{R} \times_{e^t} \mathbb{R}^n$ and $\lambda > -n(n-1) - |A|^2$, then ψ cannot be minimal.*

(b) *If $I \times_f M^n = (0, \infty) \times_t \mathbb{S}^n$ and $\lambda > -|A|^2$, then ψ cannot be minimal.*

As another application of Theorem 3.10, we also get

Corollary 3.12 *Let $(\Sigma^n, g, h = (\pi_I)|_\Sigma, \lambda)$ be a minimal gradient almost Yamabe soliton immersed into the Riemannian product manifold $I \times M^n$. If h reaches its maximum, then (Σ^n, g) is isometric to M^n.*

Remark 3.13 *Corollary 3.12, reveals that there does not exist a compact minimal gradient almost Yamabe soliton Σ^n immersed into the Riemannian product manifold $I \times M^n$ with noncompact M^n.*

The next result extends for gradient almost Yamabe solitons the result proved by Barros, Gomes and Ribeiro Jr. [4] (see Theorem 1.5).

Theorem 3.14 *Let $(\Sigma^n, g, h, \lambda)$ be a gradient almost Yamabe soliton immersed into $I \times_f M^n$ whose fiber M^n has sectional curvature $k_M \leq \inf_I (f'^2 - ff'')$.*

(a) *If $|\nabla h| \in L^1(\Sigma^n)$ and the soliton function satisfies*

$$\lambda \geq -n(n-1)\frac{f''(h)}{f(h)} + n^2 H^2,$$

then Σ^n is totally geodesic, with scalar curvature $scal_g = -n(n-1)\frac{f''}{f}$ and $k_M = f'^2 - ff''$.

(b) *If $|\nabla h| \in L^1(\Sigma^n)$ and the soliton function satisfies*

$$\lambda \geq n(n-1)\left(H^2 - \frac{f''(h)}{f(h)}\right),$$

then Σ^n is totally umbilical, with scalar curvature $scal_g = n(n-1)(H^2 - \frac{f''}{f})$ and $k_M = f'^2 - ff''$.

Remark 3.15 *The curvature assumption $k_M \leq \inf_I(f'^2 - ff'')$ in Theorem 3.14 is motivated by the ODE (2.7). As may be seen, $k_M \leq \inf_I(f'^2 - ff'')$ naturally holds on ambient manifolds $I \times_f M^n$ with constant sectional curvature (see Example 2.1).*

Proof First, note that our hypothesis under the sectional curvature k_M jointly with (2.6) implies that

$$\begin{aligned}
scal_g \leq\, & \frac{\inf_I(f'^2 - ff'')}{f^2}(n-1)\big(n - 2|\nabla h|^2\big) + n[(\log f)'(h)]^2 \left(|\nabla h|^2 - (n-1)\right) \\
& - (n-2)(\log f)''(h)|\nabla h|^2 - n\frac{f''}{f}|\nabla h|^2 + n^2H^2 - |A|^2 \\
\leq\, & -(n-1)(\log f)''(h)\big(n - 2|\nabla h|^2\big) + n[(\log f)'(h)]^2 \left(|\nabla h|^2 - (n-1)\right) \qquad (3.5) \\
& - (n-2)(\log f)''(h)|\nabla h|^2 - n\frac{f''}{f}|\nabla h|^2 + n^2H^2 - |A|^2 \\
\leq\, & -n(n-1)\frac{f''(h)}{f(h)} + n^2H^2 - |A|^2.
\end{aligned}$$

Hence, combining our assumption on the soliton function λ with the inequality (3.5), we arrive at

$$\Delta h = n(scal_g - \lambda) = n\left(-n(n-1)\frac{f''(h)}{f(h)} + n^2H^2 - \lambda - |A|^2\right) \leq 0. \tag{3.6}$$

Now, from Lemma 2.3, we derive that h is harmonic, and then from (3.6), Σ^n must be totally geodesic with $scal_g = \lambda = -n(n-1)\frac{f''}{f}$. On the other hand, from $scal_g = \lambda$, we get that $k_M = f'^2 - ff''$.

For the second assertion, note that the traceless second fundamental form of Σ^n, namely, $\Phi = A - HI$, satisfies $|\Phi|^2 = tr(\Phi^2) = |A|^2 - nH^2 \geq 0$ and equality holds if and only if, Σ^n is totally umbilical. So, from the hypothesis on λ and equation (3.6), it yields

$$\Delta h = n(scal_g - \lambda) = n\left[n(n-1)\left(-\frac{f''(h)}{f(h)} + H^2\right) - \lambda - |\Phi|^2\right] \leq 0. \tag{3.7}$$

Hence, again from Lemma 2.3, we deduce that $scal_g = \lambda = n(n-1)\left(H^2 - \frac{f''}{f}\right)$ and $|\Phi|^2 = 0$, which gives that Σ^n is totally umbilical. On the other hand, from $scal_g = \lambda$, we get that $k_M = f'^2 - ff''$. □

Proceeding, it is a well-known fact that any compact gradient almost Yamabe soliton with constant scalar curvature is isometric to the Euclidean sphere $\mathbb{S}^n$ (see [3]). From this fact, we derive the following rigidity result.

Theorem 3.16 *Let $(\Sigma^n, g, h, \lambda)$ be a compact gradient almost Yamabe soliton immersed into a space form $\overline{M}^{n+1}(c)$ of curvature c. If $\lambda \geq n(n-1)c +^2 |H|^2$, then (Σ^n, g) is isometric to the Euclidean sphere $(\mathbb{S}^n, g_1)$.*

Proof Since a warped product $I \times_f M^n$ of constant curvature c trivially fulfills the condition $k_M = \inf_I(f'^2 - ff'')$, we obtain in a similar way as in the demonstration of Theorem 3.14 that

$$\Delta h = n(scal_g - \lambda) = n\left(n(n-1)c + n^2H^2 - \lambda - |A|^2\right) \leq 0, \tag{3.8}$$

which implies that $-h$ is subharmonic, and from the maximum principle, h must be constant. Hence, from (3.8), we derive that $|A|^2 = 0$ and $scal_g = n(n-1)c$, so the result follows from Theorem 1.5 of [3]. □

A result similar to Theorem 3.16 appears in [9] (Theorem 3 (ii)), however Theorem 3.16 shows us that we can remove the assumption $|\nabla h| \in L^1(\Sigma^n)$ to get the same conclusion.

Remark 3.17 *We remark that Theorem 3.14 and Theorem 3.16 are obtained in the general case without the assumption $h = (\pi_I)|_\Sigma$. However, upon assuming this condition, we obtain from item (a) of Theorem 3.14 jointly with Lemma 2.2 that either Σ^n is a slice, or f is a constant and Σ^n is a totally geodesic hypersurface into a product manifold of zero sectional curvature.*

For item (b) of Theorem 3.14, we deduce $\nabla^2 h = (scal_g - \lambda)g = 0$, then either h is constant, which implies that Σ^n is trivial, or $|\nabla h| \neq 0$ and Σ^n splits along the gradient of h. In the last case, from Lemma 2.2, we get

$$\frac{f'(h)}{f(h)} dh \otimes dh = \left(\frac{f'(h)}{f(h)} + \theta H\right) g.$$

Hence, from Lemma 1 of [5], we obtain $\log f(h)'|\nabla h|^2 = 0$, which implies that f is a constant. So, Σ^n is a totally umbilical hypersurface into a Riemannian product manifold of zero sectional curvature. Finally, Theorem 3.16 in the particular case $h = (\pi_I)|_\Sigma$ remains the same.

4. Classification of rotational gradient almost Yamabe solitons

In this section, we present a classification of rotational gradient almost Yamabe solitons immersed into $\mathbb{R} \times_f \mathbb{R}^n$ with potential $h := (\pi_I)|_\Sigma$ and constant angle $\theta \in (0, 1)$. Following Dajczer and do Carmo [14], we shall use the terminology of rotational hypersurface in $\mathbb{R} \times_f \mathbb{R}^n$ as a hypersurface invariant by the orthogonal group $O(n)$ seen as a subgroup of the isometries group of $\mathbb{R} \times_f \mathbb{R}^n$.

Initially, consider the coordinates $(t, x_1, \ldots, x_n)$, as well as the standard orthonormal basis $\{\eta_1, \ldots, \eta_{n+1}\}$ of $\mathbb{R} \times_f \mathbb{R}^n$. Then, up to isometry, we can assume the rotation axis to be η_1. Consider an arc length parametrized curve in the tx_n plane given by

$$\gamma\colon (t_0, t_1) \longrightarrow \mathbb{R} \times_f \mathbb{R}^n \tag{4.1}$$

$$u \longmapsto (\alpha(u), 0, \ldots, 0, \beta(u)). \tag{4.2}$$

Rotating this curve around the t-axis we obtain a *rotational hypersurface* in $\mathbb{R} \times_f \mathbb{R}^n$. Now, in order to obtain a parametrization of a rotational hypersurface, consider the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n = \operatorname{span}\{\eta_2, \ldots, \eta_{n+1}\}$ with orthogonal parametrization given by

$$X_1 = \cos v_1, \quad X_2 = \sin v_1 \cos v_2, \quad X_3 = \sin v_1 \sin v_2 \cos v_3, \quad \ldots$$
$$X_{n-1} = \sin v_1 \sin v_2 \ldots \sin v_{n-2} \cos v_{n-1}, X_n = \sin v_1 \sin v_2 \ldots \sin v_{n-2} \sin v_{n-1}.$$

Therefore, a parametrization of a rotational hypersurface Σ^n with radial axis η_1 into $\mathbb{R} \times_f \mathbb{R}^n$ is given by

$$\begin{aligned} \psi\colon (t_0, t_1) \times (0, 2\pi)^{n-1} &\to \mathbb{R} \times_f \mathbb{R}^n \\ (u, v_1, \ldots, v_{n-1}) &\longmapsto \alpha(u)\eta_1 + \beta(u) X(v_1, \ldots, v_{n-1}), \end{aligned} \tag{4.3}$$

where

$$X(v_1, \ldots, v_{n-1}) = (0, X_1(v_1, \ldots, v_{n-1}), \ldots, X_n(v_1, \ldots, v_{n-1})).$$

In this setting, we provide the following classification.

Theorem 4.1 *Let $\psi : \Sigma^n \to \mathbb{R}\times_f \mathbb{R}^n$ be a rotational gradient almost Yamabe soliton hypersurface with constant angle $\theta \in (0,1)$. Then, up to constants, ψ takes the following form:*

$$\psi(u, v_1, \dots, v_{n-1}) = u\sqrt{1-\theta^2}\eta_1 + \left(\frac{\theta}{1-\theta^2}\int^{u\sqrt{1-\theta^2}} \frac{ds}{f(s)}\right) X(v_1, \dots, v_{n-1}), \quad f(t) = e^t,$$

where $\eta_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$, $-\infty < u < \infty$, $0 < v_1, \dots, v_{n-1} < 2\pi$ and X is a sphere parametrization.

Proof Since $\psi : \Sigma^n \to \mathbb{R}\times_f \mathbb{R}^n$ is a rotational hypersurface, we deduce from (4.3) that

$$\begin{aligned} \psi_u &= \alpha'(u)\eta_1 + \beta'(u)X, \\ \psi_{v_i} &= \beta(u)X_{v_i}, \quad 1 \le i \le n-1, \end{aligned} \tag{4.4}$$

and then, the first fundamental form of Σ^n takes the form

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & f(\alpha(u))^2\beta(u)^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(\alpha(u))^2\beta(u)^2 \end{bmatrix}. \tag{4.5}$$

The first fundamental equation (4.5) reveals that the induced metric on Σ^n can be expressed by the warped product metric $g = du^2 + \sigma(u)^2 dv^2$ where $\sigma(u) = f(\alpha(u))\beta(u)$. In this case, it follows from the Levi-Civita connection on the warped product metric that:

$$\begin{aligned} \nabla_{\psi_u}\psi_u &= 0, \\ \nabla_{\psi_u}\psi_{v_i} = \nabla_{\psi_{v_i}}\psi_u &= \frac{\sigma_u}{\sigma}\psi_{v_i}, \\ \nabla_{\psi_{v_i}}\psi_{v_j} &= \psi_{v_iv_j} - \sigma\sigma_u\delta_{ij}\psi_u. \end{aligned} \tag{4.6}$$

From the tangent components (4.4), we easily derive the following unit normal vector field for Σ^n

$$N = f(\alpha(u))\beta'(u)\eta_1 - \frac{\alpha'(u)}{f(\alpha(u))}X(v_1, \dots, v_{n-1}).$$

Hence, the hypersurface Σ^n determines a constant angle hypersurface with constant angle θ if, and only if,

$$\theta = \langle \partial_t, N\rangle = f(\alpha(u))\beta'(u) = \text{constant}. \tag{4.7}$$

Combining the unit condition for the rotational curve $\gamma(u) = (\alpha(u), 0, \dots, 0, \beta(u))$, i.e.,

$$\alpha'(u)^2 + f(\alpha(u))^2\beta'(u)^2 = 1,$$

and (4.7), we deduce $\alpha'(u) = \sqrt{1-\theta^2}$, whose general solution is given by

$$\alpha(u) = u\sqrt{1-\theta^2} + c_1, \quad c_1 \in \mathbb{R}. \tag{4.8}$$

Then, replacing equation (4.8) into (4.7) and solving in u, we derive the following expression

$$\beta(u) = \int^u \frac{\theta}{f(s\sqrt{1-\theta^2}+c_1)}ds + c_2 = \frac{\theta}{\sqrt{1-\theta^2}}\int^{u\sqrt{1-\theta^2}+c_1}\frac{ds}{f(s)} + c_2, \quad c_2 \in \mathbb{R}. \tag{4.9}$$

Therefore, the rotational hypersurface takes the following form

$$\psi = (u\sqrt{1-\theta^2}+c_1)\eta_1 + \left(\frac{\theta}{\sqrt{1-\theta^2}}\int^{u\sqrt{1-\theta^2}+c_1}\frac{ds}{f(s)} + c_2\right)X(v_1,\dots,v_{n-1}). \tag{4.10}$$

Now, in order to compute the Weingarten operator A_N, let us consider the following decomposition

$$\partial_t = \sqrt{1-\theta^2}\psi_u + \theta N. \tag{4.11}$$

Taking the covariant derivative of (4.11) with respect ψ_{v_i} and considering that the angle θ is constant, as well as the properties of the Levi-Civita connection of $\mathbb{R}\times_f\mathbb{R}^n$ (Proposition 7.35 in [20]), we deduce that

$$\nabla_{\psi_{v_i}}\psi_u = \frac{\theta}{\sqrt{1-\theta^2}}A_N\psi_{v_i} + \frac{1}{\sqrt{1-\theta^2}}\frac{f'(\alpha(u))}{f(\alpha(u))}\psi_{v_i}, \qquad \forall i \in \{1,\dots,n-1\}. \tag{4.12}$$

Combining (4.6) and (4.12), yields

$$\sqrt{1-\theta^2}\frac{\sigma_u}{\sigma}\psi_{v_i} = \theta A_N\psi_{v_i} + \frac{f'(\alpha(u))}{f(\alpha(u))}\psi_{v_i}, \tag{4.13}$$

and therefore, from the expression of σ, we obtain that ψ_{v_i} is an eigenvector for A_N and satisfies

$$A_N\psi_{v_i} = \left(\frac{\sqrt{1-\theta^2}}{\sigma} - \frac{f'(\alpha(u))}{f(\alpha(u))}\theta\right)\psi_{v_i}. \tag{4.14}$$

On the other hand, taking the covariant derivative of (4.11) with respect $X \in \mathfrak{X}(\Sigma^n)$ and using the Gauss-Weingarten formulas (2.1), we deduce the following implications

$$\begin{aligned}\overline{\nabla}_X\partial_t &= \sqrt{1-\theta^2}\overline{\nabla}_X\psi_u + \theta\overline{\nabla}_X N\\ &= \sqrt{1-\theta^2}\nabla_X\psi_u + \sqrt{1-\theta^2}g(A_N\psi_u,X)N - \theta A_N X,\end{aligned}$$

and, again from the properties of the Levi-Civita connection of $\mathbb{R}\times_f\mathbb{R}^n$ [20], it follows

$$\frac{f'(\alpha(u))}{f(\alpha(u))}\left(X - \sqrt{1-\theta^2}g(X,\psi_u)\partial_t\right) = \sqrt{1-\theta^2}\nabla_X\psi_u + \sqrt{1-\theta^2}g(A_N\psi_u,X)N - \theta A_N X. \tag{4.15}$$

Comparing the tangent and the normal parts of (4.15), one gets that ψ_u is an eigenvector for A_N and satisfies

$$A_N\psi_u = -\frac{f'(\alpha(u))}{f(\alpha(u))}\theta\psi_u. \tag{4.16}$$

Therefore, from (4.14) and (4.16), we conclude that $\{\psi_u, \psi_{v_1}, \dots, \psi_{v_{n-1}}\}$ form an orthogonal basis of A_N and its expression on that basis takes the form

$$A_N = \begin{bmatrix} -\frac{f'(\alpha(u))}{f(\alpha(u))}\theta & 0 & \dots & 0 \\ 0 & \frac{\sqrt{1-\theta^2}}{\sigma} - \frac{f'(\alpha(u))}{f(\alpha(u))}\theta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\sqrt{1-\theta^2}}{\sigma} - \frac{f'(\alpha(u))}{f(\alpha(u))}\theta \end{bmatrix}. \tag{4.17}$$

Now, since we are assuming that $(\Sigma^n, g, h, \lambda)$ is a gradient almost Yamabe soliton, we obtain from Lemma 2.2 that

$$(scal_g - \lambda)g(X,Y) = \frac{f'(h)}{f(h)}\left[g(X,Y) - dh \otimes dh(X,Y)\right] + \theta g(AX,Y), \qquad \forall X, Y \in \mathfrak{X}(\Sigma^n). \tag{4.18}$$

Notice that, in particular cases $X = \psi_u$, $Y = \psi_{v_i}$ and $X = \psi_{v_i}$, $Y = \psi_{v_j}$, $i \neq j$, the orthogonality of X, Y and the expression for the height function

$$h(u, v_1, \dots, v_n) = (\pi_{\mathbb{R}})|_{\Sigma^n}(u, v_1, \dots, v_n) = u\sqrt{1-\theta^2} + c_1, \quad c_1 \in \mathbb{R}, \tag{4.19}$$

implies that equation (4.18) is trivially satisfied. Hence, we need to look at equation (4.18) for a pair of fields $X = Y = \psi_u$ and $X = Y = \psi_{v_i}$.

For $X = Y = \psi_u$, we obtain

$$\begin{aligned} (scal_g - \lambda)g(\psi_u, \psi_u) &= \frac{f'(h)}{f(h)}\left[g(\psi_u, \psi_u) - dh \otimes dh(\psi_u, \psi_u)\right] + \theta g(A_N\psi_u, \psi_u) \\ &= \frac{f'(h)}{f(h)}\left[1 - (1-\theta^2)\right] - \frac{f'(h)}{f(h)}\theta^2 \\ &= 0. \end{aligned}$$

which implies that $scal_g = \lambda$.

Now, for $X = Y = \psi_{v_i}$, with $1 \le i \le n-1$, we get

$$\begin{aligned} (scal_g - \lambda)g(\psi_{v_i}, \psi_{v_i}) &= \frac{f'(h)}{f(h)}\left[g(\psi_{v_i}, \psi_{v_i}) - dh \otimes dh(\psi_{v_i}, \psi_{v_i})\right] + \theta g(A_N\psi_{v_i}, \psi_{v_i}) \\ &= \left(\frac{f'(h)}{f(h)}(1-\theta^2) + \frac{\theta\sqrt{1-\theta^2}}{\sigma}\right) g(\psi_{v_i}, \psi_{v_i}). \end{aligned}$$

Hence, since $scal_g = \lambda$, we obtain from above that

$$\frac{f'(h)}{f(h)}(1-\theta^2) + \frac{\theta\sqrt{1-\theta^2}}{\sigma} = 0, \tag{4.20}$$

and then, taking into account equation (4.13) and (4.20), it easily follows

$$\frac{\sigma_u}{\sigma}\psi_{v_i} = \frac{1}{\sqrt{1-\theta^2}}\left[\theta A_N \psi_{v_i} + \frac{f'(h)}{f(h)}\psi_{v_i}\right] = 0, \tag{4.21}$$

which implies that σ is constant. Therefore, from

$$\frac{f'(h)}{f(h)}(1-\theta^2) + \frac{\theta\sqrt{1-\theta^2}}{\sigma} = 0,$$

we deduce that

$$\frac{f'(h)}{f(h)} = \text{constant}.$$

And thus, $f(t) = c_3 e^{c_5 t}$, c_3, $c_4 \in \mathbb{R}$. Bringing together the equations (4.10), (4.19) and the expression for f we obtain the desired result. $\square$

Conflict of interest

The authors declare that there is no conflict of interest.

References

[1] Alías LJ, Dajczer M. Constant mean curvature hypersurfaces in warped product spaces. Proceedings of the Edinburgh Mathematical Society 2007; 50 (3) 511-526. https://doi.org/10.1017/S0013091505001069

[2] Aquino C, de Lima H, Gomes J. Characterizations of immersed gradient almost Ricci solitons. Pacific Journal of Mathematics 2017; 288 (2): 289-305. https://doi.org/10.2140/pjm.2017.288.289

[3] Barbosa E, Ribeiro E. On conformal solutions of the Yamabe flow. Archiv der Mathematik 2013; 101 (1): 79-89. https://doi.org/10.1007/s00013-013-0533-0

[4] Barros A, Gomes J, Ribeiro E. Immersion of almost Ricci solitons into a Riemannian manifold. Matematica Contemporanea 2011; 91-102. http://doi.org/10.21711/231766362011/rmc404

[5] Barros A, Ribeiro E. Characterizations and integral formulae for generalized m-quasi-Einstein metrics. Bulletin of the Brazilian Mathematical Society. New Series 2014; 45 (2): 325-341.https://doi.org/10.1007/s00574-014-0051-0

[6] Caminha A, de Lima HF. Complete vertical graphs with constant mean curvature in semi-Riemannian warped products. Bulletin of the Belgian Mathematical Society-Simon Stevin 2009; 16 (1): 91-105. https://doi.org/10.36045/bbms/1235574194

[7] Chen BY, Deshmukh S. Yamabe and quasi-Yamabe solitons on Euclidean submanifolds. Mediterranean Journal of Mathematics 2018; 15 (5) 194. https://doi.org/10.1007/s00009-018-1237-2

[8] Colares AG, De Lima HF. Some rigidity theorems in semi-Riemannian warped products. Kodai Mathematical Journal 2012: 35 (2) 268-282. https://doi.org/10.2996/kmj/1341401051

[9] Cunha AW, de Lima EL. Immersions of r-almost Yamabe solitons into Riemannian manifolds. Manuscripta mathematica 2022; 1-12. https://doi.org/10.1007/s00229-021-01343-1

[10] Cunha AW, de Lima EL, de Lima HF. r-almost Newton-Ricci solitons immersed into a Riemannian manifold. Journal of Mathematical Analysis and Applications 2018; 464 (1): 546-556. https://doi.org/10.1016/j.jmaa.2018.04.026

[11] Daskalopoulos P, Sesum N. The classification of locally conformally flat Yamabe solitons. Advances in Mathematics 2013; 240: 346-369. https://doi.org/10.1016/j.aim.2013.03.011

[12] de Lima EL, de Lima HF, Characterizations of minimal hypersurfaces immersed in certain warped products, Extracta mathematicae 2019; 34 (1): 123-134. https://doi:10.17398/2605-5686.34.1.123

[13] Dillen F, Munteanu MI, Van der Veken J, Vrancken L. Classification of constant angle surfaces in a warped product. Balkan Journal of Geometry and Its Applications 2011: 16 (2) 35-47.

[14] do Carmo M, Dajczer M. Rotation hypersurfaces in spaces of constant curvature. Manfredo P. do Carmo-Selected Papers, Springer 2012; pp. https://doi.org/10.1007/978-3-642-25588-5-17

[15] Gilbarg D, Trudinger NS. Elliptic partial differential equations of second order, springer, 2015.

[16] Ghosh A. Ricci almost soliton and almost Yamabe soliton on Kenmotsu manifold. Asian-European Journal of Mathematics 2020; 2150130. https://doi.org/10.1142/S1793557121501308

[17] Hamilton, R.S. The Ricci Flow on Surfaces. Contemporary Mathematics 1988; 71: 237-261. https://doi.org/10.1090/conm/071/954419

[18] Hamilton RS. Lectures on geometric flows, Unpublished manuscript 1989.

[19] Hsu SY. A note on compact gradient Yamabe solitons. Journal of Mathematical Analysis and Applications 2012; 388 (2): 725-726. https://doi.org/10.1016/j.jmaa.2011.09.062

[20] O'neill B. Semi-Riemannian geometry with applications to relativity, Vol. 03, Academic press, 1983.

[21] Pirhadi V, Razavi A. On the almost quasi-Yamabe solitons. International Journal of Geometric Methods in Modern Physics 2017; 14 (11): 1750161. https://doi.org/10.1142/S0219887817501614

[22] Seko T, Maeta S. Classification of almost Yamabe solitons in Euclidean spaces. Journal of Geometry and Physics 2019; 136: 97-103.

[23] Tokura W, Adriano L, Pina R, Barboza M. On warped product gradient Yamabe solitons. Journal of Mathematical Analysis and Applications 2019; 473 (1): 201-214. https://doi.org/10.1016/j.jmaa.2018.12.044

[24] Tokura W, Batista E. Triviality results for compact k-Yamabe solitons. Journal of Mathematical Analysis and Applications 2021; 125274. https://doi.org/10.1016/j.jmaa.2021.125274

[25] Yau ST. Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. Indiana University Mathematics Journal 1976; 25 (7): 659–670. http://doi.org/10.1512/iumj.1976.25.25051

[26] Yamabe H. On a deformation of Riemannian structures on compact manifolds . Osaka Mathematical Journal 1960 12 (1); 21-37. https://doi.org/10.18910/8577