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New oscillation criteria for first-order differential equations with general delay argument

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Abstract: This paper is concerned with the oscillation of solutions to a class of first-order differential equations with variable coefficients and general delay argument. New oscillation criteria are established, which improve and extend many known results reported in the literature. A couple of illustrative examples are given to show the efficiency of the newly obtained results. In particular, it is shown that our criteria partially fulfill a remaining gap in a recent sharp result by Pituk et al.[31].

Key words: Oscillation, Differential equation, First-order, General delay argument.

1. Introduction

In this paper, we are concerned with the oscillation of the first-order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \qquad t \geq t_0 \geq 0, \tag{1.1}$$

where $p, \tau \in C([t_0, \infty), [0, \infty))$, $\tau(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \infty$.

Equation (1.1) is termed oscillatory if each of its solutions has infinitely many zeros tending to infinity. Otherwise, Eq. (1.1) is called non-oscillatory. Throughout this paper and without further mention, we shall assume that there exists a nondecreasing continuous function $\theta(t)$ such that $\tau(t) \leq \theta(t)$ for $t \geq t_1$, $t_0 \geq t_1$. Moreover, we will make use of the following notation:

$$\delta = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(w)\ dw, \tag{1.2}$$

$$\delta^* = \liminf_{t\to\infty} \int_{\theta(t)}^{t} p(w)\ dw, \tag{1.3}$$

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and

$$\rho = \begin{cases} 1, & \delta^* = 0, \\ \lambda(\delta^*) - \epsilon, & \delta^* > 0, \qquad \epsilon \in (0, \lambda(\delta^*)), \end{cases} \tag{1.4}$$

where $\lambda(\xi)$ stands for the smaller real root of the equation $\lambda = \mathrm{e}^{\lambda\xi}$.

In dynamical models, delay and oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [24–26]. Since the pioneering work of Myshkis [28], the oscillation theory of delay differential equations has received a great deal of attention, see the monographs [1, 15, 16] as well as the papers cited in this work for a considerable account of results. In particular, oscillation properties of first-order differential equations with delayed argument have numerous applications in the study of higher-order differential equations with deviating arguments; see, e.g., the papers [3, 7, 27] for more details.

In view of the classical liminf oscillation criterion

$$\delta > \frac{1}{\mathrm{e}}$$

due to Koplatadze and Chanturija [21], it gives sense to consider only the case when

$$0 \le \delta \le \frac{1}{\mathrm{e}}.$$

Most of the research has been done in the case when the delay is non-decreasing. As a starting point, the classical limsup oscillation criterion

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(w)dw > 1 \tag{1.5}$$

due to Ladas [23] has commonly been referred. Consequently, a major research has been devoted to improve the preceding condition (1.5) so that the value at the right-hand side is as close to the treshold value 1/e as possible; see, e.g., the papers [10–13, 18–20, 22, 23, 29, 30, 32].

A sharp result in certain sense has been given in [13, Theorem 4] by Gárab, Pituk and Stavroulakis. It has been proven there that Eq. (1.1) with constant delay and $p(t)$ slowly varying at infinity is oscillatory if $\delta > 0$ and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(w)dw > \frac{1}{\mathrm{e}}.$$

For some further works on this particular class of Eq. (1.1) with $p(t)$ enjoying the slowly varying property, see [12, 14, 30].

Very recently, Pituk, Stavroulakis and Stavroulakis Jr. [31] found, for nondecreasing τ, the explicit value of the bound at the right-hand side of (1.5) depending on δ. As a result, they improved condition (1.5) and established the oscillation criterion

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(w)dw > K(\delta), \tag{1.6}$$

where $\delta \in \left[0, \frac{1}{\mathrm{e}}\right]$ and

$$K(\delta) = \begin{cases} 1, & \delta = 0, \\ 2\delta + \frac{2}{\lambda(\delta)} - 1, & \delta \in \left(0, \frac{\ln 2}{2}\right], \\ 2\delta - \frac{2}{\lambda(\delta)} - \frac{1}{\lambda(\delta)} W_{-1}\left(-\frac{\lambda(\delta)}{\mathrm{e}^2}\right), & \delta \in \left(\frac{\ln 2}{2}, \frac{1}{\mathrm{e}}\right], \end{cases}$$

where W_{-1} is the secondary real branch of the Lambert W function. It is important to notice that the constant $K(\delta)$ in (1.6) is sharp in the sense that a nonscillatory counterexample can be found if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(w)dw \le K(\delta).$$

In the paper, we confirm (see Example 3.2) that condition (1.6) is not necessary for the oscillation of Eq. (1.1) when $\delta = 0$ and that Ladas criterion (1.5) can be improved in this case. This finding points out that establishing new oscillation conditions for Eq. (1.1) is still of importance.

On the other hand, it is worth to note that the dynamics of solutions of equations with non-monotone arguments can be completely different from those with monotone ones. As a matter of fact, we recall a remarkable result due to Braverman and Karpuz [4] who showed that the well-known Ladas criterion (1.5) is no longer applicable in the non-monotone case and there is no constant L such that

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(w)dw > L$$

implies Eq. (1.1) to be oscillatory. Consequently, the oscillation problem of Eq. (1.1) with non-monotone retarded arguments has attracted the interest of many mathematicians and both iterative and non-iterative oscillation criteria have been established; see, e.g, the papers [2, 4–6, 9, 17, 22] and those cited therein. For an easy reference, we give a brief summary of some recently published oscillation results.

In 2015, Infante, Koplatadze and Stavroulakis [17] proved that Eq. (1.1) is oscillatory if

$$\limsup_{t\to\infty} \int_{\theta(t)}^{t} p(w)\, \mathrm{e}^{\int_{\tau(w)}^{\theta(t)} p(w_1)\, \mathrm{e}^{\int_{\tau(w_1)}^{w_1} p(w_2)dw_2} dw_1} dw > 1, \tag{1.7}$$

or

$$\limsup_{\epsilon\to 0^+} \left(\limsup_{t\to\infty} \int_{\theta(t)}^{t} p(w)\mathrm{e}^{(\lambda(\delta)-\epsilon) \int_{\tau(w)}^{\theta(t)} p(w_1)dw_1} dw \right) > 1. \tag{1.8}$$

In 2020, Chatzarakis and Jadlovská [5] established the condition

$$\limsup_{t\to\infty} \int_{\varphi(t)}^{t} p(w)\mathrm{e}^{\int_{\tau(w)}^{\varphi(t)} p(w_1)\, \mathrm{e}^{\int_{\tau(w_1)}^{w_1} \Psi_n(w_2)dw_2} dw_1} dw > 1, \tag{1.9}$$

where

$$\varphi(t) = \sup_{u\le t} \tau(u) \tag{1.10}$$

and

$$\Psi_0(t)=p(t)\left(1+\int_{\tau(t)}^{t}p(w)\mathrm{e}^{\int_{\tau(w)}^{t}p(w_1)\,\mathrm{e}^{\lambda(\delta)\int_{\tau(w_1)}^{w_1}p(w_2)dw_2}dw_1}dw\right),$$

$$\Psi_n(t)=p(t)\left(1+\int_{\tau(t)}^{t}p(w)\mathrm{e}^{\int_{\tau(w)}^{t}p(w_1)\,\mathrm{e}^{\int_{\tau(w_1)}^{w_1}\Psi_{n-1}(w_2)dw_2}dw_1}dw\right),\quad n=1,2,\ldots.$$

In 2022, Attia and El-Morshedy [2] obtained the condition

$$\limsup_{t\to\infty}\left(\sum_{r=1}^{n}\left(\prod_{r_1=2}^{r}C\left(\theta^{r_1-1}(t)\right)\right)\Omega_r^n(t)\right)>1-B(\delta^*),\quad n\in\mathbb{N},\tag{1.11}$$

where

$$B(\delta^*)=\frac{1-\delta^*-\sqrt{1-2\delta^*-\delta^{*2}}}{2},\quad 0\le\delta^*\le\frac{1}{\mathrm{e}},$$

$$C(t)=\frac{1}{1-\int_{\theta(t)}^{t}p(w_1)\exp\left(\int_{\tau(w_1)}^{\theta(t)}\frac{p(w_2)}{1-\Omega_1^1(w_2)}dw_2\right)dw_1}$$

and

$$\Omega_i^n(t)=\int_{\theta(t)}^{t}p(w_1)\int_{\tau(w_1)}^{\theta(t)}p(w_2)\int_{\tau(w_2)}^{\theta^2(t)}\ldots\int_{\tau(w_{i-1})}^{\theta^{i-1}(t)}p(w_i)dw_i\;dw_{i-1}\ldots\;dw_1,\quad i=1,\ldots,n-1,$$

$$\Omega_n^n(t)=\int_{\theta(t)}^{t}p(w_1)\int_{\tau(w_1)}^{\theta(t)}p(w_2)\int_{\tau(w_2)}^{\theta^2(t)}\ldots\int_{\tau(w_{n-1})}^{\theta^{n-1}(t)}p(w_n)\mathrm{e}^{\rho\int_{\tau(w_n)}^{\theta^n(t)}p(w_{n+1})dw_{n+1}}dw_n\;dw_{n-1}\ldots\;dw_1.$$

The objective of this work is to obtain new oscillation criteria for Eq. (1.1), which would improve the above mentioned ones in both cases of monotone and non-monotone arguments. Two illustrative examples are presented to demonstrate the power and efficiency of our results.

2. Main results

We start with the following lemmas, which will be of utmost importance in establishing our main results. All our results are formulated in terms of constants (1.2)–(1.4).

Lemma 2.1 (see [11, Lemma 2.1.2] and [2, Lemma 2.1]) *Assume that $x(t)$ is an eventually positive solution of Eq. (1.1). Then*

$$\frac{x(\theta(t))}{x(t)}\ge\rho\quad\text{for all sufficiently large } t.\tag{2.1}$$

Lemma 2.2 *Assume that $x(t)$ is an eventually positive solution of Eq. (1.1) and there exists a continuous positive function $Q_0(t)$ such that*

$$\frac{x(\tau(t))}{x(t)}\ge Q_0(t).\tag{2.2}$$

Then, for any $n \in \mathbb{N}$ and t sufficiently large,

$$\frac{x(\tau(t))}{x(t)} \geq Q_n(t), \tag{2.3}$$

where

$$Q_n(t) = \frac{e^{\int_{\tau(t)}^{\theta(t)} p(w)Q_{n-1}(w)dw}}{1 - \int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)Q_{n-1}(w_1)dw_1} dw}. \tag{2.4}$$

Proof Integrating (1.1) from $\theta(t)$ to t, we obtain

$$x(t) - x(\theta(t)) + \int_{\theta(t)}^{t} p(w)x(\tau(w))dw = 0. \tag{2.5}$$

Dividing (1.1) by $x(t)$ and integrating the resulting inequality from w to t, $t \geq w$, we have

$$x(w) = x(t)e^{\int_{w}^{t} p(w_1)\frac{x(\tau(w_1))}{x(w_1)}dw_1}. \tag{2.6}$$

This, together with (2.5), leads to

$$x(t) - x(\theta(t)) + x(\theta(t)) \int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)\frac{x(\tau(w_1))}{x(w_1)}dw_1} dw = 0. \tag{2.7}$$

Therefore,

$$\frac{x(\theta(t))}{x(t)} = \frac{1}{1 - \int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)\frac{x(\tau(w_1))}{x(w_1)}dw_1} dw}. \tag{2.8}$$

From (2.6), we see that

$$\frac{x(\tau(t))}{x(t)} = \frac{x(\tau(t))}{x(\theta(t))}\frac{x(\theta(t))}{x(t)} = \frac{e^{\int_{\tau(t)}^{\theta(t)} p(w)\frac{x(\tau(w))}{x(w)}dw}}{1 - \int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)\frac{x(\tau(w_1))}{x(w_1)}dw_1} dw}, \tag{2.9}$$

which in view of (2.8) leads to

$$\frac{x(\tau(t))}{x(t)} \quad \geq \quad \frac{e^{\int_{\tau(t)}^{\theta(t)} p(w)Q_0(w)dw}}{1 - \int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)Q_0(w_1)dw_1} dw} = Q_1(t). \tag{2.10}$$

Substituting again (2.10) into (2.9) we get

$$\frac{x(\tau(t))}{x(t)} \geq \frac{e^{\int_{\tau(t)}^{\theta(t)} p(w)Q_1(w)dw}}{1 - \int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)Q_1(w_1)dw_1} dw} = Q_2(t).$$

A simple induction completes the proof. $\square$

Lemma 2.3 *Assume that* $\delta^* > 0$ *and* $x(t)$ *is an eventually positive solution of Eq. (1.1). If*

$$\liminf_{t\to\infty} \int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) \mathrm{e}^{(\lambda(\delta^*)-\epsilon)\int_{\tau(w_1)}^{\theta^2(t)} p(w_2)dw_2} dw_1 dw \geq \beta > 0 \tag{2.11}$$

for some $\epsilon \in (0, \lambda(\delta^*))$, *then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(\theta(t))} \geq \frac{1-\delta^* - \sqrt{(1-\delta^*)^2 - 4\beta}}{2}. \tag{2.12}$$

Proof First, we claim that

$$\frac{x(\theta(t))}{x(\theta^2(t))} > \frac{R(t)}{1-\int_{\theta(t)}^{t} p(w)dw} \quad \text{for all sufficiently large } t, \tag{2.13}$$

where $\theta^2(t) = \theta(\theta(t))$ and

$$R(t) = \int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) \mathrm{e}^{(\lambda(\delta^*)-\epsilon)\int_{\tau(w_1)}^{\theta^2(t)} p(w_2)dw_2} dw_1 dw.$$

Integrating (1.1) from $\tau(w)$ to $\theta(t)$ for $\theta(t) \leq w \leq t$, we get

$$x(\theta(t)) - x(\tau(w)) + \int_{\tau(w)}^{\theta(t)} p(w_1) x(\tau(w_1)) dw_1 = 0.$$

Substituting this into (2.5), we obtain

$$x(t) - x(\theta(t)) + x(\theta(t)) \int_{\theta(t)}^{t} p(w)dw + \int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) x(\tau(w_1)) dw_1 dw = 0. \tag{2.14}$$

In view of $\theta^2(t) \geq \tau(w_1)$ for $\tau(w) \leq w_1 \leq \theta(t)$ and $\theta(t) \leq w \leq t$, it follows from (2.6) that

$$x(\tau(w_1)) = x(\theta^2(t)) \mathrm{e}^{\int_{\tau(w_1)}^{\theta^2(t)} p(w_2) \frac{x(\tau(w_2))}{x(w_2)} dw_2}.$$

Substituting into (2.14), we obtain

$$\begin{aligned} & x(t) - x(\theta(t)) + x(\theta(t)) \int_{\theta(t)}^{t} p(w)dw \\ & + x(\theta^2(t)) \int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) \mathrm{e}^{\int_{\tau(w_1)}^{\theta^2(t)} p(w_2) \frac{x(\tau(w_2))}{x(w_2)} dw_2} dw_1 dw = 0. \end{aligned} \tag{2.15}$$

Using Lemma 2.1 and $\delta^* > 0$, we obtain

$$\begin{aligned} x(\theta(t)) \geq {} & x(t) + x(\theta(t)) \int_{\theta(t)}^{t} p(w)dw \\ & + x(\theta^2(t)) \int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) \mathrm{e}^{\int_{\tau(w_1)}^{\theta^2(t)} p(w_2)(\lambda(\delta^*)-\epsilon)dw_2} dw_1 dw, \end{aligned} \tag{2.16}$$

where $\epsilon > 0$ is sufficiently small. Consequently,

$$\frac{x(\theta(t))}{x(\theta^2(t))} > \frac{\int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) \mathrm{e}^{\int_{\tau(w_1)}^{\theta^2(t)} p(w_2)(\lambda(\delta^*)-\epsilon)dw_2} dw_1 dw}{1-\int_{\theta(t)}^{t} p(w)dw}$$
$$= \frac{R(t)}{1-\int_{\theta(t)}^{t} p(w)dw}.$$

This completes the proof of (2.13) and so our claim holds. Now we will prove (2.12). Assume that $0 < \delta^{**} < \delta^*$ and $0 < \beta^* < \beta$ are, respectively, any two numbers arbitrarily close to δ^* and β. Then there exists T large enough so that

$$\int_{\theta(t)}^{t} p(w)dw > \delta^{**} \quad \text{for } t > T$$

and

$$\int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) \mathrm{e}^{\int_{\tau(w_1)}^{\theta^2(t)} p(w_2)(\lambda(\delta^*)-\epsilon)dw_2} dw_1 dw > \beta^* \quad \text{for } t > T.$$

Substituting both the above estimates into (2.16), we obtain

$$x(\theta(t)) > x(t) + \delta^{**}x(\theta(t)) + \beta^* x(\theta^2(t)). \tag{2.17}$$

Consequently,

$$x(\theta(t)) > b_1 x(\theta^2(t)), \tag{2.18}$$

where

$$b_1 = \frac{\beta^*}{1-\delta^{**}}.$$

Let $T_1 > T$ such that $t = \theta(T_1)$, and so

$$\int_{t}^{T_1} p(w)dw > \delta^{**}$$

and

$$\int_{t}^{T_1} p(w) \int_{\tau(w)}^{t} p(w_1) \mathrm{e}^{\int_{\tau(w_1)}^{\theta(t)} p(w_2)(\lambda(\delta)-\epsilon)dw_2} dw_1 dw > \beta^*.$$

By integrating Eq. (1.1) from t to T_1, and using the same arguments as above we obtain

$$x(t) > b_1 x(\theta(t)). \tag{2.19}$$

From this and (2.17), we get

$$x(\theta(t)) > b_2 x(\theta^2(t)),$$

where

$$b_2 = \frac{\beta^*}{1-b_1-\delta^{**}}.$$

Repeating this procedure we have

$$x(\theta(t)) > b_n x(\theta^2(t)),$$

where

$$b_n = \frac{\beta^*}{1 - b_{n-1} - \delta^{**}}.$$

Since $\{b_n\}_{n\geq 1}$ is strictly increasing and bounded, then

$$b^2 - (1 - \delta^{**})\, b + \beta^* = 0,$$

where

$$\lim_{n\to\infty} b_n = b.$$

Therefore,

$$\frac{x(\theta(t))}{x(\theta^2(t))} \geq \frac{1 - \delta^{**} - \sqrt{(1 - \delta^{**})^2 - 4\beta^*}}{2}$$

for all sufficiently large t.
Then, we see that

$$\liminf_{t\to\infty} \frac{x(\theta(t))}{x(\theta^2(t))} \geq \frac{1 - \delta^{**} - \sqrt{(1 - \delta^{**})^2 - 4\beta^*}}{2}.$$

Letting $\delta^{**} \to \delta^*$ and $\beta^* \to \beta$ the last inequality implies that

$$\liminf_{t\to\infty} \frac{x(t)}{x(\theta(t))} = \liminf_{t\to\infty} \frac{x(\theta(t))}{x(\theta^2(t))} \geq \frac{1 - \delta^* - \sqrt{(1 - \delta^*)^2 - 4\beta}}{2}.$$

The proof is complete. □

Remark 2.4 *It is clear for $\delta^* > 0$ that*

$$\int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) \mathrm{e}^{(\lambda(\delta^*) - \epsilon) \int_{\theta(w_1)}^{\theta^2(t)} p(w_2) dw_2} dw_1 dw \geq \int_{\theta(t)}^{t} p(w) \int_{\theta(w)}^{\theta(t)} p(w_1) dw_1 dw.$$

By using similar arguments as in the proof of [11, Lemma 2.1.3], we arrive at

$$\liminf_{t\to\infty} \int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) \mathrm{e}^{(\lambda(\delta^*) - \epsilon) \int_{\theta(w_1)}^{\theta^2(t)} p(w_2) dw_2} dw_1 dw \geq \frac{1}{2} \delta^{*2}.$$

As a result, in Lemma 2.3, one can choose $\beta = \frac{1}{2}\delta^{*2}$. Consequently,

$$\frac{1 - \delta^* - \sqrt{(1 - \delta^*)^2 - 4\beta}}{2} = \frac{1 - \delta^* - \sqrt{1 - 2\delta^* - \delta^{*2}}}{2}.$$

Therefore, we see that Lemma 2.3 improves [11, Lemma 2.1.3].

Now we are prepared to state the main results of the paper.

Theorem 2.5 *Assume that $\delta^*>0$ and there exists $\beta>0$ satisfying (2.20) for some $\epsilon\in(0,\lambda(\delta^*))$. If for some $n\in\mathbb{N}_0$*

$$\limsup_{t\to\infty}\int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} Q_n(w_1)p(w_1)dw_1}dw > 1-\frac{1-\delta^*-\sqrt{(1-\delta^*)^2-4\beta}}{2}, \tag{2.20}$$

where $Q_0(t)=\lambda(\delta^*)-\epsilon$ and $\{Q_n(t)\}_{n\in\mathbb{N}}$ is defined by (2.4), then Eq. (1.1) is oscillatory.

Proof Assume the contrary and let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality assume that $x(t)$ is eventually positive. By (2.7) from the proof of Lemma 2.2, we have

$$x(t)-x(\theta(t))+x(\theta(t))\int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)\frac{x(\tau(w_1))}{x(w_1)}dw_1}dw=0. \tag{2.21}$$

According to Lemma 2.1 and the non-increasing nature of $x(t)$, we have, for any $\epsilon\in(0,\lambda(\delta^*))$ and t sufficiently large,

$$\frac{x(\tau(t))}{x(t)}\geq\frac{x(\theta(t))}{x(t)}\geq\lambda(\delta^*)-\epsilon=Q_0(t).$$

By Lemma 2.2, we are led to

$$\frac{x(\tau(t))}{x(t)}\geq Q_n(t), \quad n\in\mathbb{N}_0. \tag{2.22}$$

Substituting (2.22) into (2.21), we have

$$\int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)Q_n(w_1)dw_1}dw\leq 1-\frac{x(t)}{x(\theta(t))}. \tag{2.23}$$

Therefore,

$$\limsup_{t\to\infty}\int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)Q_n(w_1)dw_1}dw\leq 1-\liminf_{t\to\infty}\frac{x(t)}{x(\theta(t))}.$$

From this and (2.12), we obtain a contradiction to (2.20). The proof of the theorem is complete. □

Theorem 2.6 *Assume that $\theta(t)$ is strictly increasing. If there exist $n\in\mathbb{N}_0$ and an unbounded sequence $\{r_l\}_{l\in\mathbb{N}_0}$ such that*

$$\int_{\theta(r_l)}^{r_l} p(w)e^{\int_{\tau(w)}^{\theta(r_l)} Q_n(w_1)p(w_1)dw_1}dw\geq 1-\frac{\int_{r_l}^{\theta^{-1}(r_l)} p(w)\int_{\tau(w)}^{r_l} p(w_1)e^{\int_{\tau(w_1)}^{\theta(r_l)} p(w_2)Q_n(w_2)dw_2}dw_1dw}{1-\int_{r_l}^{\theta^{-1}(r_l)} p(w)dw}, \tag{2.24}$$

where θ^{-1} denotes the inverse of θ, $Q_0(t)=\rho$ and $\{Q_n(t)\}_{n\in\mathbb{N}}$ is defined by (2.4), then Eq. (1.1) is oscillatory.

Proof Assume the contrary and let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality assume that $x(t)$ is eventually positive. By (2.7) from the proof of Lemma 2.2, we have

$$x(t)-x(\theta(t))+x(\theta(t))\int_{\theta(t)}^{t} p(w)e^{\int_{\tau(w)}^{\theta(t)} p(w_1)\frac{x(\tau(w_1))}{x(w_1)}dw_1}dw=0. \tag{2.25}$$

By using the non-increasing nature of $x(t)$ and Lemma 2.1, we obtain

$$\frac{x(\tau(t))}{x(t)} \geq \frac{x(\theta(t))}{x(t)} \geq \rho = Q_0(t).$$

By Lemma 2.2 , we are led to

$$\frac{x(\tau(t))}{x(t)} \geq Q_n(t), \quad n \in \mathbb{N}_0. \tag{2.26}$$

Substituting into (2.25), we get

$$\int_{\theta(t)}^{t} p(w) \mathrm{e}^{\int_{\tau(w)}^{\theta(t)} p(w_1) Q_n(w_1) dw_1} dw \leq 1 - \frac{x(t)}{x(\theta(t))} \tag{2.27}$$

for all sufficiently large t. By (2.15) and $x(t) > 0$, we have

$$\frac{x(\theta(t))}{x(\theta^2(t))} > \frac{\int_{\theta(t)}^{t} p(w) \int_{\tau(w)}^{\theta(t)} p(w_1) \mathrm{e}^{\int_{\tau(w_1)}^{\theta^2(t)} p(w_2) \frac{x(\tau(w_2))}{x(w_2)} dw_2} dw_1 dw}{1 - \int_{\theta(t)}^{t} p(w) dw}.$$

From the above inequality, (2.26) and the strictly increasing nature of $\theta(t)$, we obtain

$$\frac{x(t)}{x(\theta(t))} > \frac{\int_{t}^{\theta^{-1}(t)} p(w) \int_{\tau(w)}^{t} p(w_1) \mathrm{e}^{\int_{\tau(w_1)}^{\theta(t)} p(w_2) Q_n(w_2) dw_2} dw_1 dw}{1 - \int_{t}^{\theta^{-1}(t)} p(w) dw}.$$

This together with (2.27) implies that

$$\int_{\theta(t)}^{t} p(w) \mathrm{e}^{\int_{\tau(w)}^{\theta(t)} p(w_1) Q_n(w_1) dw_1} dw < 1 - \frac{\int_{t}^{\theta^{-1}(t)} p(w) \int_{\tau(w)}^{t} p(w_1) \mathrm{e}^{\int_{\tau(w_1)}^{\theta(t)} p(w_2) Q_n(w_2) dw_2} dw_1 dw}{1 - \int_{t}^{\theta^{-1}(t)} p(w) dw}$$

for all sufficiently large t, which contradicts (2.24) and completes the proof of the theorem. □

3. Numerical examples

In this section, we give two examples illustrating the applications of our results, showing their strength in both cases of monotone and nonmonotone delays.

Example 3.1 Consider the differential equation

$$x'(t) + p(t) x(\tau(t)) = 0, \quad t \geq 1, \tag{3.1}$$

where

$$\tau(t) = \begin{cases} t - 1 & \text{if } t \in [2l,\, 2l+1] \\ -t + 4l + 1 & \text{if } t \in [2l+1,\, 2l+1.001] \\ \frac{1001}{999} t - \frac{4}{999} l - \frac{1003}{999} & \text{if } t \in [2l+1.001,\, 2l+2] \end{cases}, \quad l \in \mathbb{N}_0,$$

and

$$p(t)=\begin{cases}\frac{1}{\mathrm{e}} & \text{if } t\in[c_i,\, d_i]\\ \left(\mu-\frac{1}{\mathrm{e}}\right)(t-d_i)+\frac{1}{\mathrm{e}} & \text{if } t\in[d_i,\, d_i+1]\\ \mu & \text{if } t\in[d_i+1,\, d_i+5]\\ \frac{\left(\frac{1}{\mathrm{e}}-\mu\right)(t-d_i-5)}{c_{i+1}-d_i-5}+\mu & \text{if } t\in[d_i+5,\, c_{i+1}]\end{cases}, \quad i\in\mathbb{N}_0,$$

where $\mu\geq\frac{1}{\mathrm{e}}$ and $\{d_i\}$ is a sequence of positive integers such that $d_i>c_i+3$, $c_{i+1}>d_i+5$ and $\lim_{i\to\infty} c_i=\infty$. Let $\theta(t)=t-1$. It is clear that

$$t-1.002\leq\tau(t)\leq t-1$$

and

$$\delta=\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(w)dw=\liminf_{t\to\infty}\int_{\theta(t)}^{t}p(w)dw=\lim_{i\to\infty}\int_{\theta(d_i)}^{d_i}p(w)dw=\frac{1}{\mathrm{e}}=\delta^*.$$

It follows that $\lambda(\delta)=\mathrm{e}$. Let

$$R(t)=\int_{\theta(t)}^{t}p(w)\int_{\tau(w)}^{\theta(t)}p(w_1)\mathrm{e}^{(\lambda(\delta^*)-\epsilon)\int_{\tau(w_1)}^{\theta^2(t)}p(w_2)dw_2}dw_1dw.$$

Therefore,

$$\begin{aligned}\liminf_{t\to\infty} R(t) &= \lim_{i\to\infty}\int_{\theta(d_i)}^{d_i}p(w)\int_{\tau(w)}^{\theta(d_i)}p(w_1)\mathrm{e}^{(\lambda(\delta)-\epsilon)\int_{\tau(w_1)}^{\theta^2(d_i)}p(w_2)dw_2}dw_1dw\\ &\geq \lim_{i\to\infty}\int_{d_i-1}^{d_i}\frac{1}{\mathrm{e}}\int_{w-1}^{\theta(d_i)}\frac{1}{\mathrm{e}}\mathrm{e}^{(\lambda(\delta)-\epsilon)\int_{w_1-1}^{d_i-2}\frac{1}{\mathrm{e}}dw_2}dw_1dw\\ &= \frac{\mathrm{e}^{\frac{\lambda(\delta)-\epsilon}{\mathrm{e}}}-\frac{\lambda(\delta)-\epsilon}{\mathrm{e}}-1}{(\lambda(\delta)-\epsilon)^2}>0.09719=\beta,\end{aligned}$$

where we put $\epsilon=0.001$. Let

$$L(t)=\int_{\theta(t)}^{t}p(w)\mathrm{e}^{\int_{\tau(w)}^{\theta(t)}Q_1(w_1)p(w_1)}dw.$$

Then

$$\begin{aligned}&L(d_i+5)\\ &\geq\int_{\theta(d_i+5)}^{d_i+5}p(w)\exp\left(\int_{w-1}^{\theta(d_i+5)}\frac{p(w_1)\exp\left(\int_{w_1-1}^{w_1-1}p(w_2)dw_2\right)}{1-\int_{w_1-1}^{w_1}p(w_2)\exp\left(\int_{w_2-1}^{w_1-1}p(w_3)Q_0(w_3)dw_3\right)dw_2}dw_1\right)dw\\ &\geq\int_{d_i+4}^{d_i+5}p(w)\exp\left(\int_{w-1}^{d_i+4}\frac{p(w_1)}{1-\int_{w_1-1}^{w_1}p(w_2)\exp\left(\int_{w_2-1}^{w_1-1}p(w_3)\left(\lambda(\delta)-\epsilon\right)dw_3\right)dw_2}dw_1\right)dw\\ &=\frac{\left(\mathrm{e}^{D\left(-(\lambda(\delta)-\epsilon)+\mathrm{e}^{(\lambda(\delta)-\epsilon)\mu}\right)}-(\lambda(\delta)-\epsilon)\mathrm{e}^{D}-\mathrm{e}^{D}-\mathrm{e}^{(\lambda(\delta)-\epsilon)\mu}+(\lambda(\delta)-\epsilon)+1\right)\mathrm{e}^{-D}}{(\lambda(\delta)-\epsilon)},\end{aligned}$$

where

$$D=\frac{(\lambda(\delta)-\epsilon)\,\mu}{\mathrm{e}^{(\lambda(\delta)-\epsilon)\mu}-(\lambda(\delta)-\epsilon)-1}.$$

Consequently,

$$\limsup_{t\to\infty} L(t)=\lim_{i\to\infty} L(d_i+5)>0.74>1-\frac{1-\delta-\sqrt{(1-\delta)^2-4\beta}}{2}$$

for $\mu=\frac{1}{\mathrm{e}}+0.01666$, which means that the condition (2.20) with $n=1$ of Theorem 2.5 is satisfied. Therefore, every solution of (3.1) is oscillatory. However, we will demonstrate that all the existed conditions mentioned in the introduction fail to do so. Let $\theta(t)=\varphi(t)$ (that is defined by (1.10)). Since

$$t-1.002\le\tau(t)\le\varphi(t)\le t-1,\quad \frac{1}{\mathrm{e}}\le p(t)\le\mu,$$

we have

$$\begin{aligned}\int_{\theta(t)}^{t} p(w)\,\mathrm{e}^{\int_{\tau(w)}^{\theta(t)} p(w_1)\ \mathrm{e}^{\int_{\tau(w_1)}^{w_1} p(w_2)dw_2} dw_1} dw &\le \int_{t-1.002}^{t} \mu\,\mathrm{e}^{\int_{w-1.002}^{t-1}\mu\ \mathrm{e}^{\int_{w_1-1.002}^{w_1}\mu dw_2} dw_1} dw\\ &\le \mathrm{e}^{-\frac{501\,\mu}{500}}\left(\mathrm{e}^{\frac{251\,\mu}{250}\mathrm{e}^{\frac{501\,\mu}{500}}}-\mathrm{e}^{\frac{\mu}{500}\mathrm{e}^{\frac{501\,\mu}{500}}}\right)<0.9999,\end{aligned}$$

for all $\mu\le\frac{1}{\mathrm{e}}+0.205$. Consequently, condition (1.7) is not satisfied for all $\mu\le\frac{1}{\mathrm{e}}+0.205$. Clearly,

$$\begin{aligned}\int_{\theta(t)}^{t} p(w)\mathrm{e}^{(\lambda(\delta)-\epsilon)\int_{\tau(w)}^{\theta(t)} p(w_1)dw_1} du &\le \int_{t-1.002}^{t} \mu\mathrm{e}^{\lambda(\delta)\int_{w-1.002}^{t-1}\mu dw_1} dw\\ &\le \mathrm{e}^{-1+\frac{251\mathrm{e}}{250}\mu}-\mathrm{e}^{-1+\frac{\mathrm{e}}{500}\mu}<0.999\end{aligned}$$

for all $\mu\le\frac{1}{\mathrm{e}}+0.113$, it follows that condition (1.8) cannot be applied for all $\mu\le\frac{1}{\mathrm{e}}+0.113$. In view of

$$\begin{aligned}\Psi_0(t) &= p(t)\left(1+\int_{\tau(t)}^{t} p(w)\mathrm{e}^{\int_{\tau(w)}^{t} p(w_1)\ \mathrm{e}^{\lambda(\delta)\int_{\tau(w_1)}^{w_1} p(w_2)dw_2} dw_1} dw\right)\\ &\le \mu\left(1+\int_{t-1.002}^{t} \mu\mathrm{e}^{\int_{w-1.002}^{t}\mu\mathrm{e}^{\lambda(\delta)\int_{w_1-1.002}^{w_1}\mu dw_2} dw_1} dw\right)<2.58535\end{aligned}$$

for all $\mu\le\frac{1}{\mathrm{e}}+0.0794$, we get

$$\limsup_{t\to\infty}\int_{\varphi(t)}^{t} p(w)\mathrm{e}^{\int_{\tau(w)}^{\varphi(t)} p(w_1)\ \mathrm{e}^{\int_{\tau(w_1)}^{w_1} p(w_2)\Psi_0(w_2)dw_2} dw_1} dw<1$$

for all $\mu\le\frac{1}{\mathrm{e}}+0.0794$. Then we conclude that condition (1.9) with $n=0$ can not be applied for $\mu\le\frac{1}{\mathrm{e}}+0.0794$. Finally, it is clear that

$$\Omega_1^1(t)\le\int_{\theta(t)}^{t} p(w)\mathrm{e}^{\lambda(\delta)\int_{\tau(w)}^{\theta(t)} p(w_1)dw_1} dw\le\int_{t-1.002}^{t} \mu\mathrm{e}^{\mathrm{e}\int_{w-1.002}^{t-1}\mu dw_1} dw<0.687061$$

for all $\mu \le \frac{1}{\mathrm{e}} + 0.0184$, so that

$$C(t) = \frac{1}{1 - \int_{\theta(t)}^{t} p(w_1) \exp\left(\int_{\tau(w_1)}^{\theta(t)} \frac{p(w_2)}{1-\Omega_1^1(w_2)} dw_2\right) dw_1} < 4.29043, \quad \text{for all } \mu \le \frac{1}{\mathrm{e}} + 0.0184.$$

Consequently,

$$\limsup_{t\to\infty} \left(\Omega_1^2(t) + C\left(\theta(t)\right)\Omega_2^2(t)\right) < 0.86157 < 1 - B(\delta) = 1 - \frac{1-\delta-\sqrt{1-2\delta-\delta^2}}{2}.$$

Therefore, condition (1.11) with $n = 2$ is not satisfied for all $\mu \le \frac{1}{\mathrm{e}} + 0.0184$.

The following example demonstrates the significance of one of our results, especially when $\delta = 0$, and shows that condition (1.6) is not necessary for the oscillation of Eq. (1.1).

Example 3.2 Consider the differential equation

$$x'(t) + p(t)x(t-1) = 0, \quad t \ge 1, \tag{3.2}$$

where

$$p(t) = \begin{cases} 0 & \text{if } t \in [c_l,\, d_l] \\ \gamma\,(t-d_l) & \text{if } t \in [d_l,\, d_l+1] \\ \gamma & \text{if } t \in [d_l+1,\, d_l+6] \\ \left(\frac{d_l-t+6}{c_{l+1}-d_l-6}+1\right)\gamma & \text{if } t \in [d_l+6,\, c_{l+1}] \end{cases}, \quad l \in \mathbb{N}_0,$$

where $\gamma \ge 0$, $d_l > c_l + 1$, $c_{l+1} > d_l + 6$ and $\lim_{l\to\infty} c_l = \infty$.

Clearly,

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(w)dw = \lim_{l\to\infty} \int_{\tau(d_l)}^{d_l} p(w)dw = \int_{d_l-1}^{d_l} p(w)dw = 0 = \delta. \tag{3.3}$$

From this and (1.4), it follows in Theorem 2.6 that $Q_0(t) = 1$.

Let $\theta(t) = \tau(t) = t-1$, $r_l = d_l + 5$,

$$I(t) = \int_{\theta(t)}^{t} p(w)\mathrm{e}^{\int_{\tau(w)}^{\theta(t)} Q_1(w_1)p(w_1)} dw_1$$

and

$$I_1(t) = \frac{\int_t^{\theta^{-1}(t)} p(w) \int_{\tau(w)}^{t} p(w_1)\mathrm{e}^{\int_{\tau(w_1)}^{\theta(t)} p(w_2)Q_1(w_2)dw_2} dw_1 dw}{1 - \int_t^{\theta^{-1}(t)} p(w)dw}.$$

Therefore,

$$I_1(r_l) \ge \frac{\int_{d_l+5}^{d_l+6} p(w) \int_{w-1}^{d_l+5} p(w_1)\mathrm{e}^{\int_{w_1-1}^{d_l+4} p(w_2)dw_2} dw_1 dw}{1 - \int_t^{t+1} p(w)dw} = \frac{\mathrm{e}^{\gamma} - \gamma - 1}{1-\gamma}.$$

Also

$$
\begin{aligned}
I(r_l) &\geq \int_{\theta(d_l+5)}^{d_l+5} p(w)\exp\left(\int_{w-1}^{\theta(d_l+5)} \frac{p(w_1)\exp\left(\int_{w_1-1}^{w_1-1} p(w_2)dw_2\right)}{1-\int_{w_1-1}^{w_1} p(w_2)\exp\left(\int_{w_2-1}^{w_1-1} p(w_3)dw_3\right)dw_2} dw_1\right) dw \\
&= \int_{d_l+4}^{d_l+5} \gamma \exp\left(\int_{w-1}^{d_l+4} \frac{\gamma}{1-\int_{w_1-1}^{w_1} \gamma\exp\left(\int_{w_2-1}^{w_1-1} \gamma dw_3\right)dw_2} dw_1\right) dw \\
&= \left(\mathrm{e}^{\frac{\gamma(\mathrm{e}^{\gamma}-1)}{-2+\mathrm{e}^{\gamma}}} - 2\,\mathrm{e}^{\frac{\gamma}{-2+\mathrm{e}^{\gamma}}} - \mathrm{e}^{\gamma} + 2\right)\mathrm{e}^{-\frac{\gamma}{-2+\mathrm{e}^{\gamma}}}.
\end{aligned}
$$

Therefore,

$$
I(r_l) + I_1(r_l) \geq \left(\mathrm{e}^{\frac{\gamma(\mathrm{e}^{\gamma}-1)}{-2+\mathrm{e}^{\gamma}}} - 2\,\mathrm{e}^{\frac{\gamma}{-2+\mathrm{e}^{\gamma}}} - \mathrm{e}^{\gamma} + 2\right)\mathrm{e}^{-\frac{\gamma}{-2+\mathrm{e}^{\gamma}}} + \frac{\mathrm{e}^{\gamma}-\gamma-1}{1-\gamma} > 1.00054,
$$

for $\gamma = 0.4488$. Then, according to Theorem 2.6, every solution of Eq. (3.2) is oscillatory for $\gamma = 0.4488$. Observe, however, that $\delta = 0$ and

$$
\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(w)dw = \lim_{l\to\infty} \int_{\tau(d_l+5)}^{d_l+5} p(w)dw = \gamma.
$$

That is, none of the results in [12–14, 18, 20, 30–32] can be applied to Eq. (3.2) with $\gamma < 1$.

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