Turk J Math
() : –
© TÜBİTAK

On the dual pseudo-spherical elastic curves

Gözde ÖZKAN TÜKEL[1,*], Ahmet YÜCESAN[2]
[1]Department of Basic Sciences, Technology Faculty, Isparta University of Applied Sciences, Isparta, Türkiye,
ORCID iD: https://orcid.org/0000-0003-1800-5718
[2]Department of Mathematics, Faculty of Engineering and Natural Sciences, Süleyman Demirel University, Isparta, Turkiye,
ORCID iD: https://orcid.org/0000-0002-5419-925X

Received: .202 • **Accepted/Published Online:** .202 • **Final Version:** ..202

Abstract: We investigate a dual bending energy functional that operates on the dual pseudo-sphere in dual Lorentzian space. For a non-null dual curve on the dual pseudo-sphere to be considered elastic, it must satisfy the conditions of a dual Euler-Lagrange equation. To solve this problem, we use Jacobi elliptic functions to approach the real part and the integral factors method to solve the dual part. Using E. Study mapping, we examine situations where every timelike or spacelike dual elastic curve on the dual pseudo-sphere matches an elastic strip with a suitable base curve in Minkowski 3-space.

Key words: calculus of variations, dual pseudo-spherical elastic curve, elastic strips, dual pseudo-sphere.

1. Introduction

Dual numbers, consisting of a real and a dual parts, establish a commutative ring with respect to addition and multiplication. A sequence of three dual numbers constitutes what is termed a dual vector. Such vectors define a module known as a dual space, denoted by $\mathbb{D}^3$ within this commutative ring. E. Study's research in line geometry and kinematics heavily relied on the application of dual numbers and dual vectors, emphasizing the representation of oriented lines through the use of dual unit vectors. His findings lead to the creation of the E. Study's theorem, establishing that the points on the dual unit sphere $\mathbb{S}^2$ in $\mathbb{D}^3$ correspond one-to-one with the oriented lines in Euclidean $3-$space $\mathbb{E}^3$; a smooth curve on $\mathbb{S}^2$ represents a ruled surface in $\mathbb{E}^3$, making this a effective subject for investigation (for details, see [6]).

An elastic curve (EC) is a solution of the variational problem that minimizes the bending energy of a thin, non-extensible wire. Mathematically, it is defined as one of the critical points of the total squared curvature functional among the family of regular curves, with the same starting and ending points and tangent vectors at these points [10]. Elastic curves, have recently been characterized in $\mathbb{D}^3$ and on $\mathbb{S}^2$ [14, 21]. One of the primary objectives of these studies is to establish a one-to-one relationship between EC, characterized on $\mathbb{S}^2$, and elastic strips (ES), a special type of ruled surface in $\mathbb{E}^3$. In particular, in [21], the authors have sought to answer the question of what type of ES in $\mathbb{E}^3$ corresponds to the dual spherical EC.

In the context of Minkowski $3-$space $\mathbb{E}^3_1$, rather than the traditional Euclidean $3-$space $\mathbb{E}^3$, E. Study's

*Correspondence: gozdetukel@isparta.edu.tr
2010 *AMS Mathematics Subject Classification:* 53A04, 53A35

mapping may be formulated as follows: The timelike dual unit vector (t.d.u.v.)and spacelike dual unit vector (s.d.u.v.) of dual pseudo-hyperbolic space $\mathbb{H}_0^2$ and dual pseudo-sphere $\mathbb{S}_1^2$ in $\mathbb{D}_1^3$ correspond one-to-one to the directed timelike and spacelike lines in $\mathbb{E}_1^3$, respectively. Then a differentiable curve on $\mathbb{H}_0^2$ relates to a timelike ruled surface in $\mathbb{E}_1^3$. Similarly, the timelike (resp. spacelike) curve on $\mathbb{S}_1^2$ corresponds to any spacelike (resp. timelike) ruled surface in $\mathbb{E}_1^3$ (see, [16, 18]). It is noteworthy that studying ruled surfaces in $\mathbb{E}_1^3$ presents a significantly richer and more complex area of investigation compared to the corresponding study in $\mathbb{E}^3$. For example, ES determined by the stationary point of the Sadowsky functional are determined by two Euler-Lagrange (E-L) equations that complement each other in $\mathbb{E}^3$, while it has provided a rich content to the literature by expressing with the different differential equation systems according to causal character of the base curve (or directrix) of the rectifying strip (RS) in $\mathbb{E}_1^3$ (for example, see [11–13]).

In this work, our primary aim is to seek the solution to a variational problem on $\mathbb{S}_1^2$. For finding solution of the problem, we establish dual E-L equation and we consider the dual and real parts of the dual E-L equation separately. The solution to the real part of the equation is recognized to be achieved through the use of Jacobi elliptic functions (see [7, 20]). We use the integral factor method to solve the dual part of the equation and then combine the results. Finally, we establish a one-to-one relationship between the timelike and spacelike dual EC on $\mathbb{S}_1^2$ and the ES with non-null base curve in $\mathbb{E}_1^3$.

2. Preliminary results

A dual number $\hat{a}$ is written as $\hat{a} = a + \xi a^*$, where ξ is the dual operator with the conditions $\xi^2 = 0$ and $\xi \neq 0$. The collection of dual numbers is represented by $\mathbb{D}$. We have the following operations:

$$\hat{a} + \hat{b} = (a+b) + \xi\left(a^* + b^*\right),$$

$$\hat{a}.\hat{b} = ab + \xi(ab^* + a^*b)$$

and

$$\frac{\hat{a}}{\hat{b}} = \frac{a}{b} + \xi\frac{a^*b - ab^*}{b^2},\ b \neq 0,$$

where $\hat{a} = a + \xi a^*$, $\hat{b} = b + \xi b^*$. $\hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ is known as a dual vector and the entire collection of dual vectors is denoted by

$$\mathbb{D}^3 = \left\{ \hat{u} \middle| \, \hat{u} = (u_1 + \xi u_1^*, u_2 + \xi u_2^*, u_3 + \xi u_3^*) = u + \xi u^*,\ u, u^* \in \mathbb{E}^3 \right\}$$

and known as dual space (see, [8, 17]). Lorentzian inner product and Lorentzian cross product are given by

$$< \hat{u}, \hat{v} > = < u, v > + \xi(< u, v^* > + < u^*, v >)$$

and

$$\hat{u} \times \hat{v} = u \times v + \xi(u \times v^* + u^* \times v),$$

for dual vector $\hat{u}$ and $\hat{v}$. Dual Lorentzian space $\mathbb{D}_1^3$ is the dual space endowed with Lorentzian inner product. The dual vector $\hat{v} = v + \xi v^*$ is called spacelike, timelike or lighlike (null) if the vector v is spacelike, timelike or lightlike (null), respectively. The norm $\|\hat{v}\|$ of $\hat{v}$ is as follows:

$$\|\hat{v}\| = \sqrt{|< \hat{v}, \hat{v} >|} = \|v\| + \xi\frac{< v, v^* >}{\|v\|}, v \neq 0.$$

A dual vector $\hat{v}$ is referred to as a dual unit vector if the norm of $\hat{v}$ equals to 1 (or $1+\xi 0$), i.e., $<v,v>=1$ and $<v,v^{*}>=0$. It follows that $\hat{v}$ is a td.u.v (resp., sd.u.v) if the relations $<v,v>=-1$ (respectively, $<v,v>=1$) and $<v,v^{*}>=0$ hold. The dual pseudo-sphere $\mathbb{S}_1^2$ or (Lorentzian dual unit sphere) and dual pseudo-hyperbolic space (hyperbolic dual unit sphere) are respectively given by

$$\mathbb{S}_1^2=\left\{\hat{v}\in\mathbb{D}_1^3 \middle| <\hat{v},\hat{v}>=1\right\}$$

and

$$\mathbb{H}_0^2=\left\{\hat{v}\in\mathbb{D}_1^3 \middle| <\hat{v},\hat{v}>=-1\right\}.$$

Let $\hat{\gamma}(t)=\gamma(t)+\xi\gamma^{*}(t)$, where $\gamma(t)=(\gamma_1(t),\gamma_2(t),\gamma_3(t))$ and $\gamma^{*}(t)=(\gamma_1^{*}(t),\gamma_2^{*}(t),\gamma_3^{*}(t))$, be a dual curve with parameter $t\in I\subset\mathbb{R}$ in $\mathbb{D}_1^3$. $\gamma(t)$ is defined the (real) indicatrix of $\hat{\gamma}(t)$. If all $\gamma_i(t)$ and $\gamma_i^{*}(t)$, $1\leq i\leq 3$, are smooth, then $\hat{\gamma}(t)$ is smooth in $\mathbb{D}_1^3$. $\hat{\gamma}(t)$ in $\mathbb{D}_1^3$ is referred to spacelike, timelike or lightlike (null) if the real part $\gamma(t)$ of $\hat{\gamma}(t)$ in $\mathbb{E}_1^3$ is spacelike, timelike or lightlike, respectively. The dual arc length of $\hat{\gamma}$ is given by

$$\hat{s}=\int_0^s\left\|\dot{\hat{\gamma}}(t)\right\|dt=\int_0^s\left\|\dot{\gamma}(t)\right\|dt+\xi\int_0^s<T,\dot{\gamma}^{*}(t)>dt=s+\xi s^{*}, \tag{2.1}$$

where s is arc length and T is the unit tangent vector (TV) to γ. Assume that $\hat{\gamma}$ is a reparametrization with s of the indicatrix. Thus,

$$\hat{\gamma}'=\dot{\hat{\gamma}}\frac{ds}{d\hat{s}}=\hat{T}$$

is defined as the dual TV to $\hat{\gamma}(s)$, where $\hat{\gamma}'=\frac{d\hat{\gamma}}{d\hat{s}}$ and $\dot{\hat{\gamma}}=\frac{d\gamma}{ds}$ and we have $\frac{d\hat{s}}{ds}=1+\xi\Delta$ from (2.1), where $\Delta=<T,\dot{\gamma}^{*}(t)>$.

$\left\{\hat{T},\hat{N},\hat{B}\right\}$ is the dual Frenet frame along $\hat{\gamma}$ with derivative equations

$$\frac{d}{d\hat{s}}\begin{pmatrix}\hat{T}\\ \hat{N}\\ \hat{B}\end{pmatrix}=\begin{pmatrix}0 & \hat{\kappa} & 0\\ -\varepsilon_T\varepsilon_N\hat{\kappa} & 0 & \hat{\tau}\\ 0 & -\varepsilon_N\varepsilon_B\hat{\tau} & 0\end{pmatrix}\begin{pmatrix}\hat{T}\\ \hat{N}\\ \hat{B}\end{pmatrix},$$

where $\hat{N}$ is the dual principle normal vector field (PNV), $\hat{B}$ is the dual binormal vector field (BV) of $\hat{\gamma}$ at the point $\hat{\gamma}(s)$, $\hat{\kappa}=\kappa+\xi\kappa^{*}$ and $\hat{\tau}=\tau+\xi\tau^{*}$ are nowhere pure dual curvature and dual torsion functions of $\hat{\gamma}$, $\varepsilon_T=<T,T>=\mp 1$, $\varepsilon_N=<N,N>=\mp 1$ and $\varepsilon_B=<B,B>=\mp 1$ (see for detail, [1, 15, 19]).

Now we recall ruled surface in $\mathbb{E}_1^3$. Let J and I be open intervals containing 0 in the real line $\mathbb{R}$. Let $\alpha(s)$ be a curve on J into $\mathbb{E}_1^3$ and $\beta(s)$ a vector field along $\alpha(s)$ orthogonal to $\alpha'(s)$. A ruled surface M in $\mathbb{E}_1^3$ is a semi-Riemannian surface swept out by the vector field $\beta(s)$ along the curve $\alpha(s)$. Such a surface has the following parametrization form

$$R(s,v)=\alpha(s)+v\beta(s), \tag{2.2}$$

for $s\in J$ and $v\in I$, where $\alpha(s)$ is called a base curve and $\beta(s)$ is called a direction curve. The causal character of the curve $\alpha(s)$ and the vector field $\beta(s)$ are important for determining the type of the ruled surface parametrized by $R(s,v)$. $R(s,v)$ is called as a spacelike ruled surface if $\alpha(s)$ is a spacelike curve and $\beta(s)$ is spacelike vector field. (2.2) is called as timelike ruled surface if $\alpha(s)$ is spacelike curve and $\beta(s)$ timelike vector field or $\alpha(s)$ is timelike curve and $\beta(s)$ spacelike vector field (see [3], for detail description)

The binormal surface (BS), which is a special ruled surface that has an important place in the future parts of our paper, is also defined as follows: Let $\alpha(s)$ be a non-null curve in $\mathbb{E}_1^3$ with the arc length parameter s and the Frenet frame $\{T(s), N(s), B(s)\}$. Then the ruled surface

$$R(s,v) = \alpha(s) + vB(s) \tag{2.3}$$

is defined as BS of the non-null curve $\alpha(s)$ [5].

E. Study's mapping allows us to rewrite a dual curve $\hat{\gamma}(s) = \gamma(s) + \xi\gamma^*(s)$ as a ruled surface Eq. (2.2) in the following form:

$$R(s,v) = \gamma(s) \times \gamma^*(s) + v\gamma(s) \tag{2.4}$$

[16, 18].

3. Setting of the E-L equation

Let $\hat{\gamma}$ be a non-null dual pseudo-spherical curve, that is, a non-null dual curve on $\mathbb{S}_1^2$. Suppose that $\hat{\gamma}$ is a reparametrization curve with the parametrization s of the indicatrix, $\hat{T}$ is called the dual TV to $\hat{\gamma}$ and $\hat{g} = \varepsilon_g \hat{\gamma} \times \hat{T}$ at the point $\hat{\gamma}(s)$, where $\varepsilon_g =< g, g >= \pm 1$ such that $\hat{g} = g + \varepsilon g^*$. Since $\hat{\gamma}$ is a non-null dual pseudo-spherical curve, we know that $\varepsilon_g = -\varepsilon_T$. Thus, we get the orthonormal frame $\{\hat{\gamma}, \hat{T}, \hat{g}\}$ with the following fundamental relations

$$\frac{d}{d\hat{s}} \begin{pmatrix} \hat{\gamma} \\ \hat{T} \\ \hat{g} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\varepsilon_T & 0 & \hat{\kappa}_g \\ 0 & \hat{\kappa}_g & 0 \end{pmatrix} \begin{pmatrix} \hat{\gamma} \\ \hat{T} \\ \hat{g} \end{pmatrix} \tag{3.1}$$

(see, [1, 2]). We have the following relation between dual geodesic curvature $\hat{\kappa}_g$ and dual curvature $\hat{\kappa}$ of $\hat{\gamma}$:

$$\hat{\kappa}^2 = \left|1 - \varepsilon_T \hat{\kappa}_g^2\right|. \tag{3.2}$$

Therefore, we can define dual pseudo-spherical EC as a stationary point of the dual bending energy

$$\int\limits_{\hat{\gamma}} \left(\hat{\kappa}_g^2 + \hat{\sigma}\right) d\hat{s} \tag{3.3}$$

in the space $\Phi = \left\{\hat{\gamma} : [0,\ell] \to \mathbb{S}_1^2 \subset \mathbb{D}_1^3,\ \hat{\gamma}(i\ell) = \hat{p}_i,\ \hat{\gamma}'(i\ell) = \hat{v}_i,\ i = 0,1\right\}$ for fixed dual constant $\hat{\sigma} = \sigma + \xi\sigma^*$.

One may clearly check the following equality from (3.2) and (3.1):

$$\left\|\hat{T}'\right\|^2 = \epsilon - \epsilon\varepsilon_T \hat{\kappa}_g^2,$$

where $\epsilon = \left|1 - \varepsilon_T \hat{\kappa}_g^2\right| / \left(1 - \varepsilon_T \hat{\kappa}_g^2\right)$. So, (3.3) can be rewritten as follows

$$\int\limits_{\hat{\gamma}} \left(-\epsilon\varepsilon_T \left\|\hat{T}'\right\|^2 + \hat{\rho}\right) d\hat{s},$$

where $\hat{\rho} = \hat{\sigma} + \varepsilon_T$ under the constrain $\hat{T} = \hat{\gamma}'$. As a result, we get

$$\hat{F} = -\epsilon\varepsilon_T \left| < \hat{T}, \hat{T} > \right| + \hat{\rho} + \hat{\lambda} \left(< \hat{T}, \hat{T} > -\varepsilon_T \right) + \hat{\mu} \left(< \hat{\gamma}, \hat{\gamma} > -1 \right) + 2 < \hat{\Lambda}, \hat{\gamma}' - \hat{T} > .$$

The following equations satisfy if $\hat{\gamma}$ is a critical value for $\hat{F}$,

$$\frac{\partial \hat{F}}{\partial \hat{\gamma}} - \frac{d}{d\hat{s}} \left(\frac{\partial \hat{F}}{\partial \hat{\gamma}'} \right) = 0, \quad \frac{\partial \hat{F}}{\partial \hat{T}} - \frac{d}{d\hat{s}} \left(\frac{\partial \hat{F}}{\partial \hat{T}'} \right) = 0.$$

Thus, we have

$$\hat{\mu}\hat{\gamma} - \hat{\Lambda}' = 0 \tag{3.4}$$

and

$$\hat{\lambda}\hat{T} + \varepsilon_T \hat{T}'' = \hat{\Lambda}. \tag{3.5}$$

Taking into consideration (3.4) and (3.5), we arrive at

$$\hat{\lambda}'\hat{T} + \hat{\lambda}\hat{T}' + \varepsilon_T \hat{T}''' = \hat{\mu}\hat{\gamma}. \tag{3.6}$$

We get the following derivatives from (3.1):

$$\hat{T}' = -\varepsilon_T \hat{\gamma} + \hat{\kappa}_g \hat{g}, \tag{3.7}$$

$$\hat{T}'' = \hat{\kappa}_g' \hat{g} - \left(\varepsilon_T - \hat{\kappa}_g \right) \hat{T}, \tag{3.8}$$

$$\hat{T}''' = \left(1 - \varepsilon_T \hat{\kappa}_g \right) \hat{\gamma} + 3\hat{\kappa}_g \hat{\kappa}_g' \hat{T} + \left(\hat{\kappa}_g'' - \left(\varepsilon_T - \hat{\kappa}_g^2 \right) \hat{\kappa}_g \right) \hat{g}. \tag{3.9}$$

Using (3.7), (3.8) and (3.9) in (3.6), we find

$$-\left(\varepsilon_T \hat{\lambda} + \hat{\mu} + \left(\hat{\kappa}_g^2 - \varepsilon_T \right) \right) \hat{\gamma} + \left(\hat{\lambda}' + 3\varepsilon_T \hat{\kappa}_g \hat{\kappa}_g' \right) \hat{T} + \left(\hat{\lambda}\hat{\kappa}_g + \varepsilon_T \hat{\kappa}_g'' + \varepsilon_T \hat{\kappa}_g^3 - \hat{\kappa}_g \right) \hat{g} = 0.$$

Because the dual vectors $\hat{\gamma}$, $\hat{T}$ and $\hat{g}$ are linearly independent, we obtain

$$\varepsilon_T \hat{\lambda} + \hat{\mu} + \hat{\kappa}_g^2 - \varepsilon_T = 0, \tag{3.10}$$

$$\hat{\lambda} = -\frac{3}{2} \varepsilon_T \hat{\kappa}_g^2 + \hat{C}, \tag{3.11}$$

where $\hat{C} = C + \xi C^*$ is a dual constant and

$$\hat{\lambda}\hat{\kappa}_g + \varepsilon_T \hat{\kappa}_g'' + \varepsilon_T \hat{\kappa}_g^3 - \hat{\kappa}_g = 0. \tag{3.12}$$

Substituting (3.11) into (3.12), we get

$$\hat{\kappa}_g'' - \frac{1}{2} \hat{\kappa}_g^3 - \varepsilon_T \left(1 - \hat{C} \right) \hat{\kappa}_g = 0. \tag{3.13}$$

We address the boundary condition for $\hat{\gamma}$ to find $\hat{C}$ with regard to $\hat{\sigma}$:

$$\hat{F}(\ell) - \frac{\partial \hat{F}}{\partial \hat{\gamma}'}(\ell)\,\hat{\gamma}'(\ell) - \frac{\partial \hat{F}}{\partial \hat{T}'}(\ell)\,\hat{T}'(\ell) = 0.$$

Then we have

$$\varepsilon_T\left(1 - \varepsilon_T \hat{\kappa}_g^2(\ell)\right) - 2 < \hat{\Lambda}(\ell), \hat{\gamma}'(\ell) > + \hat{\rho} = 0. \tag{3.14}$$

Using (3.5), we calculate

$$< \hat{\Lambda}(\ell), \hat{\gamma}'(\ell) >= -\frac{1}{2}\hat{\kappa}_g^2(\ell) + \varepsilon_T \hat{C} - \varepsilon_T. \tag{3.15}$$

Substituting (3.15) into (3.14), we have

$$-\varepsilon_T\left(1 - \hat{C}\right) = \varepsilon_T + \frac{1}{2}\hat{\sigma}.$$

Thus, we can rewrite equation (3.13) as follows

$$\hat{\kappa}_g'' - \frac{1}{2}\hat{\kappa}_g^3 + \left(\varepsilon_T + \frac{1}{2}\hat{\sigma}\right)\hat{\kappa}_g = 0. \tag{3.16}$$

This leads us to the next theorem.

Theorem 1. A non-null dual pseudo-spherical EC can be characterized by the dual E-L equation (3.16).

4. Solutions of the dual E-L equation

In this section we solve the dual E-L equation (3.16). If the dual geodesic curvature $\hat{\kappa}_g$ is a dual constant value satisfying (3.16), then Eq. (3.1) is a system of linear ordinary differential equations with constant coefficients. Hence, it can be directly resolved.

Now, suppose that $\hat{\kappa}_g$ has a non dual constant. Thus, (3.16) may be integrated to

$$\left(\hat{\kappa}_g'\right)^2 = \hat{C}_1 + \frac{1}{4}\hat{\kappa}_g^4 - \left(\varepsilon_T + \frac{1}{2}\hat{\sigma}\right)\hat{\kappa}_g^2, \tag{4.1}$$

where $\hat{C}_1 = C_1 + \xi C_1^*$ is a dual constant. We consider the solution of (4.1) separately, depending on whether the dual pseudo-spherical curve is timelike or spacelike.

Case 1: We consider to the solution of the problem for timelike dual pseudo-spherical EC. In this case the real and the dual parts of Eq. (4.1) are respectively as follows;

$$\left(\dot{\kappa}_g\right)^2 = \frac{1}{4}\kappa_g^4 - \frac{1}{2}(\sigma - 2)\,\kappa_g^2 + C_1 \tag{4.2}$$

and

$$\dot{\kappa}_g^* + \frac{\kappa_g}{2\dot{\kappa}_g}\left(\sigma - \kappa_g^2 - 2\right)\kappa_g^* = \frac{1}{2\dot{\kappa}_g}\left(C_1^* - \kappa_g^2\sigma^*\right), \tag{4.3}$$

(4.2) may be regarded as a cubic polynomial, and subsequently, it is solved by using Jacobi elliptic functions for $\kappa_{g_0}^2 < 2(\sigma - 2)$ as follows

$$\kappa_g = \kappa_{g_0} k sn\left(\frac{\kappa_{g_0}}{2}(s - s_0)\,|k\right),$$

where κ_{g_0} stands for the maximal geodesic curvature, k is the real parameter related to σ and κ_{g_0} such that

$$k^2 = \frac{2(\sigma-2)-\kappa_{g_0}^2}{\kappa_{g_0}^2}$$

and

$$C_1 = -\frac{1}{4}\kappa_{g_0}^4 + \frac{1}{2}(\sigma-2)\kappa_{g_0}^2$$

(see, [7, 9, 20]). Eq. (4.3) may be solved by integral factor method. The integral factor is calculated as follows

$$\mu = e^{\int \frac{\kappa_g}{2\dot{\kappa}_g}\left(\sigma-\kappa_g^2-2\right)ds}. \tag{4.4}$$

Multiplying by μ of (4.3), we arrive at

$$\left(\mu\kappa_g^*\right)^{\cdot} = \frac{\mu}{2\dot{\kappa}_g}\left(C_1^* - \kappa_g^2\sigma^*\right).$$

So, we obtain

$$\kappa_g^* = \frac{1}{\mu}\left[\int \frac{\mu}{2\dot{\kappa}_g}\left(C_1^* - \kappa_g^2\sigma^*\right)ds + C_2\right],$$

where C_2 is the integration constant. So, any timelike dual pseudo-spherical EC is determined by the following dual geodesic curvature:

$$\kappa_{g_0} k sn\left(\frac{\kappa_{g_0}}{2}(s-s_0)\,|k\right) + \xi\frac{1}{\mu}\left[\int \frac{\mu}{2\dot{\kappa}_g}\left(C_1^* - \kappa_g^2\sigma^*\right)ds + C_2\right]$$

for $\kappa_{g_0}^2 < 2(\sigma-2)$.

Case 2: We consider to the solution of the problem for spacelike dual pseudo-spherical EC. In this case the real and dual parts of Eq. (4.1) can be rewritten as follows;

$$\left(\dot{\kappa}_g\right)^2 = \frac{1}{4}\kappa_g^4 - \frac{1}{2}(\sigma+2)\kappa_g^2 + C_1 \tag{4.5}$$

and

$$\dot{\kappa}_g^* + \frac{\kappa_g}{2\dot{\kappa}_g}\left(\sigma-\kappa_g^2+2\right)\kappa_g^* = \frac{1}{2\dot{\kappa}_g}\left(C_1^* - \kappa_g^2\sigma^*\right), \tag{4.6}$$

respectively. Similarly to the case of the timelike dual pseudo-spherical EC, Eq. (4.5) may be solved by Jacobi elliptic functions for $\kappa_{g_0}^2 < 2(\sigma+2)$ as follows

$$\kappa_g = \kappa_{g_0} k sn\left(\frac{\kappa_{g_0}}{2}(s-s_0)\,|k\right),$$

where

$$k^2 = \frac{2(\sigma+2)-\kappa_{g_0}^2}{\kappa_{g_0}^2}$$

and

$$C_1 = -\frac{1}{4}\kappa_{g_0}^4 + \frac{1}{2}(\sigma+2)\kappa_{g_0}^2$$

(see, [7, 9, 20]). Eq. (4.6) may be solved by integral factor method. The integral factor is calculated as follows

$$\mu = e^{\int \frac{\kappa_g}{2\dot{\kappa}_g}\left(\sigma-\kappa_g^2+2\right)ds}. \tag{4.7}$$

Then, the solution of Eq. (4.6) is found in the following

$$\kappa_g^* = \frac{1}{\mu}\left[\int \frac{\mu}{2\dot{\kappa}_g}\left(C_1^* - \kappa_g^2\sigma^*\right)ds + C_3\right],$$

where C_3 is the integration constant. So, any spacelike dual pseudo-spherical EC is determined by the following dual geodesic curvature:

$$\kappa_{g_0} ksn\left(\frac{\kappa_{g_0}}{2}(s-s_0)\,|k\right) + \xi\frac{1}{\mu}\left[\int \frac{\mu}{2\dot{\kappa}_g}\left(C_1^* - \kappa_g^2\sigma^*\right)ds + C_3\right]$$

for $\kappa_{g_0}^2 < 2(\sigma+2)$.

5. Geometric interpretations of results

We know that a non-null dual curve $\hat{\gamma}$ on $\mathbb{S}_1^2$ corresponds a ruled surface written by a form (2.4) in $\mathbb{E}_1^3$. Because ES are special ruled surfaces, we can get a relationship between non-null dual EC on $\mathbb{S}_1^2$ and ES with non-null base curve in $\mathbb{E}_1^3$ in this section.

ES with non-null base curve in $\mathbb{E}_1^3$ is a developable ruled surface (or Minkowski RS) denoted by

$$R(t,\delta) = \gamma(t) + \delta\left(\omega(t)T(t) + B(t)\right) \tag{5.1}$$

if γ is an extremal of the modified Sadowsky functional

$$S_\eta(\gamma) = \int_0^\ell (\kappa^2(1+\omega^2)^2 - \eta)v\,dt,$$

where η is Lagrange multiplier, T is TV, B is BV of γ and $\omega = \frac{\tau}{\kappa}$ is the modified torsion of γ such that κ is the curvature and τ is the torsion of γ. An ES with non-null base curve γ, parametrized by its arc length s , is characterized by the E-L equations

$$r_1 = r_2 = 0, \tag{5.2}$$

where

$$\begin{aligned} r_1 := {} & \frac{\varepsilon_N d\left(\frac{d\kappa}{ds}\left(1+\omega^2\right)^2 + 2\kappa\left(1+\omega^2\right)\omega\frac{d\omega}{ds}\right)}{ds} \\ & + \frac{\kappa}{2}\left(\kappa^2\left(1+\omega^2\right)\left(\varepsilon_T + \left(5\varepsilon_T - 4\varepsilon_B\right)\omega^2\right) + \varepsilon_T\eta\right) \\ & + \omega\kappa(\varepsilon_B\kappa^2\left(1+\omega^2\right)^2\omega + \frac{d\left(\frac{2\varepsilon_N}{\kappa}\frac{d\kappa}{ds}\left(1+\omega^2\right)\omega\right)}{ds} + \frac{d^2\left(2\varepsilon_N\left(1+\omega^2\right)\omega\right)}{ds^2}) \end{aligned}$$

and

$$\begin{aligned} r_2 := {} & -\frac{d\left(\kappa^2\left(1+\omega^2\right)^2\omega + \left(\left(\varepsilon_T\varepsilon_B - 1\right)2\kappa^2\omega\left(1+\omega^2\right)\right) + \frac{d\left(\frac{2\varepsilon_N\varepsilon_B}{\kappa}\frac{d\kappa}{ds}\left(1+\omega^2\right)\omega\right)}{ds} + \frac{d^2\left(2\varepsilon_N\varepsilon_B\left(1+\omega^2\right)\omega\right)}{ds^2}\right)}{ds} \\ & + \omega\kappa(\frac{d\kappa}{ds}\left(1+\omega^2\right)^2 + 2\kappa\left(1+\omega^2\right)\omega\frac{d\omega}{ds}), \end{aligned}$$

where $\varepsilon_T, \varepsilon_N$ and ε_B are the sign of T, N and B of γ [4, 11–13].

By applying the E. Study mapping, we now derive the ensuing results

Conclusion 1. We suppose that a non-null dual curve $\hat{\gamma} = \gamma + \xi\gamma^*$ on $\mathbb{S}^2_1 \subset \mathbb{D}^3_1$ corresponds to the Minkowski RS with non-null base curve $\gamma \times \gamma^*$. Thus, we present the following claims:

i) If $\hat{\gamma}$ is a dual timelike curve and PNV of the curve $\gamma \times \gamma^*$ is timelike, consequently, the associated Minkowski RS corresponds to a spacelike BS.

ii) If $\hat{\gamma}$ is a dual timelike curve and PNV of the curve $\gamma \times \gamma^*$ is spacelike, thus the corresponding spacelike Minkowski RS is formed by spacelike cylindirical helix.

iii) If $\hat{\gamma}$ is a dual spacelike curve, so the corresponding Minkowski RS corresponds to a timelike BS.

Proof. *i*) Let $\hat{\gamma} = \gamma + \xi\gamma^*$ be a timelike dual curve on $\mathbb{S}^2_1$. From E. Study mapping, we know that the corresponding Minkowski RS is spacelike ruled surface in $\mathbb{E}^3_1$. Then Minkowski RS must be in the form of (5.1), i.e., the parametrization of Minkowski RS is given by

$$R(t,\delta) = \gamma(t) \times \gamma^*(t) + \delta\left(\omega(t)T(t) + B(t)\right),$$

where $\omega(t)T(t) + B(t) = \gamma(t)$, κ, τ, T and B are the curvature, torsion, TV and BV of $\gamma \times \gamma^*$ at the point $(\gamma \times \gamma^*)(t)$, respectively. Since $\hat{\gamma}$ is a timelike dual curve on $\mathbb{S}^2_1$, we have for all $t \in \mathbb{R}$

$$1 = < \hat{\gamma}, \hat{\gamma} > = \omega^2 + \varepsilon_B. \tag{5.3}$$

We may see from (5.3), ω is zero if N of $\gamma \times \gamma^*$ is timelike vector field. Also, we may see from (2.3), it is a BS. *ii*) If N of $\gamma \times \gamma^*$ is spacelike vector field, then $\omega^2 = 2$ and $\gamma \times \gamma^*$ is a spacelike cylindiral helix. Therefore, spacelike Minkowski RS is formed by spacelike cylindirical helix. Similarly, we can show the condition (iii).

As is commonly understood, geodesics frequently serve as prime examples of EC. With that in mind, we can present the subsequent finding.

Conclusion 2. Let $\hat{\gamma} = \gamma + \xi\gamma^*$ be a non-null dual curve on $\mathbb{S}^2_1$ and R the corresponding Minkowski RS with non-null base curve. We present the following claims:

i) If $\hat{\gamma}$ is a timelike dual curve, then the base curve of R is a spacelike EC with timelike PNV in $\mathbb{E}^3_1$.

ii) If $\hat{\gamma}$ is a spacelike dual curve, then the base curve of R is a timelike EC in $\mathbb{E}^3_1$.

Proof. Suppose that R with spacelike base curve is the spacelike BS corresponding to a timelike dual curve $\hat{\gamma} = \gamma + \xi\gamma^*$ on $\mathbb{S}^2_1$ and PNV of the spacelike curve $\gamma \times \gamma^*$ is timelike. Since the base curve of BS is a geodesic and any geodesic is EC, the base curve $\gamma \times \gamma^*$ of R is a spacelike EC in $\mathbb{E}^3_1$. Similarly, we can show the assertion (ii).

A non-null EC with modified torsion $\omega = 0$ satisfies the E-L equations (5.2), that is a Minkowski RS formed by a non-null EC with zero modified torsion is ES with non-null base curve. In such a scenario, we reach a certain conclusion, the evidence for which is clear-cut.

Conclusion 3. A timelike dual curve on $\mathbb{S}^2_1$ corresponds to ES with spacelike base curve formed by spacelike EC with the timelike PNV and zero modifed torsion in $\mathbb{E}^3_1$.

Conclusion 3 shows that a timelike dual EC on $\mathbb{S}^2_1$ corresponds to ES with spacelike base curve formed by a spacelike EC with timelike PNV and zero modified torsion in $\mathbb{E}^3_1$.

Conclusion 4. A spacelike dual curve on $\mathbb{S}^2_1$ corresponds to ES with timelike base curve formed by timelike EC with zero modifed torsion in $\mathbb{E}_1^3$. Thus, a spacelike dual pseudo-spherical EC corresponds to ES with timelike base curve constituted by a timelike EC in $\mathbb{E}_1^3$.

The following result can be seen from Conclusion 1 and E-L equations (5.2).

Conclusion 5. A timelike dual curve on $\mathbb{S}_1^2$ corresponds to ES with spacelike base curve formed by a spacelike cylindirical helix with the spacelike PNV in which the curvature κ satisfies the differential equation

$$9\frac{d^2\kappa}{ds^2} - \frac{9}{2}\kappa^3 + \left(\frac{\eta}{2} + \sqrt{2}C_4\right) = 0,\ C_4 \in \mathbb{R},$$

in $\mathbb{E}_1^3$.

References

[1] Ayyıldız N, Çöken AC, Yücesan A. A characterization of dual Lorentzian spherical curves in the dual Lorentzian space. Taiwanese Journal of Mathematics 2007; 11 (4): 999-1018. https://doi.org/10.11650/twjm/1500404798

[2] Abdel-Baky R. Evolutes of hyperbolic dual spherical curve in dual Lorentzian $3-$space. International Journal of Analysis and Applications 2017; 15 (2): 114-124. http://https//10.28924/2291-8639-15-2017-114

[3] Choi SM. On the Gauss map of ruled surfaces in a $3-$dimensional Minkowski space. Tsukuba Journal of Mathematics 1995; 19 (2): 285-304. https://doi.org/10.21099/tkbjm/1496162870

[4] Chubelaschwili D, Pinkall U. Elastic strips. Manuscripta Mathematica 2010; 133: 307-326. https://doi.org/10.1007/s00229-010-0369-x

[5] Liu H, Yuan Y. Pitch functions of ruled surfaces and B-scrolls in Minkowski $3-$space. Journal of Geometry and Physics 2012; 62 (1): 47-52. https://doi.org/10.1016/j.geomphys.2011.09.007

[6] Guggenheimer HW. Differential Geometry, Dover Publications, 1977.

[7] Oral M. Elastic curves on hyperquadrics in Minkowski $3-$space., MSc, Süleyman Demirel University, Isparta, Türkiye, 2010.

[8] Scheaf JA. Curvature theory of line trajectories in spatial kinematics. PhD, University of California, San Diego, USA, 1988.

[9] Schwalm WA. Lectures on Selected Topics in Mathematical Physics: Elliptic Functions and Elliptic Integrals, IOP Publishing, Morgan-Claypool Publishers, 2015.

[10] Singer DA. Lectures On elastic curves and rods, AIP Conference Proceedings, 1002, Amer. Inst. Phys., Melville, NY, 2008.

[11] Özkan G. Elastic strips in Minkowski $3-$space, PhD, Süleyman Demirel University, Isparta, Türkiye, 2014.

[12] Tükel GÖ, Yücesan A. Elastic strips with timelike directrix. Mathematical Reports 2019; 21 (1): 67-83.

[13] Tükel GÖ, Yücesan A. Elastic strips with spacelike directrix. Bulletin of the Malaysian Mathematical Sciences Society 2019; 42: 2623-2638. https://doi.org/10.1007/s40840-018-0622-0

[14] Tükel GÖ, Yücesan A. Dual elastica. In: International Hazar Scientific Research Conference-I Conference Proceedings; Bakü, Azerbaijan; 2020. 950-957.

[15] Tükel GÖ, Yücesan A. A variational problem on the dual pseudo-sphere. In: 2nd International E-conference on Mathematical and Statistical Sciences: a Selcuk Meeting Abstract Book; Konya, Türkiye; 2023.

[16] Uğurlu HH, Çalişkan A. The study mapping for directed space-like and time-like in Minkowski 3-space R_1^3. Mathematical and Computational Applications 1996; 1 (2): 142-148. https://doi.org/10.3390/mca1020142

[17] Veldkamp GR. On the use of dual numbers, vectors and matrices in instantaneous, spatial kinematics. Mechanism and Machine Theory 1976; 11: 141-156. https://doi.org/10.1016/0094-114X(76)90006-9

[18] Yaylı Y, Çalişkan A, Uğurlu HH. The E. Study maps of circles on dual hyperbolic and Lorentzian unit spheres H_0^2 and S_1^2. Mathematical Proceedings of the Royal Irish Academy 2002; 102A (1): 37-47. https://doi.org/10.1353/mpr.2002.0013

[19] Yücesan A, Tükel GÖ. Elastic curves in the dual Lorentzian space. In: International Modern Scientific Research Congress Conference Proceedings; İstanbul, Türkiye; 2021. 884-892

[20] Yücesan A, Oral M. Elastica on 2-dimensional anti-de Sitter space. International Joournal of Geometric Methods in Modern Physics 2011; 8 (1): 107-113. https://doi.org/10.1142/S0219887811005002

[21] Yücesan A, Tükel GÖ. Dual spherical elastica. Filomat 2023; 37 (8): 2483-2493. https://doi.org/10.2298/FIL2308483Y