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Hyers-Ulam stability of a certain Fredholm integral equation

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Abstract: In this paper, by using fixed point theorem we establish the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of certain homogeneous Fredholm Integral equation of the second kind

$$\varphi(x) = \lambda \int_0^1 (1 + x + t)\, \varphi(t)\, dt$$

and the nonhomogeneous equation

$$\varphi(x) = x + \lambda \int_0^1 (1 + x + t)\, \varphi(t)\, dt$$

for all $x \in [0, 1]$ and $0 < \lambda < \frac{2}{5}$.

Key words: Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Fredholm integral equation of second kind, fixed point theorem

1. Introduction

The Ulam stability problem for various functional equation was initiated by S.M. Ulam [31] in 1940. Then, in the next year, D.H. Hyers [16] solved the Ulam problem for Cauchy additive functional equation on Banach spaces. After that Aoki [3], Bourgin [6] and Rassias [25] have generalized the Hyers result. These days the Hyers-Ulam stability for different functional equations was proved by many mathematicians (see [4, 5, 11, 26]). A generalization Ulam problem was recently proposed by replacing functional equations with differential equations. In 1998, Alsina et al., [1] proved the Hyers-Ulam stability of differential equation of first order of the form $y'(t) = y(t)$. This result was generalized by Takahasi [30] for Banach space valued differential equation $y'(t) = \lambda y(t)$. Then several researchers have studied the Hyers-Ulam stability of differential equations in various directions, for example (see [7, 10, 17–24, 29, 32]).

Nowadays, the Hyers-Ulam stability of integral equations has been given attention. In 2015, L. Hua et al., [15] studied the Hyers-Ulam stability of some kinds of Fredholm integral equations. Also, in 2015, Z. Gu

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and J. Huang [14] investigated the Hyers-Ulam stability of the Fredholm integral equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,s)\,\varphi(s)\,ds$$

by fixed point theorem. Recently, only few authors are investigating the Hyers-Ulam stability of the various integral equations (see [2, 8, 9, 12, 13, 27, 28]). Motivated by the above ideas, our foremost aim is to study the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the certain Fredholm integral equations of second kind

$$\varphi(x) = \lambda \int_0^1 (1+x+t)\,\varphi(t)\,dt \tag{1.1}$$

and

$$\varphi(x) = x + \lambda \int_0^1 (1+x+t)\,\varphi(t)\,dt \tag{1.2}$$

for all $x \in [0,1]$ and $0 < \lambda < \frac{2}{5}$ in the sense of Z. Gu and J. Huang [14].

2. Preliminaries

The following theorems and definitions are very useful to prove our main results.

Theorem 2.1 *(fixed point theorem) Let (X,ρ) be a complete metric space. Assume that $T : X \to X$ is a strictly contractive operator with $\rho(Tx,Ty) \leq \theta\ \rho(x,y)$ where $0 < \theta < 1$. Then*

(i) there exists an unique fixed point x^* of T;

(ii) the sequence $\{T^n\ x\}_{n\in\mathbb{N}}$ converges to the fixed point x^* of T.

Theorem 2.2 *(Hölder's inequality) Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in L^p(E)$ and $y \in L^q(E)$. Then $xy \in L(E)$ and*

$$\int_E |x(t)y(t)|\ dt \leq \left(\int_E |x^p(t)|\ dt\right)^{\frac{1}{p}} \left(\int_E |y^q(t)|\ dt\right)^{\frac{1}{q}}.$$

Now, we give the definition of Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the Fredholm integral equations (1.1) and (1.2).

Definition 2.3 *We say that the Fredholm integral equations (1.1) has the Hyers-Ulam stability, if there exists a real constant S which satisfies the following conditions: For every $\epsilon > 0$, and for each solution $\varphi : [0,1] \to \mathbb{R}$ satisfying the inequation*

$$\left|\varphi(x) - \lambda \int_0^1 (1+x+t)\,\varphi(t)\,dt\right| \leq \epsilon,$$

then there is some $\psi : [0,1] \to \mathbb{R}$ satisfying the integral equation (1.1) such that

$$|\varphi(x) - \psi(x)| \leq S\,\epsilon, \quad \forall\, x \in [0,1].$$

Definition 2.4 *We say that the Fredholm integral equations (1.2) have the Hyers-Ulam stability, if there exists a real constant S which satisfies the following conditions: For every $\epsilon > 0$, and for each solution $\varphi : [0,1] \to \mathbb{R}$ satisfying the inequality*

$$\left|\varphi(x) - x - \lambda \int_0^1 (1 + x + t)\, \varphi(t)\, dt\right| \leq \epsilon,$$

then there exists a solution $\psi : [0,1] \to \mathbb{R}$ satisfies the integral equation (1.2) such that

$$|\varphi(x) - \psi(x)| \leq S\, \epsilon, \quad \forall\, x \in [0,1].$$

Definition 2.5 *The Fredholm integral equations (1.1) are said to have the Hyers-Ulam-Rassias stability, if there exists a real constant S which fulfills the following: For every $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$, and for each solution $\varphi : [0,1] \to \mathbb{R}$ satisfying the inequality*

$$\left|\varphi(x) - \lambda \int_0^1 (1 + x + t)\, \varphi(t)\, dt\right| \leq \theta(x),$$

then there is a solution $\psi : [0,1] \to \mathbb{R}$ satisfying the integral equation (1.1) such that

$$|\varphi(x) - \psi(x)| \leq S\, \theta(x), \quad \forall\, x \in [0,1].$$

Definition 2.6 *We say that the Fredholm integral equations (1.2) have the Hyers-Ulam-Rassias stability, if there exists a real constant S which fulfills the following properties: For every $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$, and for each solution $\varphi : [0,1] \to \mathbb{R}$ satisfying the inequation*

$$\left|\varphi(x) - x - \lambda \int_0^1 (1 + x + t)\, \varphi(t)\, dt\right| \leq \theta(x),$$

then there exists some $\psi : [0,1] \to \mathbb{R}$ satisfying the integral equation (1.2) such that

$$|\varphi(x) - \psi(x)| \leq S\, \theta(x), \quad \forall\, x \in [0,1].$$

3. Main results

In this section, we are going to prove the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the homogeneous and nonhomogeneous Fredholm integral equations of second kind (1.1) and (1.2) with $\lambda < \frac{2}{5}$. First, we investigate the two stabilities of the homogeneous Fredholm integral equation of second kind (1.1).

Theorem 3.1 *Consider H a fixed real number such that $H \geq \frac{5}{2}$ and $\lambda H < 1$. Let $\varphi : [0,1] \to \mathbb{R}$ a continuous function and the kernel $K : [0,1] \times [0,1] \to \mathbb{R}$ defined by $K(x,t) = 1 + x + t$. If φ is such that*

$$\left|\varphi(x) - \lambda \int_0^1 (1 + x + t)\, \varphi(t)\, dt\right| \leq \epsilon, \tag{3.1}$$

where $\epsilon \geq 0$ then there exists a solution $\psi : [0,1] \to \mathbb{R}$ of the Fredholm integral equation (1.1) and a real constant S such that $|\varphi(x) - \psi(x)| \leq S\, \epsilon$ for all $x \in [0,1]$.

Proof Firstly, we define an operator T by,

$$(T\varphi)(x) = \lambda \int_0^1 (1+x+t)\,\varphi(t)\,dt, \quad \varphi \in L^2([0,1]). \tag{3.2}$$

We have for each $x \in [0,1]$,

$$\left|\int_0^1 (1+x+t)\,dt\right| \le H \quad \text{and} \quad \left|\left(\int_0^1\int_0^1 (1+x+t)^2\,dtdx\right)^{\frac{1}{2}}\right| \le H,$$

for any $H \ge \frac{5}{2}$.

Now, we define a metric ρ as follows,

$$\rho(\varphi_1,\varphi_2) = \left\{\left(\int_0^1 \left|\frac{\varphi_1(x)-\varphi_2(x)}{\lambda H}\right|^2 dx\right)^{\frac{1}{2}} : \varphi_1,\varphi_2 \in L^2([0,1]),\, \lambda H < 1\right\}.$$

By using the Hölder's inequality, we obtain that

$$\begin{aligned}\int_0^1 \left|\int_0^1 (1+x+t)\,\varphi(t)\,dt\right|^2 dx &\le \int_0^1 \left(\int_0^1 (1+x+t)^2\,dt \int_0^1 \varphi^2(t)\,dt\right) dx \\ &\le \int_0^1 \varphi^2(t)\,dt \int_0^1\int_0^1 (1+x+t)^2\,dtdx < \infty.\end{aligned}$$

This implies that $T\varphi \in L^2([0,1])$ and T is a self–mapping of $L^2([0,1])$. Thus, the solution of the equation (3.2) is the fixed point of T. So,

$$\begin{aligned}\rho(T\varphi_1,T\varphi_2) &= \left(\int_0^1 \left|\frac{(T\varphi_1)(x)-(T\varphi_2)(x)}{\lambda H}\right|^2 dx\right)^{\frac{1}{2}} \\ &= \frac{1}{H}\left(\int_0^1 \left|\int_0^1 (1+x+t)\,(\varphi_1(t)-\varphi_2(t))\,dt\right|^2 dx\right)^{\frac{1}{2}} \\ &\le \frac{1}{H}\left(\int_0^1\int_0^1 (1+x+t)^2\,dtdx\right)^{\frac{1}{2}}\left(\int_0^1 |\varphi_1(t)-\varphi_2(t)|^2\,dt\right)^{\frac{1}{2}} \\ &\le \left(\int_0^1 |\varphi_1(t)-\varphi_2(t)|^2\,dt\right)^{\frac{1}{2}} \\ &= \lambda H\left(\int_0^1 \left|\frac{\varphi_1(t)-\varphi_2(t)}{\lambda H}\right|^2 dt\right)^{\frac{1}{2}} \\ &= \lambda H \rho(\varphi_1,\varphi_2).\end{aligned}$$

Since $\lambda H < 1$, T is a strictly contractive operator. Then by Theorem 2.1 the equation (3.2) has a unique solution $\varphi^* \in L^2([0,1])$, where $\varphi^* = \lim_{r\to\infty}\varphi_r$ for

$$\varphi_r(x) = \lambda \int_0^1 (1+x+t)\,\varphi_{r-1}(t)\,dt$$

and $\varphi_0 \in L^2([0,1])$ is an arbitrary function.

Let $\psi \in L^2([0,1])$ be a solution of inequality (3.1) and

$$\psi(x) - \lambda \int_0^1 (1+x+t)\,\psi(t)\,dt =: h(x). \tag{3.3}$$

Obviously, we have $|h(x)| \leq \epsilon$ for all $x \in [0,1]$. Then we can conclude that the solution of equation

$$\psi(x) = h(x) + \lambda \int_0^1 (1+x+t)\,\psi(t)\,dt$$

is $\psi^* \in L^2([0,1])$, where $\psi^* = \lim_{r\to\infty} \psi_r$ for

$$\psi_r(x) = h(x) + \lambda \int_0^1 (1+x+t)\,\psi_{r-1}(t)\,dt$$

and $\psi_0 \in L^2([0,1])$ is an arbitrary function.

For $\varphi_0(x) = \psi_0(x) = 0$, we get,

$$\begin{aligned}
|\varphi_1(x) - \psi_1(x)| &= |h(x)| \leq \epsilon,\\
|\varphi_2(x) - \psi_2(x)| &= \left|h(x) + \lambda \int_0^1 (1+x+t)(\psi_1(t) - \varphi_1(t))dt\right| \leq \epsilon\left(1 + \lambda \int_0^1 |1+x+t|\,dt\right)\\
|\varphi_3(x) - \psi_3(x)| &= \left|h(x) + \lambda \int_0^1 (1+x+t_2)(\psi_2(t_2) - \varphi_2(t_2))dt_2\right|\\
&\leq \epsilon + \epsilon\,\lambda \int_0^1 |1+x+t_2| \left(1 + \lambda \int_0^1 |1+t_2+t_1|dt_1\right) dt_2\\
&\leq \epsilon\left(1 + \lambda \int_0^1 |1+x+t_2|dt_2 + \lambda^2 \int_0^1 |1+x+t_2| \int_0^1 |1+t_2+t_1|dt_1 dt_2\right)\\
&\cdots\cdots\cdots
\end{aligned}$$

$$\begin{aligned}
|\varphi_r(x) - \psi_r(x)| &= \left|h(x) + \lambda \ \int_0^1 (1+x+t) \ (\psi_{r-1}(x) - \varphi_{r-1}(x)) \ dt\right|\\
&\leq \epsilon\Bigg(1 + \lambda \int_0^1 |1+x+t_{r-1}|dt_{r-1}\\
&\quad + \lambda^2 \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}|dt_{r-2}dt_{r-1} + \cdots\\
&\quad \cdots + \lambda^{r-1} \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}| \int_0^1 |1+t_{r-2}+t_{r-3}| \cdots\\
&\quad \cdots \int_0^1 |1+t_2+t_1|dt_1 \cdots dt_{r-3}dt_{r-2}dt_{r-1}\Bigg)\\
&\leq \epsilon\left(1 + \lambda H + (\lambda H)^2 + ... + (\lambda H)^{r-1}\right) = \epsilon\left(\frac{1-(\lambda H)^r}{1-\lambda H}\right),
\end{aligned}$$

as $r \to \infty$, we obtain

$$|\varphi^*(x) - \psi^*(x)| \leq \frac{1}{1-\lambda H}\,\epsilon.$$

Let us choose $S = \dfrac{1}{1-\lambda H}$, hence $|\varphi^*(x) - \psi^*(x)| \leq S\epsilon$, and $0 < \lambda H < 1$, where S is the Hyers-Ulam stability constant for (1.1). Hence, by the virtue of Definition 2.3 the Fredholm integral equation (1.1) has the Hyers-Ulam stability. □

The following theorem shows the Hyers-Ulam-Rassias stability of the homogeneous Fredholm integral equation of second kind (1.1).

Theorem 3.2 *Consider H a fixed real number such that $H \geq \frac{5}{2}$ and $\lambda H < 1$. Let $\varphi : [0,1] \to \mathbb{R}$ a continuous function and the kernel $K : [0,1] \times [0,1] \to \mathbb{R}$ defined by $K(x,t) = 1 + x + t$ such that*

$$\int_0^1 |1 + x + t|\theta(t)dt \leq \theta(x) \int_0^1 |1 + x + t|dt,$$

for all $x \in [0,1]$, where $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$. If φ is such that

$$\left|\varphi(x) - \lambda \int_0^1 (1 + x + t)\,\varphi(t)\,dt\right| \leq \theta(x), \tag{3.4}$$

then there exists a solution $\psi : [0,1] \to \mathbb{R}$ of the Fredholm integral equation (1.1) and a real constant S such that $|\varphi(x) - \psi(x)| \leq S\,\theta(x)$ for all $x \in [0,1]$.

Proof By a similar procedure to the previous we define a strictly contractive operator T as in (3.2) since $\lambda H < 1$. By (3.3) we have $|h(x)| \leq \theta(x)$ for all $x \in [0,1]$. As in the previous proof, for $\varphi_0(x) = \psi_0(x) = 0$, we get,

$$\begin{aligned}
|\varphi_1(x) - \psi_1(x)| &= |h(x)| \leq \theta(x), \\
|\varphi_2(x) - \psi_2(x)| &= \left|h(x) + \lambda \int_0^1 (1 + x + t)(\psi_1(t) - \varphi_1(t))dt\right| \leq \theta(x)\left(1 + \lambda \int_0^1 |1 + x + t|\,dt\right) \\
|\varphi_3(x) - \psi_3(x)| &= \left|h(x) + \lambda \int_0^1 (1 + x + t_2)(\psi_2(t_2) - \varphi_2(t_2))dt_2\right| \\
&\leq \theta(x) + \theta(x)\ \lambda \int_0^1 |1 + x + t_2| \left(1 + \lambda \int_0^1 |1 + t_2 + t_1|dt_1\right) dt_2 \\
&\leq \theta(x)\left(1 + \lambda \int_0^1 |1 + x + t_2|dt_2 + \lambda^2 \int_0^1 |1 + x + t_2| \int_0^1 |1 + t_2 + t_1|dt_1 dt_2\right) \\
&\cdots\cdots\cdots
\end{aligned}$$

$$
\begin{aligned}
|\varphi_r(x) - \psi_r(x)| &= \left| h(x) + \lambda \int_0^1 (1+x+t) \; (\psi_{r-1}(x) - \varphi_{r-1}(x)) \; dt \right| \\
&\le \theta(x) \Bigg(1 + \lambda \int_0^1 |1+x+t_{r-1}| dt_{r-1} \\
&\quad + \lambda^2 \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}| dt_{r-2} dt_{r-1} + \cdots \\
&\quad \cdots + \lambda^{r-1} \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}| \int_0^1 |1+t_{r-2}+t_{r-3}| \cdots \\
&\quad \cdots \int_0^1 |1+t_2+t_1| dt_1 \cdots dt_{r-3} dt_{r-2} dt_{r-1} \Bigg) \\
&\le \theta(x) \left(1 + \lambda H + (\lambda H)^2 + ... + (\lambda H)^{r-1}\right) = \theta(x) \left(\frac{1-(\lambda H)^r}{1-\lambda H} \right),
\end{aligned}
$$

as $r \to \infty$, we obtain

$$
|\varphi^*(x) - \psi^*(x)| \le \frac{1}{1-\lambda H}\, \theta(x)
$$

for all $x \in [0,1]$. Let us choose $S = \dfrac{1}{1-\lambda H}$, hence $|\varphi^*(x) - \psi^*(x)| \le S\theta(x)$, and $0 < \lambda H < 1$. Hence, by the virtue of Definition 2.5 the Fredholm integral equation (1.1) has the Hyers-Ulam-Rassias stability. □

Now, we are going to establish the Hyers-Ulam stability of the nonhomogeneous Fredholm integral equation of second kind (1.2).

Theorem 3.3 *Consider H a fixed real number such that $H \ge \frac{5}{2}$ and $\lambda H < 1$. Let $\varphi : [0,1] \to \mathbb{R}$ a continuous function and the kernel $K : [0,1] \times [0,1] \to \mathbb{R}$ defined by $K(x,t) = 1 + x + t$. If φ is such that*

$$
\left| \varphi(x) - x - \lambda \int_0^1 (1+x+t)\, \varphi(t)\, dt \right| \le \epsilon, \tag{3.5}
$$

where $\epsilon \ge 0$ then there exists a solution $\psi : [0,1] \to \mathbb{R}$ of the nonhomogeneous Fredholm integral equation (1.2) and a real constant S such that $|\varphi(x) - \psi(x)| \le S\,\epsilon$ for all $x \in [0,1]$.

Proof Let us define an operator T as

$$
(T\varphi)(x) = x + \lambda \int_0^1 (1+x+t) \; \varphi(t) \; dt, \quad \varphi \in L^2([0,1]). \tag{3.6}
$$

We have $T\varphi \in L^2([0,1])$ and T a self–mapping of $L^2([0,1])$. The solution of the equation (3.6) is the fixed point of the strictly contractive operator T since $\lambda H < 1$. By Theorem 2.1 the equation (3.6) has a unique solution $\varphi^* \in L^2([0,1])$, where $\varphi^* = \lim\limits_{r\to\infty} \varphi_r$ for

$$
\varphi_r(x) = x + \lambda \int_0^1 (1+x+t)\, \varphi_{r-1}(t)\, dt
$$

and $\varphi_0 \in L^2([0,1])$ is an arbitrary function.

Let $\psi \in L^2([0,1])$ be a solution of inequality (4) and

$$\psi(x) - x - \lambda \int_0^1 (1+x+t)\,\psi(t)\,dt =: h(x).$$

We have $|h(x)| \leq \epsilon$ for all $x \in [0,1]$. Then we can conclude that the solution of equation

$$\psi(x) = h(x) + x + \lambda \int_0^1 (1+x+t)\,\psi(t)\,dt$$

is $\psi^* \in L^2([0,1])$, where $\psi^* = \lim_{r\to\infty} \psi_r$ for

$$\psi_r(x) = h(x) + x + \lambda \int_0^1 (1+x+t)\,\psi_{r-1}(t)\,dt$$

and $\psi_0 \in L^2([0,1])$ is an arbitrary function.

For $\varphi_0(x) = \psi_0(x) = 0$, we get,

$$\begin{aligned}
|\varphi_r(x) - \psi_r(x)| &= \left| h(x) + \lambda \int_0^1 (1+x+t)\ \left(\psi_{r-1}(x) - \varphi_{r-1}(x)\right)\ dt \right| \\
&\leq \epsilon \Bigg(1 + \lambda \int_0^1 |1+x+t_{r-1}| dt_{r-1} \\
&\quad + \lambda^2 \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}| dt_{r-2} dt_{r-1} + \cdots \\
&\quad \cdots + \lambda^{r-1} \int_0^1 |1+x+t_{r-1}| \int_0^1 |1+t_{r-1}+t_{r-2}| \int_0^1 |1+t_{r-2}+t_{r-3}| \cdots \\
&\quad \cdots \int_0^1 |1+t_2+t_1| dt_1 \cdots dt_{r-3} dt_{r-2} dt_{r-1} \Bigg) \\
&\leq \epsilon \left(1 + \lambda H + (\lambda H)^2 + \ldots + (\lambda H)^{r-1}\right) = \epsilon \left(\frac{1-(\lambda H)^r}{1-\lambda H} \right),
\end{aligned}$$

as $r \to \infty$, we obtain

$$|\varphi^*(x) - \psi^*(x)| \leq \frac{1}{1-\lambda H}\,\epsilon.$$

Let us choose $S = \dfrac{1}{1-\lambda H}$, hence $|\varphi^*(x) - \psi^*(x)| \leq S\epsilon$, and $0 < \lambda H < 1$, where S is the Hyers-Ulam stability constant for (1.2). Hence, by the virtue of Definition 2.4 the nonhomogeneous Fredholm integral equation (1.2) has the Hyers-Ulam stability. □

Finally, the following corollary proves the Hyers-Ulam-Rassias stability of the nonhomogeneous Fredholm integral equation of second kind (1.2).

Corollary 3.4 *Consider H a fixed real number such that $H \geq \frac{5}{2}$ and $\lambda H < 1$. Let $\varphi : [0,1] \to \mathbb{R}$ a continuous function and the kernel $K : [0,1] \times [0,1] \to \mathbb{R}$ defined by $K(x,t) = 1 + x + t$ such that*

$$\int_0^1 |1+x+t|\theta(t)dt \leq \theta(x) \int_0^1 |1+x+t|dt,$$

for all $x \in [0,1]$, where $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$. If φ is such that

$$\left|\varphi(x) - x - \lambda \int_0^1 (1+x+t)\,\varphi(t)\,dt\right| \leq \theta(x), \tag{3.7}$$

then there exists a solution $\psi : [0,1] \to \mathbb{R}$ of the nonhomogeneous Fredholm integral equation (1.2) and a real constant S such that $|\varphi(x) - \psi(x)| \leq S\,\theta(x)$ for all $x \in [0,1]$.

4. Examples

In order to illustrate our results we will present some examples.

Let us consider the nonhomogeneous Fredholm integral equation of second kind (1.2) defined by

$$\varphi(x) = x + \lambda \int_0^1 (1+x+t)\,\varphi(t)\,dt$$

for all $x \in [0,1]$ and $\lambda = \frac{1}{5}$. Let $H = \frac{13}{5}$ and the perturbation of the solution $\varphi(x) = \frac{587}{500}x + \frac{28}{100}$.

We realize that all conditions of Theorem 3.3 are satisfied. In fact $\lambda H = \frac{13}{25} < 1$ and φ is a continuous function such that

$$\left|\varphi(x) - x - \lambda \int_0^1 (1+x+t)\,\varphi(t)\,dt\right| = \left|\frac{3}{5000}x + \frac{1}{3000}\right| \leq \frac{7}{7500} := \epsilon.$$

By the exact solution $\psi(x) = \frac{210}{179}x + \frac{50}{179}$, we realize that

$$|\varphi(x) - \psi(x)| = \left|\frac{73}{89500}x + \frac{3}{4475}\right| \leq \frac{1}{1-\lambda H}\,\epsilon = \frac{7}{3600}. \tag{4.1}$$

To illustrate the inequality (4.1), we have the Figure 1.

Let us consider the same nonhomogeneous Fredholm integral equation of second kind (1.2) but now with $\lambda = \frac{1}{100}$. Let $H = 3$ and the perturbation of the solution $\varphi(x) = \frac{10052}{10000}x + \frac{851}{100000}$. We have $\lambda H = \frac{3}{100} < 1$ and φ a continuous function such that

$$\left|\varphi(x) - x - \lambda \int_0^1 (1+x+t)\,\varphi(t)\,dt\right| = \left|\frac{5334}{60000000}x + \frac{341}{60000000}\right| \leq \frac{227}{2400000} := \epsilon.$$

By the exact solution $\psi(x) = \frac{118200}{117599}x + \frac{1000}{117599}$, we realize that

$$|\varphi(x) - \psi(x)| = \left|\frac{26287}{293997500}x + \frac{76749}{11759900000}\right| \leq \frac{1}{1-\lambda H}\,\epsilon = \frac{227}{2328000}. \tag{4.2}$$

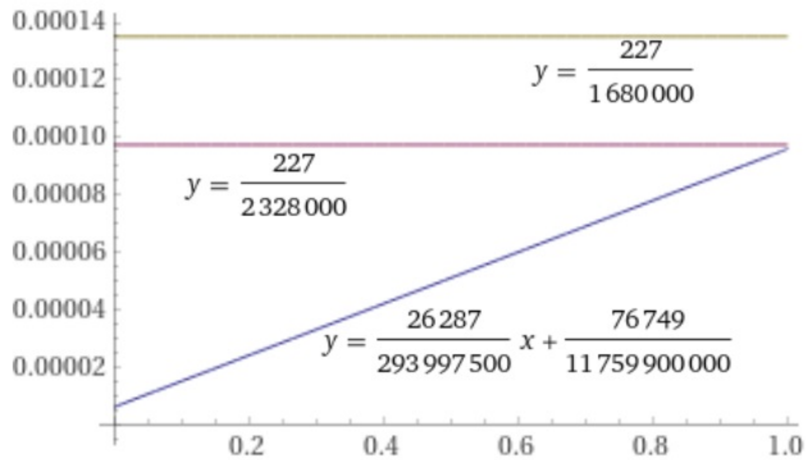

Figure 1.

If we consider $H = 30$, we get a worse result but still acceptable. We get,

$$|\varphi(x) - \psi(x)| = \left| \frac{26287}{293997500}x + \frac{76749}{11759900000} \right| \leq \frac{1}{1 - \lambda H}\epsilon = \frac{227}{1680000}. \tag{4.3}$$

Therefore, we have that the nonhomogeneous Fredholm integral equation of second kind (1.2) has the Hyers-Ulam stability.

To illustrate the inequalities (4.2) and (4.3), we have the Figure 2.

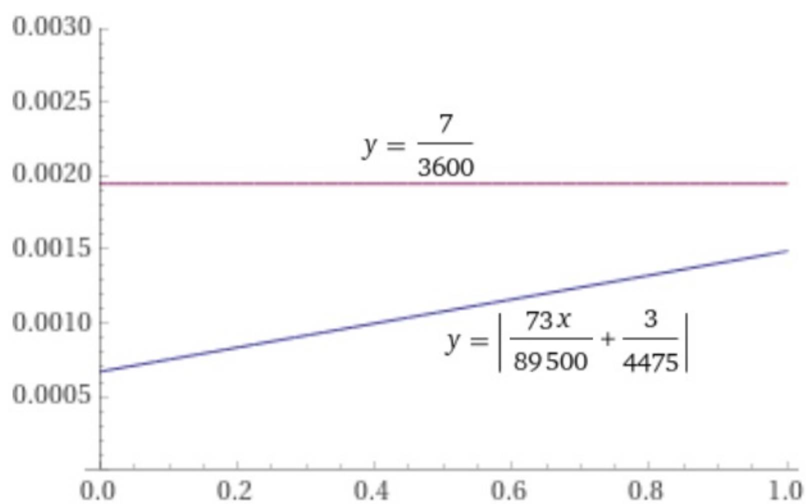

Figure 2.

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