Hyers-Ulam stability of a certain Fredholm integral equation

Alberto SIMÕES¹ ², Ponmana SELVAN³

¹CMA-UBI – Center of Mathematics and Applications, Department of Mathematics, University of Beira Interior, Covilhã, Portugal,
²CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro, Portugal, ORCID iD: https://orcid.org/0000-0002-4772-4300
³Department of Mathematics, Sri Sai Ram Institute of Technology, Tamil Nadu, Chennai, India, https://orcid.org/0000-0002-6594-4913

Abstract: In this paper, by using Fixed point Theorem we establish the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of certain homogeneous Fredholm Integral equation of the second kind

\[ \varphi(x) = \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \]

and the non-homogeneous equation

\[ \varphi(x) = x + \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \]

for all \( x \in [0,1] \) and \( 0 < \lambda < \frac{2}{5} \).

Key words: Hyers-Ulam Stability, Hyers-Ulam-Rassias stability, Fredholm Integral equation of second kind, Fixed point Theorem.

1. Introduction

The Ulam stability problem for various functional equation was initiated by S.M. Ulam [31] in 1940. Then, in the next year, D.H. Hyers [16] was solved the Ulam problem for Cauchy additive functional equation on Banach spaces. After that Aoki [3], Bourgin [6] and Rassias [25] have generalized the Hyers result. These days the Hyers-Ulam stability for different functional equations was proved by many mathematicians (see [4, 5, 11, 26]). A generalization Ulam problem was recently proposed by replacing functional equations with differential equations. In 1998, Alsina et al., [1] was proved the Hyers-Ulam stability of differential equation of first order of the form \( y'(t) = y(t) \). This result was generalized by Takahasi [30] for Banach space valued differential equation \( y'(t) = \lambda y(t) \). Then several researchers have studied the Hyers-Ulam stability of differential equations in various directions, for example (see [7, 10, 17–24, 29, 32]).

Now a days, the Hyers-Ulam stability of integral equations has been given attention. In 2015, L. Hua et al., [15] studied the Hyers-Ulam stability of some kinds of Fredholm integral equations. Also, in 2015, Z. Gu

Correspondence: asimoes@ubi.pt

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and J. Huang [14] are investigated the Hyers-Ulam stability of the Fredholm integral equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,s) \varphi(s) \, ds$$

by fixed point Theorem. Recently, only few authors are investigating the Hyers-Ulam stability of the various integral equations (see [2, 8, 9, 12, 13, 27, 28]). Motivated by the above ideas, our foremost aim is to study the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the certain Fredholm Integral equations of second kind

$$\varphi(x) = \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \quad (1.1)$$

and

$$\varphi(x) = x + \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \quad (1.2)$$

for all $x \in [0, 1]$ and $0 < \lambda < \frac{2}{5}$ in the sense of Z. Gu and J. Huang [14].

2. Preliminaries

The following Theorems and Definitions are very useful to prove our main results.

Theorem 2.1 (Fixed Point Theorem) Let $(X, \rho)$ be a complete metric space. Assume that $T : X \to X$ is a strictly contractive operator with $\rho(Tx, Ty) \leq \theta \rho(x, y)$ where $0 < \theta < 1$. Then

(i) there exists an unique fixed point $x^*$ of $T$;

(ii) the sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges to the fixed point $x^*$ of $T$.

Theorem 2.2 (Hölder’s Inequality) Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1, x \in L^p(E)$ and $y \in L^q(E)$. Then $xy \in L(E)$ and

$$\int_E |x(t)y(t)| \, dt \leq \left( \int_E |x^p(t)| \, dt \right)^{\frac{1}{p}} \left( \int_E |y^q(t)| \, dt \right)^{\frac{1}{q}}.$$

Now, we give the definition of Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the Fredholm integral equations (1.1) and (1.2).

Definition 2.3 We say that the Fredholm integral equations (1.1) has the Hyers-Ulam stability, if there exists a real constant $S$ which satisfies the following conditions: For every $\epsilon > 0$, and for each solution $\varphi : [0, 1] \to \mathbb{R}$ satisfying the inequation

$$\left| \varphi(x) - \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \right| \leq \epsilon,$$

then there is some $\psi : [0, 1] \to \mathbb{R}$ satisfying the integral equation (1.1) such that

$$|\varphi(x) - \psi(x)| \leq S \epsilon, \quad \forall x \in [0, 1].$$
Definition 2.4 We say that the Fredholm integral equations (1.2) has the Hyers-Ulam stability, if there exists a real constant $S$ which satisfies the following conditions: For every $\epsilon > 0$, and for each solution $\varphi : [0, 1] \to \mathbb{R}$ satisfying the inequality

$$\left| \varphi(x) - x - \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \right| \leq \epsilon,$$

then there exists a solution $\psi : [0, 1] \to \mathbb{R}$ satisfies the integral equation (1.2) such that

$$|\varphi(x) - \psi(x)| \leq S \epsilon, \quad \forall \ x \in [0, 1].$$

Definition 2.5 The Fredholm integral equations (1.1) is said to have the Hyers-Ulam-Rassias stability, if there exists a real constant $S$ which fulfill the following: For every $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$, and for each solution $\varphi : [0, 1] \to \mathbb{R}$ satisfying the inequality

$$\left| \varphi(x) - x - \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \right| \leq \theta(x),$$

then there is a solution $\psi : [0, 1] \to \mathbb{R}$ satisfying the integral equation (1.1) such that

$$|\varphi(x) - \psi(x)| \leq S \theta(x), \quad \forall \ x \in [0, 1].$$

Definition 2.6 We say that the Fredholm integral equations (1.2) has the Hyers-Ulam-Rassias stability, if there exists a real constant $S$ which fulfill the following properties: For every $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$, and for each solution $\varphi : [0, 1] \to \mathbb{R}$ satisfying the inequation

$$\left| \varphi(x) - x - \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \right| \leq \theta(x),$$

then there exists some $\psi : [0, 1] \to \mathbb{R}$ satisfying the integral equation (1.2) such that

$$|\varphi(x) - \psi(x)| \leq S \theta(x), \quad \forall \ x \in [0, 1].$$

3. Main Results

In this section, we are going to prove the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the homogeneous and non-homogeneous Fredholm integral equations of second kind (1.1) and (1.2) with $\lambda < \frac{5}{8}$.

First, we investigate the two stabilities of the homogeneous Fredholm integral equation of second kind (1.1).

Theorem 3.1 Consider $H$ a fixed real number such that $H \geq \frac{5}{2}$ and $\lambda H < 1$. Let $\varphi : [0, 1] \to \mathbb{R}$ a continuous function and the kernel $K : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by $K(x, t) = 1 + x + t$. If $\varphi$ is such that

$$\left| \varphi(x) - x - \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \right| \leq \epsilon,$$  \hspace{1cm} (3.1)

then there exists a solution $\psi : [0, 1] \to \mathbb{R}$ of the Fredholm integral equation (1.1) and a real constant $S$ such that $|\varphi(x) - \psi(x)| \leq S \epsilon$ for all $x \in [0, 1]$. 

\hspace{1cm}
Proof  Firstly, we define an operator $T$ by,

$$
(T\varphi)(x) = \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt, \quad \varphi \in L^2([0, 1]).
$$

(3.2)

We have for each $x \in [0, 1],$

$$
\left| \int_0^1 (1 + x + t) \, dt \right| \leq H \quad \text{and} \quad \left| \int_0^1 \int_0^1 (1 + x + t)^2 \, dt \, dx \right|^{\frac{1}{2}} \leq H,
$$

for any $H \geq \frac{5}{2}.$

Now, we define a metric $\rho$ as follows,

$$
\rho(\varphi_1, \varphi_2) = \left\{ \left( \int_0^1 \left| \frac{\varphi_1(x) - \varphi_2(x)}{\lambda H} \right|^2 \, dx \right)^{\frac{1}{2}} : \varphi_1, \varphi_2 \in L^2([0, 1]), \lambda H < 1 \right\}.
$$

By using the Hölder’s inequality, we obtain that

$$
\int_0^1 \left( \int_0^1 (1 + x + t) \varphi(t) \, dt \right)^2 \, dx \leq \int_0^1 \left( \int_0^1 (1 + x + t)^2 \, dt \right) \left( \int_0^1 \varphi^2(t) \, dt \right) \, dx
$$

$$
\leq \int_0^1 \varphi^2(t) \, dt \int_0^1 \int_0^1 (1 + x + t)^2 \, dt \, dx < \infty.
$$

This implies that $T\varphi \in L^2([0, 1])$ and $T$ is a self–mapping of $L^2([0, 1])$. Thus, the solution of the equation (3.2) is the fixed point of $T$. So,

$$
\rho(T\varphi_1, T\varphi_2) = \left( \int_0^1 \left| \frac{(T\varphi_1)(x) - (T\varphi_2)(x)}{\lambda H} \right|^2 \, dx \right)^{\frac{1}{2}}
$$

$$
= \frac{1}{H} \left( \int_0^1 \left| \int_0^1 (1 + x + t) (\varphi_1(t) - \varphi_2(t)) \, dt \right|^2 \, dx \right)^{\frac{1}{2}}
$$

$$
\leq \frac{1}{H} \left( \int_0^1 \int_0^1 (1 + x + t)^2 \, dt \, dx \right)^{\frac{1}{2}} \left( \int_0^1 |\varphi_1(t) - \varphi_2(t)|^2 \, dt \right)^{\frac{1}{2}}
$$

$$
\leq \left( \int_0^1 |\varphi_1(t) - \varphi_2(t)|^2 \, dt \right)^{\frac{1}{2}}
$$

$$
= \lambda H \left( \int_0^1 \left| \frac{\varphi_1(t) - \varphi_2(t)}{\lambda H} \right|^2 \, dt \right)^{\frac{1}{2}}
$$

$$
= \lambda H \rho(\varphi_1, \varphi_2).
$$

Since $\lambda H < 1$, $T$ is a strictly contractive operator. Then by Theorem 2.1 the equation (3.2) has a unique solution $\varphi^* \in L^2([0, 1])$, where $\varphi^* = \lim_{r \to \infty} \varphi_r$ for

$$
\varphi_r(x) = \lambda \int_0^1 (1 + x + t) \varphi_{r-1}(t) \, dt
$$
Let $\psi \in L^2([0,1])$ be a solution of inequality (3.1) and

$$
\psi(x) - \lambda \int_0^1 (1 + x + t) \psi(t) \, dt =: h(x).
$$

(3.3)

Obviously, we have $|h(x)| < \epsilon$ for all $x \in [0,1]$. Then we can conclude that the solution of equation

$$
\psi(x) = h(x) + \lambda \int_0^1 (1 + x + t) \psi(t) \, dt
$$

is $\psi^* \in L^2([0,1])$, where $\psi^* = \lim_{r \to \infty} \psi_r$ for

$$
\psi_r(x) = h(x) + \lambda \int_0^1 (1 + x + t) \psi_{r-1}(t) \, dt
$$

and $\psi_0 \in L^2([0,1])$ is an arbitrary function.

For $\varphi_0(x) = \psi_0(x) = 0$, we get,

$$
|\varphi_1(x) - \psi_1(x)| = |h(x)| < \epsilon,
$$

$$
|\varphi_2(x) - \psi_2(x)| = \left| h(x) + \lambda \int_0^1 (1 + x + t) (\psi_1(t) - \varphi_1(t)) \, dt \right| \leq \epsilon \left( 1 + \lambda \int_0^1 |1 + x + t| \, dt \right)
$$

$$
|\varphi_3(x) - \psi_3(x)| = \left| h(x) + \lambda \int_0^1 (1 + x + t_2) (\psi_2(t_2) - \varphi_2(t_2)) \, dt_2 \right|
$$

$$
\leq \epsilon + \epsilon \lambda \int_0^1 |1 + x + t_2| \left( 1 + \lambda \int_0^1 |1 + t_2 + t_1| \, dt_1 \right) \, dt_2
$$

$$
\leq \epsilon \left( 1 + \lambda \int_0^1 |1 + x + t_2| \, dt_2 + \lambda^2 \int_0^1 |1 + x + t_2| \int_0^1 |1 + t_2 + t_1| \, dt_1 \, dt_2 \right)
$$

$$
\cdots \cdots
$$

$$
|\varphi_r(x) - \psi_r(x)| = \left| h(x) + \lambda \int_0^1 (1 + x + t) (\psi_{r-1}(x) - \varphi_{r-1}(x)) \, dt \right|
$$

$$
\leq \epsilon \left( 1 + \lambda \int_0^1 |1 + x + t_{r-1}| \, dt_{r-1} \right)
$$

$$
+ \lambda^2 \int_0^1 |1 + x + t_{r-1}| \int_0^1 |1 + t_{r-1} + t_{r-2}| \, dt_{r-2} \, dt_{r-1} + \cdots
$$

$$
\cdots + \lambda^{r-1} \int_0^1 |1 + x + t_{r-1}| \int_0^1 |1 + t_{r-1} + t_{r-2}| \int_0^1 |1 + t_{r-2} + t_{r-3}| \cdots
$$

$$
\cdots \int_0^1 |1 + t_2 + t_1| \, dt_1 \cdots dt_{r-3} dt_{r-2} dt_{r-1}
$$

$$
\leq \epsilon \left( 1 + \lambda H + (\lambda H)^2 + \cdots + (\lambda H)^{r-1} \right) = \epsilon \left( \frac{1 - (\lambda H)^r}{1 - \lambda H} \right),
$$
as \( r \to \infty \), we obtain
\[
|\varphi^*(x) - \psi^*(x)| \leq \frac{1}{1 - \lambda H} \epsilon.
\]

Let us choose \( S = \frac{1}{1 - \lambda H} \), hence \( |\varphi^*(x) - \psi^*(x)| \leq S \epsilon \), and \( 0 < \lambda H < 1 \), where \( S \) is the Hyers-Ulam stability constant for (1.1). Hence, by the virtue of Definition 2.3 the Fredholm integral equation (1.1) has the Hyers-Ulam stability.

The following theorem shows the Hyers-Ulam-Rassias stability of the homogeneous Fredholm integral equation of second kind (1.1).

**Theorem 3.2** Consider \( H \) a fixed real number such that \( H \geq \frac{5}{2} \) and \( \lambda H < 1 \). Let \( \varphi : [0, 1] \to \mathbb{R} \) a continuous function and the kernel \( K : [0, 1] \times [0, 1] \to \mathbb{R} \) defined by \( K(x, t) = 1 + x + t \) such that
\[
\int_0^1 |1 + x + t| \theta(t) dt \leq \theta(x) \int_0^1 |1 + x + t| dt,
\]
for all \( x \in [0, 1] \), where \( \theta \in C([0, 1]) \). If \( \varphi \) is such that
\[
|\varphi(x) - \lambda \int_0^1 (1 + x + t) \varphi(t) dt| \leq \theta(x), \tag{3.4}
\]
then there exists a solution \( \psi : [0, 1] \to \mathbb{R} \) of the Fredholm integral equation (1.1) and a real constant \( S \) such that \( |\varphi(x) - \psi(x)| \leq S \theta(x) \) for all \( x \in [0, 1] \).

**Proof** By a similar procedure to the previous we define a strictly contractive operator \( T \) as in (3.2) since \( \lambda H < 1 \). By (3.3) we have \(|h(x)| \leq \theta(x)\) for all \( x \in [0, 1] \). As in the previous proof, for \( \varphi_0(x) = \psi_0(x) = 0 \), we get,
\[
|\varphi_1(x) - \psi_1(x)| = |h(x)| \leq \theta(x),
\]
\[
|\varphi_2(x) - \psi_2(x)| = |h(x) + \lambda \int_0^1 (1 + x + t) \varphi_1(t) dt| \leq \theta(x) \left( 1 + \lambda \int_0^1 |1 + x + t| dt \right)
\]
\[
|\varphi_3(x) - \psi_3(x)| = |h(x) + \lambda \int_0^1 (1 + x + t_2) \varphi_2(t_2) dt_2| \leq \theta(x) + \theta(x) \lambda \int_0^1 |1 + x + t_2| \left( 1 + \lambda \int_0^1 |1 + t_2 + t_1| dt_1 \right) dt_2
\]
\[
\leq \theta(x) \left( 1 + \lambda \int_0^1 |1 + x + t_2| dt_2 + \lambda^2 \int_0^1 |1 + x + t_2| \int_0^1 |1 + t_2 + t_1| dt_1 dt_2 \right)
\]
\[
\cdots.
\]
\[ |\varphi_r(x) - \psi_r(x)| = \left| h(x) + \lambda \int_0^1 (1 + x + t) (\psi_{r-1}(x) - \varphi_{r-1}(x)) \, dt \right| \]
\[ \leq \theta(x) \left( 1 + \lambda \int_0^1 |1 + x + t_r| dt_r \right. \]
\[ + \lambda^2 \int_0^1 |1 + x + t_{r-1}| \int_0^1 |1 + t_{r-1} + t_{r-2}| dt_{r-2} dt_{r-1} + \cdots \]
\[ \cdots + \lambda^{r-1} \int_0^1 |1 + x + t_{r-1}| \int_0^1 |1 + t_{r-1} + t_{r-2}| \int_0^1 |1 + t_{r-2} + t_{r-3}| \cdots \]
\[ \left. \cdots \int_0^1 |1 + t_2 + t_1| dt_1 \cdots dt_{r-3} dt_{r-2} dt_{r-1} \right) \]
\[ \leq \theta(x) \left( 1 + \lambda H + (\lambda H)^2 + \cdots + (\lambda H)^{r-1} \right) = \theta(x) \left( \frac{1 - (\lambda H)^r}{1 - \lambda H} \right), \]
as \( r \to \infty \), we obtain
\[ |\varphi^*(x) - \psi^*(x)| \leq \frac{1}{1 - \lambda H} \theta(x) \]
for all \( x \in [0, 1] \). Let us choose \( S = \frac{1}{1 - \lambda H} \), hence \( |\varphi^*(x) - \psi^*(x)| \leq S \theta(x) \), and \( 0 < \lambda H < 1 \). Hence, by the virtue of Definition 2.5 the Fredholm integral equation (1.1) has the Hyers-Ulam-Rassias stability.

Now, we are going to establish the Hyers-Ulam stability of the non-homogeneous Fredholm integral equation of second kind (1.2).

**Theorem 3.3** Consider \( H \) a fixed real number such that \( H \geq \frac{5}{2} \) and \( \lambda H < 1 \). Let \( \varphi : [0, 1] \to \mathbb{R} \) a continuous function and the kernel \( K : [0, 1] \times [0, 1] \to \mathbb{R} \) defined by \( K(x, t) = 1 + x + t \). If \( \varphi \) is such that
\[
\left| \varphi(x) - x - \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt \right| \leq \epsilon, \tag{3.5}
\]
where \( \epsilon \geq 0 \) then there exists a solution \( \psi : [0, 1] \to \mathbb{R} \) of the non-homogeneous Fredholm integral equation (1.2) and a real constant \( S \) such that \( |\varphi(x) - \psi(x)| \leq S \epsilon \) for all \( x \in [0, 1] \).

**Proof** Let us define an operator \( T \) as
\[
(T\varphi)(x) = x + \lambda \int_0^1 (1 + x + t) \varphi(t) \, dt, \quad \varphi \in L^2([0, 1]). \tag{3.6}
\]
We have \( T\varphi \in L^2([0, 1]) \) and \( T \) a self–mapping of \( L^2([0, 1]) \). The solution of the equation (3.6) is the fixed point of the strictly contractive operator \( T \) since \( \lambda H < 1 \). By Theorem 2.1 the equation (3.6) has a unique solution \( \varphi^* \in L^2([0, 1]) \), where \( \varphi^* = \lim_{r \to \infty} \varphi_r \) for
\[
\varphi_r(x) = x + \lambda \int_0^1 (1 + x + t) \varphi_{r-1}(t) \, dt
\]
and \( \varphi_0 \in L^2([0,1]) \) is an arbitrary function.

Let \( \psi \in L^2([0,1]) \) be a solution of inequality (4) and

\[
\psi(x) - x - \lambda \int_0^1 (1 + x + t) \psi(t) \, dt =: h(x).
\]

We have \( |h(x)| \leq \epsilon \) for all \( x \in [0,1] \). Then we can conclude that the solution of equation

\[
\psi(x) = h(x) + x + \lambda \int_0^1 (1 + x + t) \psi(t) \, dt
\]

is \( \psi^* \in L^2([0,1]) \), where \( \psi^* = \lim_{r \to \infty} \psi_r \) for

\[
\psi_r(x) = h(x) + x + \lambda \int_0^1 (1 + x + t) \psi_{r-1}(t) \, dt
\]

and \( \psi_0 \in L^2([0,1]) \) is an arbitrary function.

For \( \varphi_0(x) = \psi_0(x) = 0 \), we get,

\[
|\varphi_r(x) - \psi_r(x)| = \left| h(x) + \lambda \int_0^1 (1 + x + t) (\psi_{r-1}(x) - \varphi_{r-1}(x)) \, dt \right|
\]

\[
\leq \epsilon \left( 1 + \lambda \int_0^1 |1 + x + t_{r-1}| \, dt_{r-1} + \lambda^2 \int_0^1 |1 + x + t_{r-1}| \int_0^1 |1 + t_{r-1} + t_{r-2}| \, dt_{r-2} \, dt_{r-1} + \cdots 
\right.
\]

\[
\cdots + \lambda^{r-1} \int_0^1 |1 + x + t_{r-1}| \left( \int_0^1 |1 + t_{r-1} + t_{r-2}| \, dt_{r-2} \right) \, dt_{r-3} \cdots 
\]

\[
\left. \cdots \int_0^1 |1 + t_2 + t_1| \, dt_1 \cdots dt_{r-3} \, dt_{r-2} \, dt_{r-1} \right)
\]

\[
\leq \epsilon \left( 1 + \lambda H + (\lambda H)^2 + \cdots + (\lambda H)^{r-1} \right) = \epsilon \left( 1 - (\lambda H)^r \right)
\]

as \( r \to \infty \), we obtain

\[
|\varphi^*(x) - \psi^*(x)| \leq \frac{1}{1 - \lambda H} \epsilon.
\]

Let us choose \( S = \frac{1}{1 - \lambda H} \), hence \( |\varphi^*(x) - \psi^*(x)| \leq S \epsilon \), and \( 0 < \lambda H < 1 \), where \( S \) is the Hyers-Ulam stability constant for (1.2). Hence, by the virtue of Definition 2.4 the non-homogeneous Fredholm integral equation (1.2) has the Hyers-Ulam stability. \( \square \)

Finally, the following corollary proves the Hyers-Ulam-Rassias stability of the non-homogeneous Fredholm integral equation of second kind (1.2).
Corollary 3.4 Consider $H$ a fixed real number such that $H \geq \frac{5}{2}$ and $\lambda H < 1$. Let $\varphi : [0, 1] \rightarrow \mathbb{R}$ a continuous function and the kernel $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ defined by $K(x, t) = 1 + x + t$ such that

$$
\int_0^1 |1 + x + t| \theta(t) dt \leq \theta(x) \int_0^1 |1 + x + t| dt,
$$

for all $x \in [0, 1]$, where $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$. If $\varphi$ is such that

$$
\left| \varphi(x) - x - \lambda \int_0^1 (1 + x + t) \varphi(t) dt \right| \leq \theta(x),
$$

(3.7)

then there exists a solution $\psi : [0, 1] \rightarrow \mathbb{R}$ of the non-homogeneous Fredholm integral equation (1.2) and a real constant $S$ such that $|\varphi(x) - \psi(x)| \leq S \theta(x)$ for all $x \in [0, 1]$.

4. Examples

In order to illustrate our results we will present some examples.

Let us consider the non-homogeneous Fredholm integral equation of second kind (1.2) defined by

$$
\varphi(x) = x + \lambda \int_0^1 (1 + x + t) \varphi(t) dt
$$

for all $x \in [0, 1]$ and $\lambda = \frac{1}{5}$. Let $H = \frac{13}{5}$ and the perturbation of the solution $\varphi(x) = \frac{587}{500} x + \frac{28}{100}$.

We realize that all conditions of Theorem 3.3 are satisfied. In fact $\lambda H = \frac{13}{50} < 1$ and $\varphi$ is a continuous function such that

$$
\left| \varphi(x) - x - \lambda \int_0^1 (1 + x + t) \varphi(t) dt \right| = \left| \frac{3}{5000} x + \frac{1}{300} \right| \leq \frac{7}{7500} := \epsilon.
$$

(4.1)

By the exact solution $\psi(x) = \frac{210}{179} x + \frac{50}{179}$, we realize that

$$
|\varphi(x) - \psi(x)| = \left| \frac{73}{89500} x + \frac{3}{4475} \right| \leq \frac{1}{1 - \lambda H} \epsilon = \frac{7}{3600}.
$$

To illustrate the inequality (4.1), we have the Figure 1.

Let us consider the same non-homogeneous Fredholm integral equation of second kind (1.2) but now with $\lambda = \frac{1}{100}$. Let $H = 3$ and the perturbation of the solution $\varphi(x) = \frac{10052}{10000} x + \frac{851}{100000}$. We have $\lambda H = \frac{3}{100} < 1$ and $\varphi$ a continuous function such that

$$
\left| \varphi(x) - x - \lambda \int_0^1 (1 + x + t) \varphi(t) dt \right| = \left| \frac{5334}{60000000} x + \frac{341}{6000000} \right| \leq \frac{227}{2400000} := \epsilon.
$$

(4.2)

By the exact solution $\psi(x) = \frac{118200}{117509} x + \frac{1000}{117509}$, we realize that

$$
|\varphi(x) - \psi(x)| = \left| \frac{26287}{293997500} x + \frac{76749}{11759900000} \right| \leq \frac{1}{1 - \lambda H} \epsilon = \frac{227}{2328000}.
$$

(4.2)
If we consider $H = 30$, we get a worse result but still acceptable. We get,

$$|\varphi(x) - \psi(x)| = \left| \frac{26287}{293997500} x + \frac{76749}{11759900000} \right| \leq \frac{1 \cdot 227}{1680000} = \frac{227}{1680000}. \quad (4.3)$$

So we have that the non-homogeneous Fredholm integral equation of second kind (1.2) has the Hyers-Ulam stability.

To illustrate the inequalities (4.2) and (4.3), we have the Figure 2.

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References


