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Finite ordered Γ-hypersemigroups constructed by ordered Γ-semigroups

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Abstract: In the investigation of ordered Γ-hypersemigroups we often need counterexamples (of finite order) given by a table of multiplication and a figure that are impossible to make by hand and very difficult to write programs as well. So it is useful to have examples of ordered Γ-semigroups for which is much more easier to write programs and then from these examples to obtain corresponding examples of ordered Γ-hypersemigroups. In this respect we show that from every example of a regular, intra-regular, right (left) regular, right (left) quasi-regular, semisimple, right (left) simple, simple, strongly regular ordered Γ-semigroup given by a table of multiplication and an order, a corresponding example on ordered Γ-hypersemigroups can be obtained. From examples of different kind of ideals of ordered Γ-semigroups, corresponding examples of ordered Γ-hypersemigroups can be obtained. Examples illustrate the results.

Key words: ordered Γ-semigroup, ordered Γ-hypersemigroup, ideal, regular, intra-regular, semisimple

1. Introduction-Prerequisites

In the theory of ordered hypersemigroups the regularity, intra-regularity and related topics are essential for the investigation; in many of them, different kind of ideals play an essential role. We often need (counter)examples that clearly are impossible to make by hand and difficult to write programs. To overcome this difficulty we use examples from ordered semigroups [5]. As a continuation of the paper in [5], we show here that from (finite) examples of ordered Γ-semigroups given by a table of multiplication and an order, examples of ordered Γ-hypersemigroups can be obtained.

For nonempty sets A, B and Γ, denote by $A\Gamma B$ the set defined by

$$A\Gamma B := \{a\gamma b \mid a \in A,\, b \in B,\, \gamma \in \Gamma\}.$$

Let M and Γ be two nonempty sets. Then M is called a Γ-semigroup if the following assertions are satisfied:

(1) $M\Gamma M \subseteq M$

(2) if $a, b, c, d \in M$ and $\gamma, \mu \in \Gamma$ such that $a = c$, $\gamma = \mu$ and $b = d$, then $a\gamma b = c\mu d$

(3) $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in M$ and all $\gamma, \mu \in \Gamma$ [3, 10].

Condition (1) can be also written as follows:

$$\text{For every } a, b \in M \text{ and every } \gamma \in \Gamma, \text{ we have } a\gamma b \in M.$$

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If we have only the conditions (1) and (2), then this is the definition of a Γ-groupoid.

In other words, a Γ-semigroup is a nonempty set M with a set Γ of binary operations on M, satisfying the associativity condition $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a,b,c \in M$ and all $\gamma,\mu \in \Gamma$. A Γ-semigroup M is called an ordered Γ-semigroup (shortly po-Γ-semigroup) if there exists an order relation $\leq$ on M such that $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for every $c \in M$ and every $\gamma \in \Gamma$ [11].

For the sake of completeness, we will give some definitions already given in [6].

Let M and Γ be two nonempty sets. The set M is called a Γ-hypergroupoid if the following assertions are satisfied:

(1) if $a,b \in M$ and $\gamma \in \Gamma$, then $\emptyset \neq a\gamma b \subseteq M$ and

(2) if $a,b,c,d \in M$ and $\gamma,\mu \in \Gamma$ such that $a = c$, $\gamma = \mu$ and $b = d$, then $a\gamma b = c\mu d$.

In other words, a Γ-hypergroupoid is a nonempty set M with a set Γ of binary hyperoperations on M.

According to [6, Definition 3.2] and [6, Definition 3.3], if M is a Γ-hypergroupoid, then for every $\gamma \in \Gamma$ we denote by $\overline{\gamma}$ the operation on $\mathcal{P}^*(M)$ (induced by the hyperoperation γ) defined by

$$A\overline{\gamma}B := \bigcup_{a \in A, b \in B} a\gamma b$$

and by Γ the operation on $\mathcal{P}^*(M)$ defined by

$$A\Gamma B := \bigcup_{\gamma \in \Gamma} A\overline{\gamma}B$$

respectively ($\mathcal{P}^*(M)$ is the set of all nonempty subsets of M).

As one can easily see, $A\Gamma B = \bigcup\limits_{a \in A, b \in B, \gamma \in \Gamma} a\gamma b$ [6, Remark 3.4]. As a consequence, if M is a Γ-hypergroupoid, $x \in M$ and A,B nonempty subsets of M, then $x \in A\Gamma B$ if and only if $x \in a\gamma b$ for some $a \in A$, $b \in B$, $\gamma \in \Gamma$ (and thus, $a \in A$, $b \in B$, $\gamma \in \Gamma$, implies $a\gamma b \subseteq A\Gamma B$) (see also [6, Lemma 3.7]).

We also have $\{x\}\overline{\gamma}\{y\} = x\gamma y$ for every $x \in M$ [6, Lemma 3.5] which we often use.

In addition, if M is a Γ-hypergroupoid, A,B,C,D subsets of M such that $A \neq \emptyset$, $C \neq \emptyset$ and $A \subseteq B$ and $C \subseteq D$, then $A\overline{\gamma}C \subseteq B\overline{\gamma}D$ [6, Lemma 3.6] and $A\Gamma C \subseteq B\Gamma D$ [6, Lemma 3.8].

For two nonempty subsets A and B of M, we write $A \preceq B$ if for any $a \in A$ there exists $b \in B$ such that $a \leq b$. If $A \preceq B$ and $B \preceq C$, then $A \preceq C$. Indeed: Let $a \in A$. Since $A \preceq B$, we have $a \leq b$ for some $b \in B$. Since $B \preceq C$ and $b \in B$, we have $b \leq c$ for some $c \in C$. For the element $c \in C$, we have $a \leq c$ and so $A \preceq C$.

A Γ-hypergroupoid M is called a Γ-hypersemigroup [6, Definition 3.14] if

$$\{a\}\overline{\gamma}(b\mu c) = (a\gamma b)\overline{\mu}\{c\}$$

for every $a,b,c \in M$ and every $\gamma,\mu \in \Gamma$.

For the concept of ordered Γ-hypersemigroups we refer to [9, Definition 2.3] and [9, Definition 2.4]; though the concept of Γ-hypersemigroup should be corrected in it. Thus, a Γ-hypersemigroup M is called an ordered Γ-hypersemigroup [7] if there exists an order relation $\leq$ on M such that $a \leq b$ implies $a\gamma c \preceq b\gamma c$ and $c\gamma a \preceq c\gamma b$ for every $c \in M$ and every $\gamma \in \Gamma$. If $u \leq a$ and $v \leq b$, then $u\gamma v \preceq a\gamma v$ and $a\gamma v \preceq a\gamma b$ and so $u\gamma v \preceq a\gamma b$.

According to [7, Lemma 2], if M is a Γ-hypersemigroup then, for any nonempty subsets A, B, C of M and any $\gamma, \mu \in \Gamma$, we have
$$(A\overline{\gamma}B)\overline{\mu}C = A\overline{\gamma}(B\overline{\mu}C).$$

According to [6, Proposition 3.17], for any nonempty subsets A, B, C of a Γ-hypersemigroup M, we have
$$(A\Gamma B)\Gamma C = A\Gamma(B\Gamma C).$$

As a consequence we can write expressions of the form, say $A\overline{\gamma}B\overline{\mu}C$, $A\overline{\gamma}B\overline{\mu}C\overline{\rho}D\overline{\varpi}E$ or $A\Gamma B\Gamma C$, $A\Gamma B\Gamma C\Gamma D\Gamma E$ etc., without using parentheses.

For a nonempty subset A of M, we denote by $(A]$ the subset of M defined by
$$(A] := \{t \in M \mid t \leq a \text{ for some } a \in A\}.$$

Clearly, $(M] = M$ and $\big((A]\big] = (A]$ for any nonempty subset A of M. We also have the following
$$(A] = A \Longleftrightarrow \text{ if } a \in A \text{ and } M \ni b \leq a, \text{ then } b \in A.$$

2. Main results

Theorem 2.1 *Let $(M, \widetilde{\Gamma}, \leq)$ be an ordered Γ-groupoid, the elements of $\widetilde{\Gamma}$ denoted by symbols like $\widetilde{\gamma}$, $\widetilde{\mu}$, $\widetilde{\rho}$, $\widetilde{\omega}$ etc. For each $a, b \in M$ and each $\widetilde{\gamma} \in \widetilde{\Gamma}$ we consider the (hyper)operation on M defined by*
$$\gamma : M \times M \to \mathcal{P}^*(M) \mid (a, b) \to a\gamma b := \{t \in M \mid t \leq a\widetilde{\gamma}b\}.$$

Then the set M with the same order $\leq$ and the set of hyperoperations $\Gamma = \{\gamma \mid \widetilde{\gamma} \in \widetilde{\Gamma}\}$ is an ordered Γ-hypergroupoid; denoted by $(M, \Gamma, \leq)$. In particular, if $(M, \widetilde{\Gamma}, \leq)$ is an ordered Γ-semigroup, then $(M, \Gamma, \leq)$ is an ordered Γ-hypersemigroup as well.

Proof If $(M, \widetilde{\Gamma}, \leq)$ is an ordered Γ-groupoid, then $(M, \Gamma, \leq)$ is an ordered Γ-hypergroupoid. In fact: If $a, b \in M$ and $\gamma \in \Gamma$, then $\emptyset \neq a\gamma b \subseteq M$. If $a, b, c, d \in M$, $\gamma, \mu \in \Gamma$, $a = c$, $\gamma = \mu$, $b = d$ and $x \in a\gamma b$, then $x \leq a\widetilde{\gamma}b = c\widetilde{\mu}d$, then $x \in c\mu d$ and so $a\gamma b \subseteq c\mu d$. Similarly, $c\mu d \subseteq a\gamma b$ and so $a\gamma b = c\mu d$. If $a \leq b$, $c \in M$ and $\gamma \in \Gamma$, then $a\gamma c \preceq b\gamma c$ and $c\gamma a \preceq c\gamma b$. Indeed: Let $x \in a\gamma c$. Then $x \leq a\widetilde{\gamma}c$. Since $\leq$ is an order on $(M, \widetilde{\Gamma}, \leq)$ and $a \leq b$, we have $a\widetilde{\gamma}c \leq b\widetilde{\gamma}c$. Then $x \leq b\widetilde{\gamma}c$, from which $x \in b\gamma c$ and so $a\gamma c \preceq b\gamma c$. If $x \in c\gamma a$, then $x \leq c\widetilde{\gamma}a \leq c\widetilde{\gamma}b$, then $x \in c\gamma b$ and so $c\gamma a \preceq c\gamma b$.

Let now $(M, \widetilde{\Gamma}, \leq)$ be an ordered Γ-semigroup. To prove that $(M, \Gamma, \leq)$ is an ordered Γ-hypersemigroup, is enough to prove that $\{a\}\overline{\gamma}(b\mu c) = (a\gamma b)\overline{\mu}\{c\}$ for every $a, b, c \in M$ and every $\gamma, \mu \in \Gamma$ (see [6, Definition 3.14]). In this respect, let $a, b, c \in M$, $\gamma, \mu \in \Gamma$ and $t \in \{a\}\overline{\gamma}(b\mu c)$. By [6, Definition 3.2], we have $t \in a\gamma u$ for some $u \in b\mu c$. Then $t \leq a\widetilde{\gamma}u$ and $u \leq b\widetilde{\mu}c$ from which $t \leq a\widetilde{\gamma}(b\widetilde{\mu}c) = (a\widetilde{\gamma}b)\widetilde{\mu}c$. Then we have $t \in (a\widetilde{\gamma}b)\mu c = \{a\widetilde{\gamma}b\}\overline{\mu}\{c\}$ (by [6, Lemma 3.5]). Since $a\widetilde{\gamma}b \leq a\widetilde{\gamma}b$, we have $a\widetilde{\gamma}b \in a\gamma b$, then $\{a\widetilde{\gamma}b\} \subseteq a\gamma b$. Since $\{a\widetilde{\gamma}b\} \subseteq a\gamma b$ and $\{c\} \subseteq \{c\}$, by [6, Lemma 3.6], we have $\{a\widetilde{\gamma}b\}\overline{\mu}\{c\} \subseteq (a\gamma b)\overline{\mu}\{c\}$. Thus we get $t \in (a\gamma b)\overline{\mu}\{c\}$. Let now $t \in (a\gamma b)\overline{\mu}\{c\}$. Then $t \in u\mu c$ for some $u \in a\gamma b$. Then $t \leq u\widetilde{\mu}c$ and $u \leq a\widetilde{\gamma}b$ from which $t \leq (a\widetilde{\gamma}b)\widetilde{\mu}c = a\widetilde{\gamma}(b\widetilde{\mu}c)$. Then we have $t \in a\gamma(b\widetilde{\mu}c) = \{a\}\overline{\gamma}\{b\widetilde{\mu}c\}$. Since $b\widetilde{\mu}c \leq b\widetilde{\mu}c$, we have $b\widetilde{\mu}c \in b\mu c$, then $\{b\widetilde{\mu}c\} \subseteq b\mu c$ and so $t \in \{a\}\overline{\gamma}(b\mu c)$. Thus we obtain $\{a\}\overline{\gamma}(b\mu c) = (a\gamma b)\overline{\mu}\{c\}$ and the proof is complete. $\square$

Example 2.2 *The set $M = \{a, b, c, d\}$ with the operations $\widetilde{\gamma}$ and $\widetilde{\mu}$ defined by Tables 1, 2 and Figure 1 is an ordered Γ-semigroup. Applying Theorem 2.1, from this ordered Γ-semigroup, the ordered Γ-hypersemigroup defined by Tables 3 and 4 and the same order (given by Figure 1) can be obtained.*

Table 1. The operation $\widetilde{\gamma}$ of the Γ-semigroup of Example 2.2.

$\widetilde{\gamma}$	a	b	c	d
a	a	b	c	a
b	c	c	c	b
c	c	c	c	c
d	a	b	c	d

Table 2. The operation $\widetilde{\mu}$ of the Γ-semigroup of Example 2.2.

$\widetilde{\mu}$	a	b	c	d
a	a	b	c	a
b	c	c	c	c
c	c	c	c	c
d	a	b	c	a

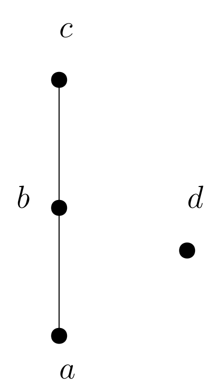

Figure 1. Figure 1 of Example 2.2.

Table 3. The operation γ of the Γ-hypersemigroup of Example 2.2.

γ	a	b	c	d
a	{a}	{a,b}	{a,b,c}	{a}
b	{a,b,c}	{a,b,c}	{a,b,c}	{a,b}
c	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}
d	{a}	{a,b}	{a,b,c}	{d}

If we keep the Tables 1 and 2 and change the order to anyone of Figures 2, 3 or 4, we still get an ordered Γ-semigroup. From Tables 1, 2 and Figure 2, the ordered Γ-hypersemigroup given by Tables 5, 6 and the same

Table 4. The operation μ of the Γ-hypersemigroup of Example 2.2.

μ	a	b	c	d
a	{a}	{a,b}	{a,b,c}	{a}
b	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}
c	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}
d	{a}	{a,b}	{a,b,c}	{a}

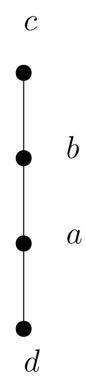

Figure 2. Figure 2 of Example 2.2.

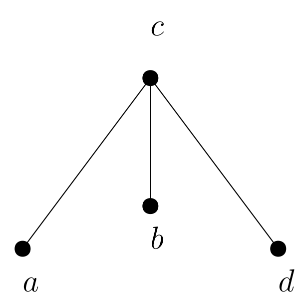

Figure 3. Figure 3 of Example 2.2.

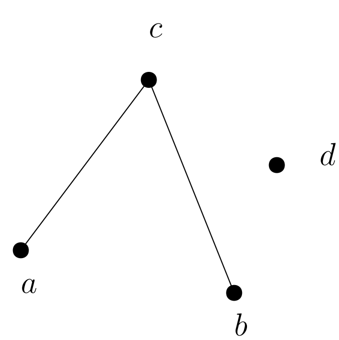

Figure 4. Figure 4 of Example 2.2.

order can be obtained. From Tables 1, 2 and Figure 3, the ordered Γ-hypersemigroup given by Tables 7, 8 and the same order can be obtained. From Tables 1, 2 and Figure 4, the ordered Γ-hypersemigroup given by Tables 9, 10 and the same order can be obtained.

Table 5. The operation γ of the Γ-hypersemigroup of Example 2.2 related to Figure 2.

γ	a	b	c	d
a	{d,a}	{d,a,b}	{d,a,b,c}	{d,a}
b	{d,a,b,c}	{d,a,b,c}	{d,a,b,c}	{d,a,b}
c	{d,a,b,c}	{d,a,b,c}	{d,a,b,c}	{d,a,b,c}
d	{d,a}	{d,a,b}	{d,a,b,c}	{d}

Table 6. The operation μ of the Γ-hypersemigroup of Example 2.2 related to Figure 2.

μ	a	b	c	d
a	{d,a}	{d,a,b}	{d,a,b,c}	{d,a}
b	{d,a,b,c}	{d,a,b,c}	{d,a,b,c}	{d,a,b,c}
c	{d,a,b,c}	{d,a,b,c}	{d,a,b,c}	{d,a,b,c}
d	{d,a}	{d,a,b}	{d,a,b,c}	{d,a}

Table 7. The operation γ of the Γ-semigroup of Example 2.2 related to Figure 3.

γ	a	b	c	d
a	{a}	{b}	{a,b,c,d}	{a}
b	{a,b,c,d}	{a,b,c,d}	{a,b,c,d}	{b}
c	{a,b,c,d}	{a,b,c,d}	{a,b,c,d}	{a,b,c,d}
d	{a}	{b}	{a,b,c,d}	{d}

Table 8. The operation μ of the Γ-hypersemigroup of Example 2.2 related to Figure 3.

μ	a	b	c	d
a	{a}	{b}	{a,b,c,d}	{a}
b	{a,b,c,d}	{a,b,c,d}	{a,b,c,d}	{a,b,c,d}
c	{a,b,c,d}	{a,b,c,d}	{a,b,c,d}	{a,b,c,d}
d	{a}	{b}	{a,b,c,d}	{a}

Table 9. The operation γ of the Γ-hypersemigroup of Example 2.2 related to Figure 4.

γ	a	b	c	d
a	{a}	{b}	{a,b,c}	{a}
b	{a,b,c}	{a,b,c}	{a,b,c}	{b}
c	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}
d	{a}	{b}	{a,b,c}	{d}

In the following, when we refer to $(M, \widetilde{\Gamma}, \leq)$ this is the ordered Γ-semigroup (or Γ-groupoid) considered in Theorem 2.1. When we refer to $(M, \Gamma, \leq)$, this is the ordered Γ-hypersemigroup (or Γ-hypergroupoid) constructed by the ordered Γ-semigroup $(M, \widetilde{\Gamma}, \leq)$ in the way indicated in Theorem 2.1.

As an application of Theorem 2.1, we get the theorems of this section.

Table 10. The operation μ of the Γ-hypersemigroup of Example 2.2 related to Figure 4.

μ	a	b	c	d
a	{a}	{b}	{a,b,c}	{a}
b	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}
c	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}
d	{a}	{b}	{a,b,c}	{a}

A nonempty subset A of the ordered groupoid $(M, \widetilde{\Gamma}, \leq)$ is called a right ideal of M if (1) $A\widetilde{\Gamma}M \subseteq A$; that is, if $a \in A$, $\widetilde{\gamma} \in \widetilde{\Gamma}$, $u \in M$, then $a\widetilde{\gamma}u \in A$ and (2) if $a \in A$ and $M \ni b \leq a$, then $b \in A$ i.e. $(A] = A$; it is called a left ideal of M if (1) $M\widetilde{\Gamma}A \subseteq A$; that is, if $u \in M$, $\widetilde{\gamma} \in \widetilde{\Gamma}$, $a \in A$, then $a\widetilde{\gamma}u \in A$ and (2) if $a \in A$ and $M \ni b \leq a$, then $b \in A$. If A is both a right and a left ideal of M, then it is called an ideal of M. A nonempty subset Q of $(M, \widetilde{\Gamma}, \leq)$ is called a quasi-ideal of M if (1) $(Q\widetilde{\Gamma}M] \cap (M\widetilde{\Gamma}Q] \subseteq Q$; that is if $t \leq q\widetilde{\gamma}m$ for some $q \in Q$, $\widetilde{\gamma} \in \widetilde{\Gamma}$, $m \in M$ and $t \leq n\widetilde{\mu}g$ for some $n \in M$, $\widetilde{\mu} \in \widetilde{\Gamma}$, $g \in Q$, then $t \in Q$ and (2) if $a \in Q$ and $M \ni b \leq a$, then $b \in Q$.

These concepts can be naturally extended to the ordered Γ-hypergroupoid $(M, \Gamma, \leq)$ as follows.

Definition 2.3 *[8, Definition 2.5] A nonempty subset A of the ordered Γ-hypergroupoid $(M, \Gamma, \leq)$ is called a right ideal of M if (1) $A\Gamma M \subseteq A$; that is, if $u \in a\gamma m$ for some $a \in A$, $\gamma \in \Gamma$, $m \in M$, then $u \in A$ and (2) if $a \in A$ and $M \ni b \leq a$, then $b \in A$; it is called a left ideal of M if (1) $M\Gamma A \subseteq A$; that is, if $u \in m\gamma a$ for some $m \in M$, $\gamma \in \Gamma$, $a \in A$, then $u \in A$ and (2) if $a \in A$ and $M \ni b \leq a$, then $b \in A$. If A is both a right and a left ideal of M, then it is called an ideal of M.*

Definition 2.4 *[7] A nonempty subset A of the ordered Γ-hypergroupoid $(M, \Gamma, \leq)$ is called a quasi-ideal of M if (1) $(Q\Gamma M] \cap (M\Gamma Q] \subseteq Q$; that is if $t \leq x$ for some $x \in q\gamma m$; $q \in Q$, $\gamma \in \Gamma$, $m \in M$ and $t \leq y$ for some $y \in n\mu g$; $n \in M$, $\mu \in \Gamma$, $g \in Q$, then $t \in Q$ and (2) if $a \in Q$ and $M \ni b \leq a$, then $b \in Q$.*

Theorem 2.5 *The set A is a right (resp. left) ideal of the ordered Γ-groupoid $(M, \widetilde{\Gamma}, \leq)$ if and only if it is a right (resp. left) ideal of the ordered Γ-hypergroupoid $(M, \Gamma, \leq)$. A set Q is a quasi-ideal of the ordered Γ-groupoid $(M, \widetilde{\Gamma}, \leq)$ if and only if it is a quasi-ideal of the ordered Γ-hypergroupoid $(M, \Gamma, \leq)$.*

Proof Let A be a right ideal of $(M, \widetilde{\Gamma}, \leq)$ and let $x \in A\Gamma M$. By [6, Remark 3.4], we have $x \in a\gamma m$ for some $a \in A$, $\gamma \in \Gamma$, $m \in M$, then we have $x \leq a\widetilde{\gamma}m \in A\widetilde{\Gamma}M$. Since A is a right ideal of $(M, \widetilde{\Gamma}, \leq)$,we have $A\widetilde{\Gamma}M \subseteq A$ and $x \in A$. Thus we get $A\Gamma M \subseteq A$ and so A is a right ideal of $(M, \Gamma, \leq)$. Conversely, let A be a right ideal of $(M, \Gamma, \leq)$ and let $x \in A\widetilde{\Gamma}M$. Then $x = a\widetilde{\gamma}m$ for some $a \in A$, $\widetilde{\gamma} \in \widetilde{\Gamma}$, $m \in M$. Since $x \leq a\widetilde{\gamma}m$, we have $x \in a\gamma m \subseteq A\Gamma M \subseteq A$ and so $x \in A$. Thus we have $A\widetilde{\Gamma}M \subseteq A$ and so A is a right ideal of $(M, \widetilde{\Gamma}, \leq)$.

Let Q be a quasi-ideal of $(M, \widetilde{\Gamma}, \leq)$, $t \leq x$ for some $x \in q\gamma m$; $q \in Q$, $\gamma \in \Gamma$, $m \in M$ and $t \leq y$ for some $y \in n\mu g$; $n \in M$, $\mu \in \widetilde{\mu}$, $g \in Q$. Since $x \in q\gamma m$, we have $x \leq q\widetilde{\gamma}m$. Since $y \in n\mu g$, we have $y \leq n\widetilde{\mu}g$. We have $t \leq q\widetilde{\gamma}m$ for some $q \in Q$, $\widetilde{\gamma} \in \widetilde{\Gamma}$, $m \in M$ and $t \leq n\widetilde{\mu}g$ for some $n \in M$, $\widetilde{\mu} \in \widetilde{\Gamma}$, $g \in Q$. Since Q is a quasi-ideal of $(M, \widetilde{\Gamma}, \leq)$, we have $t \in Q$ and so Q is a quasi-ideal of $(M, \Gamma, \leq)$. Conversely, let Q be a quasi-ideal of $(M,\Gamma,\leq)$ and let $t\leq q\widetilde{\gamma}m$ for some $q\in Q$, $\widetilde{\gamma}\in\widetilde{\Gamma}$, $m\in M$ and $t\leq n\widetilde{\mu}g$ for some $n\in M$, $\widetilde{\mu}\in\widetilde{\Gamma}$, $g\in Q$. Then $t\leq t$, $t\in q\gamma m$; $q\in Q$, $\gamma\in\Gamma$, $m\in M$ and $t\in n\mu g$; $n\in M$, $\mu\in\Gamma$, $g\in Q$. Since Q is a quasi-ideal of $(M,\Gamma,\leq)$, we have $t\in Q$ and so Q is a right ideal of $(M,\widetilde{\Gamma},\leq)$. □

A nonempty subset B of $(M,\widetilde{\Gamma},\leq)$ is called a bi-ideal of M if (1) $B\widetilde{\Gamma}B\widetilde{\Gamma}B\subseteq B$; that is, if $x=a\widetilde{\gamma}b\widetilde{\mu}c$ for some $a,b,c\in B$, $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$, then $x\in B$ and (2) if $a\in B$ and $M\ni b\leq a$, then $b\in B$. A nonempty subset D of $(M,\widetilde{\Gamma},\leq)$ is called an interior ideal of M (see also [2]) if (1) $M\widetilde{\Gamma}D\widetilde{\Gamma}M\subseteq D$; that is if $x=a\widetilde{\gamma}d\widetilde{\mu}b$ for some $a,b\in M$, $d\in D$, $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$, then $x\in D$ and (2) if $a\in D$ and $M\ni b\leq a$, then $b\in D$.

The corresponding concepts for the ordered Γ-hypersemigroup $(M,\Gamma,\leq)$ are as follows.

Definition 2.6 *[7] A nonempty subset B of the ordered Γ-hypersemigroup $(M,\Gamma,\leq)$ is called a bi-ideal of M if (1) $B\Gamma B\Gamma B\subseteq B$; that is, if $x\in(a\gamma b)\overline{\mu}\{c\}$ for some $a,b,c\in B$, $\gamma,\mu\in\Gamma$, then $x\in B$ and (2) if $a\in B$ and $M\ni b\leq a$, then $b\in B$.*

Definition 2.7 *A nonempty subset D of the ordered Γ-hypersemigroup $(M,\Gamma,\leq)$ is called an interior ideal of M if (1) $M\Gamma D\Gamma M\subseteq D$; that is if $x\in u\gamma m$ and $u\in n\mu d$ for some $m,n\in M$, $\gamma,\mu\in\Gamma$, $d\in D$, then $x\in D$ and (2) if $a\in D$ and $M\ni b\leq a$, then $b\in D$.*

Theorem 2.8 *A set B is a bi-ideal of the ordered Γ-semigroup $(M,\widetilde{\Gamma},\leq)$ if and only if it is a bi-ideal of the ordered Γ-hypersemigroup $(M,\Gamma,\leq)$. A set D is an interior ideal of $(M,\widetilde{\Gamma},\leq)$ if and only if it is an interior ideal of $(M,\Gamma,\leq)$.*

Proof Let B be a bi-ideal of $(M,\widetilde{\Gamma},\leq)$ and let $x\in B\Gamma B\Gamma B$. Then $x\in(a\gamma b)\overline{\mu}\{c\}$ for some $a,b,c\in B$, $\gamma,\mu\in\Gamma$. Then there exists $u\in a\gamma b$ such that $x\in u\mu c$ (by [6, Definition 3.2]). Then $u\leq a\widetilde{\gamma}b$ and $x\leq u\widetilde{\mu}c$, and thus $x\leq a\widetilde{\gamma}b\widetilde{\mu}c\in B\widetilde{\Gamma}B\widetilde{\Gamma}B$. Since B is a bi-ideal of $(M,\widetilde{\Gamma},\leq)$, we have $B\widetilde{\Gamma}B\widetilde{\Gamma}B\subseteq B$ and $x\in B$. Thus we have $B\Gamma B\Gamma B\subseteq B$ and so B is a bi-ideal of $(M,\Gamma,\leq)$. Let now B be a bi-ideal of $(M,\Gamma,\leq)$ and let $x=a\widetilde{\gamma}b\widetilde{\mu}c$ for some $a,b,c\in B$, $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$. Since $x\leq a\widetilde{\gamma}b\widetilde{\mu}c$, we have $x\in(a\widetilde{\gamma}b)\mu c=\{a\widetilde{\gamma}b\}\overline{\mu}\{c\}$ (by [6, Lemma 3.5]). Since $a\widetilde{\gamma}b\in a\gamma b$, we have $\{a\widetilde{\gamma}b\}\subseteq a\gamma b$. Then $x\in(a\gamma b)\overline{\mu}\{c\}$ (by [6, Lemma 3.6]). Then, since B be a bi-ideal of $(M,\Gamma,\leq)$, we have $x\in B$ and so B is a bi-ideal of $(M,\widetilde{\Gamma},\leq)$.

Let D be an interior ideal of $(M,\widetilde{\Gamma},\leq)$ and let $x\in u\gamma m$ and $u\in n\mu d$ for some $m,n\in M$, $\gamma,\mu\in\Gamma$, $d\in D$. Then $x\leq u\widetilde{\gamma}m$ and $u\leq n\widetilde{\mu}d$, and thus $x\leq n\widetilde{\mu}d\widetilde{\gamma}m\in M\widetilde{\Gamma}D\widetilde{\Gamma}M$. Since D is an interior ideal of $(M,\widetilde{\Gamma},\leq)$, we have $M\widetilde{\Gamma}D\widetilde{\Gamma}M\subseteq D$ and $x\in D$. Thus D is an interior ideal of $(M,\Gamma,\leq)$. Conversely, let D be an interior ideal of $(M,\Gamma,\leq)$ and let $x=a\widetilde{\gamma}d\widetilde{\mu}b$ for some $a,b\in M$, $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$, $d\in D$. Then $x\in(a\widetilde{\gamma}d)\mu b=\{a\widetilde{\gamma}d\}\overline{\mu}\{b\}$ (by [6, Lemma 3.5]). Since $a\widetilde{\gamma}d\in a\gamma d$, we have $\{a\widetilde{\gamma}d\}\subseteq a\gamma d$ and so $x\in(a\gamma d)\overline{\mu}\{b\}$ (by [6, Lemma 3.6]). Then $x\in u\mu b$ for some $u\in a\gamma d$ (by [6, Definition 3.2]). Since $x\in u\mu b$, $u\in a\gamma d$, $a,b\in M$, $\gamma,\mu\in\Gamma$, $d\in D$ and D is an interior ideal of $(M,\Gamma,\leq)$, we have $x\in D$. Thus D is an interior ideal of $(M,\widetilde{\Gamma},\leq)$. □

The ordered Γ-semigroup $(M,\widetilde{\Gamma},\leq)$ is called regular if for every $a\in M$ there exist $x\in M$ and $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$ such that $a\leq a\widetilde{\gamma}x\widetilde{\mu}a$. It is called intra-regular if for every $a\in M$ there exist $x,y\in M$ and $\widetilde{\gamma},\widetilde{\mu},\widetilde{\rho}\in\widetilde{\Gamma}$ such that $a\leq x\widetilde{\gamma}a\widetilde{\mu}a\widetilde{\rho}y$.

Definition 2.9 *[8, Definition 2.3] The ordered* Γ*-hypersemigroup* $(M,\Gamma,\le)$ *is called regular if for every* $a\in M$ *there exist* $x\in M$ *and* $\gamma,\mu\in\Gamma$ *such that* $\{a\}\preceq(a\gamma x)\overline{\mu}\{a\}$*; in other words, for every* $a\in M$*, there exist* $x,t\in M$ *and* $\gamma,\mu\in\Gamma$ *such that* $t\in(a\gamma x)\overline{\mu}\{a\}$ *and* $a\le t$*.*

Definition 2.10 *[8, Definition 2.10] The ordered* Γ*-hypersemigroup* $(M,\Gamma,\le)$ *is called intra-regular if for every* $a\in M$ *there exist* $x,y\in M$ *and* $\gamma,\mu,\rho\in\Gamma$ *such that* $\{a\}\preceq(x\gamma a)\overline{\mu}(a\rho y)$*; in other words, there exist* $x,y,t\in M$ *and* $\gamma,\mu,\rho\in\Gamma$ *such that* $t\in(x\gamma a)\overline{\mu}(a\rho y)$ *and* $a\le t$*.*

Theorem 2.11 *The ordered* Γ*-semigroup* $(M,\widetilde{\Gamma},\le)$ *is regular if and only if the ordered* Γ*-hypersemigroup* $(M,\Gamma,\le)$ *is regular. The ordered* Γ*-semigroup* $(M,\widetilde{\Gamma},\le)$ *is intra-regular if and only if the ordered* Γ*-hypersemigroup* $(M,\Gamma,\le)$ *is intra-regular.*

Proof Let $(M,\widetilde{\Gamma},\le)$ be regular and $a\in M$. Then there exist $x\in M$ and $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$ such that $a\le a\widetilde{\gamma}x\widetilde{\mu}a$. Then we have $a\in(a\widetilde{\gamma}x)\mu a=\{a\widetilde{\gamma}x\}\overline{\mu}\{a\}$ (by [6, Lemma 3.5]). Since $a\widetilde{\gamma}x\in a\gamma x$, we have $\{a\widetilde{\gamma}x\}\subseteq a\gamma x$. Then we have $a\in\{a\widetilde{\gamma}x\}\overline{\mu}\{a\}\subseteq(a\gamma x)\overline{\mu}\{a\}$ (by [6, Lemma 3.6]). Since $a\in(a\gamma x)\overline{\mu}\{a\}$ and $a\le a$, we have $\{a\}\preceq(a\gamma x)\overline{\mu}\{a\}$ and so $(M,\Gamma,\le)$ is regular. Let now $(M,\Gamma,\le)$ be regular and $a\in M$. Then there exist $x,t\in M$ and $\gamma,\mu\in\Gamma$ such that $t\in(a\gamma x)\overline{\mu}\{a\}$ and $a\le t$. Since $t\in(a\gamma x)\overline{\mu}\{a\}$, we have $t\in u\mu a$ for some $u\in a\gamma x$ (by [6, Definition 3.2]). Then we have $t\le u\widetilde{\mu}a$ and $u\le a\widetilde{\gamma}x$ from which $t\le(a\widetilde{\gamma}x)\widetilde{\mu}a$ and so $a\le a\widetilde{\gamma}x\widetilde{\mu}a$; thus $(M,\widetilde{\Gamma},\le)$ is regular.

Let $(M,\widetilde{\Gamma},\le)$ be intra-regular and $a\in M$. Then there exist $x,y\in M$ and $\widetilde{\gamma},\widetilde{\mu},\widetilde{\rho}\in\widetilde{\Gamma}$ such that $a\le(x\widetilde{\gamma}a)\widetilde{\mu}(a\widetilde{\rho}y)$. Then we have $a\in(x\widetilde{\gamma}a)\mu(a\widetilde{\rho}y)=\{x\widetilde{\gamma}a\}\overline{\mu}\{a\widetilde{\rho}y\}$ (by [6, Lemma 3.5]). Since $x\widetilde{\gamma}a\in x\gamma a$ and $a\widetilde{\rho}y\in a\rho y$, we have $\{x\widetilde{\gamma}a\}\subseteq x\gamma a$ and $\{a\widetilde{\rho}y\}\subseteq a\rho y$. By [6, Lemma 3.6], we have $\{x\widetilde{\gamma}a\}\overline{\mu}\{a\widetilde{\rho}y\}\subseteq(x\gamma a)\overline{\mu}(a\rho y)$, and so $a\in(x\gamma a)\overline{\mu}(a\rho y)$. Since $a\in(x\gamma a)\overline{\mu}(a\rho y)$ and $a\le a$, $(M,\Gamma,\le)$ is intra-regular. Let now $(M,\Gamma,\le)$ be intra-regular and $a\in M$. Then there exist $x,y,t\in M$ and $\gamma,\mu,\rho\in\Gamma$ such that $t\in(x\gamma a)\overline{\mu}(a\rho y)$ and $a\le t$. Since $t\in(x\gamma a)\overline{\mu}(a\rho y)$, we have $t\in u\mu v$ for some $u\in x\gamma a$, $v\in a\rho y$ (by [6, Definition 3.2]). Then we have $t\le u\widetilde{\mu}v$, $u\le x\widetilde{\gamma}a$, $v\le a\widetilde{\rho}y$, and then $t\le x\widetilde{\gamma}a\widetilde{\mu}a\widetilde{\rho}y$. Then $a\le x\widetilde{\gamma}a\widetilde{\mu}a\widetilde{\rho}y$ and so $(M,\widetilde{\Gamma},\le)$ is intra-regular. $\square$

The ordered Γ-semigroup $(M,\widetilde{\Gamma},\le)$ is called right regular if for every $a\in M$ there exist $x\in M$ and $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$ such that $a\le a\widetilde{\gamma}a\widetilde{\mu}x$; it is called left regular if for every $a\in M$ there exist $x\in M$ and $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$ such that $a\le x\widetilde{\gamma}a\widetilde{\mu}a$.

Definition 2.12 *[8, Definition 2.13] The ordered* Γ*-hypersemigroup* $(M,\Gamma,\le)$ *is called right regular if for every* $a\in M$ *there exist* $x\in M$ *and* $\gamma,\mu\in\Gamma$ *such that* $\{a\}\preceq(a\gamma a)\overline{\mu}\{x\}$*; in other words, there exist* $x,t\in M$ *and* $\gamma,\mu\in\Gamma$ *such that* $t\in(a\gamma a)\overline{\mu}\{x\}$ *and* $a\le t$*. It is called left regular if for every* $a\in M$ *there exist* $x\in M$ *and* $\gamma,\mu\in\Gamma$ *such that* $\{a\}\preceq\{x\}\overline{\gamma}(a\mu a)$*; in other words, there exist* $x,t\in M$ *and* $\gamma,\mu\in\Gamma$ *such that* $t\in\{x\}\overline{\gamma}(a\mu a)$ *and* $a\le t$*.*

Theorem 2.13 *The ordered* Γ*-semigroup* $(M,\widetilde{\Gamma},\le)$ *is right (resp. left) regular if and only if the ordered* Γ*-hypersemigroup* $(M,\Gamma,\le)$ *is right (resp. left) regular.*

Proof Let $(M,\widetilde{\Gamma},\le)$ be right regular and $a\in M$. Then there exist $x\in M$ and $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$ such that $a\le(a\widetilde{\gamma}a)\widetilde{\mu}x$. Then $a\in(a\widetilde{\gamma}a)\mu x=\{a\widetilde{\gamma}a\}\overline{\mu}\{x\}$ (by [6, Lemma 3.5]). Since $a\widetilde{\gamma}a\in a\gamma a$, we have $\{a\widetilde{\gamma}a\}\subseteq a\gamma a$

and so $\{a\widetilde{\gamma}a\}\overline{\mu}\{x\} \subseteq (a\gamma a)\overline{\mu}\{x\}$ (by [6, Lemma 3.6]). We have $a \in (a\gamma a)\overline{\mu}\{x\}$ and $a \le a$ and so $(M, \Gamma, \le)$ is right regular. Let now $(M, \Gamma, \le)$ be right regular and $a \in M$. Then there exist $x, t \in M$ and $\gamma, \mu \in \Gamma$ such that $t \in (a\gamma a)\overline{\mu}\{x\}$ and $a \le t$. Since $t \in (a\gamma a)\overline{\mu}\{x\}$, we have $t \in u\mu x$ for some $u \in a\gamma a$ (by [6, Definition 3.2]). Then we have $t \le u\widetilde{\mu}x$ and $u \le a\widetilde{\gamma}a$, then $t \le a\widetilde{\gamma}a\widetilde{\mu}x$ and so $a \le a\widetilde{\gamma}a\widetilde{\mu}x$; thus $(M, \widetilde{\Gamma}, \le)$ is right regular. $\square$

The ordered Γ-semigroup $(M, \widetilde{\Gamma}, \le)$ is called right (resp. left) quasi-regular [4] if for every $a \in M$ there exist $x, y \in M$ and $\widetilde{\gamma}, \widetilde{\mu}, \widetilde{\rho} \in \widetilde{\Gamma}$ such that $a \le a\widetilde{\gamma}x\widetilde{\mu}a\widetilde{\rho}y$ (resp. $a \le x\widetilde{\gamma}a\widetilde{\mu}y\widetilde{\rho}a$).

Definition 2.14 *The ordered Γ-hypersemigroup $(M, \Gamma, \le)$ is called right quasi-regular if for every $a \in M$ there exist $x, y \in M$ and $\gamma, \mu, \rho \in \Gamma$ such that $\{a\} \preceq (a\gamma x)\overline{\mu}(a\rho y)$; in other words there exist $x, y, t \in M$ and $\gamma, \mu, \rho \in \Gamma$ such that $t \in (a\gamma x)\overline{\mu}(a\rho y)$ and $a \le t$. It is called left quasi-regular if for every $a \in M$ there exist $x, y \in M$ and $\gamma, \mu, \rho \in \Gamma$ such that $\{a\} \preceq (x\gamma a)\overline{\mu}(y\rho a)$; in other words, there exist $x, y, t \in M$ and $\gamma, \mu, \rho \in \Gamma$ such that $t \in (x\gamma a)\overline{\mu}(y\rho a)$ and $a \le t$.*

Theorem 2.15 *The ordered Γ-semigroup $(M, \widetilde{\Gamma}, \le)$ is right (resp. left) quasi-regular if and only if the ordered Γ-hypersemigroup $(M, \Gamma, \le)$ is right (resp. left) quasi-regular.*

Proof Let $(M, \widetilde{\Gamma}, \le)$ be right quasi-regular and $a \in M$. Then there exist $x, y \in M$ and $\widetilde{\gamma}, \widetilde{\mu}, \widetilde{\rho} \in \widetilde{\Gamma}$ such that $a \le (a\widetilde{\gamma}x)\widetilde{\mu}(a\widetilde{\rho}y)$. Then $a \in (a\widetilde{\gamma}x)\mu(a\widetilde{\rho}y) = \{a\widetilde{\gamma}x\}\overline{\mu}\{a\widetilde{\rho}y\}$ (by [6, Lemma 3.5]). Since $a\widetilde{\gamma}x \in a\gamma x$ and $a\widetilde{\rho}y \in a\rho y$, we have $\{a\widetilde{\gamma}x\} \subseteq a\gamma x$ and $\{a\widetilde{\rho}y\} \subseteq a\rho y$, hence $\{a\widetilde{\gamma}x\}\overline{\mu}\{a\widetilde{\rho}y\} \subseteq (a\gamma x)\overline{\mu}(a\rho y)$ (by [6, Lemma 3.6]). Since $a \in (a\gamma x)\overline{\mu}(a\rho y)$ and $a \le a$, $(M, \Gamma, \le)$ is right quasi-regular. Let now $(M, \Gamma, \le)$ be right quasi-regular and $a \in M$. Then there exist $x, y, t \in M$ and $\gamma, \mu, \rho \in \Gamma$ such that $t \in (a\gamma x)\overline{\mu}(a\rho y)$ and $a \le t$. Then $t \in u\mu v$ for some $u \in a\gamma x$, $v \in a\rho y$ (by [6, Definition 3.2]). Thus we have $t \le u\widetilde{\mu}v$, $u \le a\widetilde{\gamma}x$ and $v \le a\widetilde{\rho}y$; therefore $t \le a\widetilde{\gamma}x\widetilde{\mu}a\widetilde{\rho}y$. Then $a \le a\widetilde{\gamma}x\widetilde{\mu}a\widetilde{\rho}y$ and so $(M, \widetilde{\Gamma}, \le)$ is right quasi-regular. $\square$

The ordered Γ-semigroup $(M, \widetilde{\Gamma}, \le)$ is called semisimple [4] if for every $a \in M$ there exist $x, y, z \in M$ and $\widetilde{\gamma}, \widetilde{\mu}, \widetilde{\rho}, \widetilde{\omega} \in \widetilde{\Gamma}$ such that $a \le x\widetilde{\gamma}a\widetilde{\mu}y\widetilde{\rho}a\widetilde{\omega}z$.

Definition 2.16 *The ordered Γ-hypersemigroup $(M, \Gamma, \le)$ is called semisimple if for every $a \in M$ there exist $x, y, z \in M$ and $\gamma, \mu, \rho, \omega \in \Gamma$ such that $\{a\} \preceq (x\gamma a)\overline{\mu}(y\rho a)\overline{\omega}\{z\}$; in other words there exist $x, y, z, t \in M$ and $\gamma, \mu, \rho, \omega \in \Gamma$ such that $t \in (x\gamma a)\overline{\mu}(y\rho a)\overline{\omega}\{z\}$ and $a \le t$.*

Theorem 2.17 *The ordered Γ-semigroup $(M, \widetilde{\Gamma}, \le)$ is semisimple if and only if the ordered Γ-hypersemigroup $(M, \Gamma, \le)$ is so.*

Proof $\Longrightarrow$. Let $(M, \widetilde{\Gamma}, \le)$ be semisimple and $a \in M$. Then there exist $x, y, z \in M$ and $\widetilde{\gamma}, \widetilde{\mu}, \widetilde{\rho}, \widetilde{\omega} \in \widetilde{\Gamma}$ such that $a \le (x\widetilde{\gamma}a)\widetilde{\mu}(y\widetilde{\rho}a\widetilde{\omega}z)$. Then

$$a \in (x\widetilde{\gamma}a)\mu(y\widetilde{\rho}a\widetilde{\omega}z) = \{x\widetilde{\gamma}a\}\overline{\mu}\{y\widetilde{\rho}a\widetilde{\omega}z\} \text{ (by [6, Lemma 3.5])}.$$

Since $x\widetilde{\gamma}a \in x\gamma a$, we have $\{x\widetilde{\gamma}a\} \subseteq x\gamma a$. We also have $y\widetilde{\rho}a\widetilde{\omega}z = (y\widetilde{\rho}a)\widetilde{\omega}z \in (y\widetilde{\rho}a)\omega z$ and so

$$\{y\widetilde{\rho}a\widetilde{\omega}z\} \subseteq (y\widetilde{\rho}a)\omega z = \{y\widetilde{\rho}a\}\overline{\omega}\{z\} \text{ (by [6, Lemma 3.5])}.$$

Since $y\widetilde{\rho}a \in y\rho a$, we have $\{y\widetilde{\rho}a\} \subseteq y\rho a$. Then $\{y\widetilde{\rho}a\}\overline{\omega}\{z\} \subseteq (y\rho a)\overline{\omega}\{z\}$ (by [6, Lemma 3.6]). Thus we have $\{y\widetilde{\rho}a\widetilde{\omega}z\} \subseteq (y\rho a)\overline{\omega}\{z\}$. Hence we obtain

$$\begin{aligned} a \in \{x\widetilde{\gamma}a\}\overline{\mu}\{y\widetilde{\rho}a\widetilde{\omega}z\} &\subseteq (x\gamma a)\overline{\mu}\Big((y\rho a)\overline{\omega}\{z\}\Big) \text{ (by [6, Lemmas 3.5 and 3.6])} \\ &= (x\gamma a)\overline{\mu}(y\rho a)\overline{\omega}\{z\} \text{ (by [7, Lemma 2]).} \end{aligned}$$

Then $\{a\} \preceq (x\gamma a)\overline{\mu}(y\rho a)\overline{\omega}\{z\}$ and so $(M,\Gamma,\leq)$ is semisimple.

$\Longleftarrow$. Let $(M,\Gamma,\leq)$ be semisimple and $a \in M$. Then there exist $x,y,z,t \in M$ and $\gamma,\mu,\rho,\omega \in \Gamma$ such that $t \in (x\gamma a)\overline{\mu}(y\rho a)\overline{\omega}\{z\}$ and $a \leq t$. By [7, Lemma 2]), $t \in \Big((x\gamma a)\overline{\mu}(y\rho a)\Big)\overline{\omega}\{z\}$. Since $(x\gamma a)\overline{\mu}(y\rho a) \subseteq M\overline{\mu}M \subseteq M$, by [6, Definition 3.2], we have $t \in h\omega z$ for some $h \in (x\gamma a)\overline{\mu}(y\rho a)$, and $h \in u\mu v$ for some $u \in x\gamma a$, $v \in y\rho a$. Then

$$t \leq h\widetilde{\omega}z,\ h \leq u\widetilde{\mu}v,\ u \leq x\widetilde{\gamma}a,\ v \leq y\widetilde{\rho}a.$$

We have $t \leq (u\widetilde{\mu}v)\widetilde{\omega}z$ and $u\widetilde{\mu}v \leq x\widetilde{\gamma}a\widetilde{\mu}y\widetilde{\rho}a$. Then $t \leq (x\widetilde{\gamma}a\widetilde{\mu}y\widetilde{\rho}a)\widetilde{\omega}z$ and $a \leq x\widetilde{\gamma}a\widetilde{\mu}y\widetilde{\rho}a\widetilde{\omega}z$. Thus $(M,\widetilde{\Gamma},\leq)$ is semisimple. □

The ordered Γ-groupoid $(M,\widetilde{\Gamma},\leq)$ is called right (resp. left) simple if M is the only right (resp. left) ideal of M. This is equivalent to saying that for every $a,b \in M$ there exist $x \in M$ and $\widetilde{\gamma} \in \widetilde{\Gamma}$ such that $b \leq a\widetilde{\gamma}x$ (resp. $b \leq x\widetilde{\gamma}a$). An ordered Γ-semigroup $(M,\widetilde{\Gamma},\leq)$ is called simple if M is the only ideal of M; equivalently, for every $a,b \in M$ there exist $x,y \in M$ and $\widetilde{\gamma},\widetilde{\mu} \in \widetilde{\Gamma}$ such that $b \leq x\widetilde{\gamma}a\widetilde{\mu}y$.

Definition 2.18 *(see also [8, Definition 2.27]) An ordered Γ-hypergroupoid $(M,\Gamma,\leq)$ is called right (resp. left) simple if for every $a,b \in M$ there exist $x \in M$ and $\gamma \in \Gamma$ such that $\{b\} \preceq a\gamma x$ (resp. $\{b\} \preceq x\gamma a$); in other words, for every $a,b \in M$ there exist $x,t \in M$ and $\gamma \in \Gamma$ such that $t \in a\gamma x$ (resp. $t \in x\gamma a$) and $b \leq t$; (this is equivalent to saying that M is the only right (resp. left) ideal of M.*

Remark 2.19 *Let us prove that the following are equivalent: (1) M is the only right ideal of M (2) For every $a,b \in M$ there exist $x \in M$ and $\gamma \in \Gamma$ such that $\{b\} \preceq a\gamma x$. We prove first that, for any nonempty subsets A,B of M, we have $(A]\Gamma(B] \subseteq (A\Gamma B]$. We note that $A\Gamma B = \bigcup\limits_{a\in A,\gamma\in\Gamma,b\in B} a\gamma b \subseteq M$. Let now $x \in (A]\Gamma(B]$. Then $x \in u\gamma v$ for some $u \in (A]$, $\gamma \in \Gamma$, $v \in (B]$. Since $u \in (A]$, we have $u \leq a$ for some $a \in A$. Since $v \in (B]$, we have $v \leq b$ for some $b \in B$. Since $u \leq a$ and $v \leq b$, we have $u\gamma v \preceq a\gamma b$. Since $x \in u\gamma v$, there exists $y \in a\gamma b$ such that $x \leq y$. We have $x \leq y \in a\gamma b \subseteq A\Gamma B$ and so $x \in (A\Gamma B]$. $(1) \Rightarrow (2)$. Let $a,b \in M$. The set $(a\Gamma M]$ is a right ideal of M. Indeed: $(a\Gamma M]\Gamma M = (a\Gamma M]\Gamma(M] \subseteq (a\Gamma M\Gamma M] \subseteq (a\Gamma M]$ and $\big((a\Gamma M]\big] = (a\Gamma M]$ (as it holds for any nonempty subset X of M). By hypothesis we have $(a\Gamma M] = M$. Since $b \in M$, we have $b \leq t$ for some $t \in a\Gamma M$. Since $t \in a\Gamma M$, we have $t \in a\gamma x$ for some $\gamma \in \Gamma$, $x \in M$. We have $t \in a\gamma x$ and $b \leq t$ and so $\{b\} \preceq a\gamma x$. $(2) \Rightarrow (1)$. Let A be a right ideal of M. Then $A = M$. Indeed: Let $b \in M$. Take an element $a \in A$ $(A \neq \emptyset)$. Since $a,b \in M$, by (2), there exist $x,t \in M$ and $\gamma \in \Gamma$ such that $t \in a\gamma x$ and $b \leq t$. We have $t \in a\gamma x \subseteq A\Gamma M \subseteq A$ and so $t \in A$. Since $b \leq t \in A$, we have $b \in A$. Thus we have $M \subseteq A$ and so $A = M$.*

Definition 2.20 *An ordered Γ-hypersemigroup is called simple if for every $a,b \in M$ there exist $x,y \in M$ and*

$\gamma,\mu\in\Gamma$ such that $\{b\}\preceq(x\gamma a)\overline{\mu}\{y\}$; in other words, for every $a,b\in M$ there exist $x,y,t\in M$ and $\gamma,\mu\in\Gamma$ such that $t\in(x\gamma a)\overline{\mu}\{y\}$ and $b\le t$ (this is equivalent to saying that M is the only ideal of M).

Remark 2.21 *Let us prove that, for an ordered Γ-hypersemigroup M, the following conditions are equivalent: (1) M is the only ideal of M (2) For every $a,b\in M$ there exist $x,y\in M$ and $\gamma,\mu\in\Gamma$ such that $\{b\}\preceq(x\gamma a)\overline{\mu}\{y\}$. (1) $\Rightarrow$ (2). Let $a,b\in M$. The set $(M\Gamma a\Gamma M]$ is an ideal of M. Indeed: $(M\Gamma a\Gamma M]\Gamma M=(M\Gamma a\Gamma M]\Gamma(M]\subseteq\big(M\Gamma a\Gamma(M\Gamma M)\big]\subseteq(M\Gamma a\Gamma M]$ also $M\Gamma(M\Gamma a\Gamma M]\subseteq(M\Gamma a\Gamma M]$ and $\big((M\Gamma a\Gamma M]\big]=(M\Gamma a\Gamma M]$. Since M is the only ideal of M, we have $(M\Gamma a\Gamma M]=M$. Since $b\in M$, $b\le t$ for some $t\in M\Gamma a\Gamma M$. Since $t\in(M\Gamma a)\Gamma M$, by [6, Remark 3.4], $t\in u\mu y$ for some $u\in M\Gamma a$, $\mu\in\Gamma$, $y\in M$. Since $u\in M\Gamma a$, $u\in x\gamma a$ for some $x\in M$, $\gamma\in\Gamma$. We have $t\in u\mu y=\{u\}\overline{\mu}\{y\}\subseteq(x\gamma a)\overline{\mu}\{y\}$. Since $t\in(x\gamma a)\overline{\mu}\{y\}$ and $b\le t$, we have $\{b\}\preceq(x\gamma a)\overline{\mu}\{y\}$. (2) $\Rightarrow$ (1). Let A be an ideal of M. Then $A=M$. Indeed: Let $b\in M$. Take an element $a\in A$ $(A\neq\emptyset)$. Since $a,b\in M$, by (2), there exist $x,y,t\in M$ and $\gamma,\mu\in\Gamma$ such that $t\in(x\gamma a)\overline{\mu}\{y\}$ and $b\le t$. We have $x\gamma a\subseteq M\Gamma A\subseteq A$ (by [6, Remark 3.4]) and $(x\gamma a)\overline{\mu}\{y\}\subseteq A\overline{\mu}M\subseteq A\Gamma M\subseteq A$ (see [6, Lemma 3.6 and Definition 3.3]). Thus we have $t\in A$. Since $b\le t\in A$, we have $b\in A$. Thus we $M\subseteq A$ and so $A=M$.*

Theorem 2.22 *The ordered Γ-groupoid $(M,\widetilde{\Gamma},\le)$ is right (resp. left) simple if and only if the ordered Γ-hypergroupoid $(M,\Gamma,\le)$ is so. The ordered Γ-semigroup $(M,\widetilde{\Gamma},\le)$ is simple if and only if the ordered Γ-hypersemigroup $(M,\Gamma,\le)$ is so.*

Proof Let $(M,\widetilde{\Gamma},\le)$ be right simple and $a,b\in M$. Then there exist $x\in M$ and $\widetilde{\gamma}\in\widetilde{\Gamma}$ such that $b\le a\widetilde{\gamma}x$. Then we have $b\in a\gamma x$ and so $\{b\}\preceq a\gamma x$ and $(M,\Gamma,\le)$ is right simple. Let now $(M,\Gamma,\le)$ be right simple and $a,b\in M$. Then there exist $x,t\in M$ and $\gamma\in\Gamma$ such that $t\in a\gamma x$ and $b\le t$. Then we have $t\le a\widetilde{\gamma}x$ and so $b\le a\widetilde{\gamma}x$; thus $(M,\widetilde{\Gamma},\le)$ is right simple.

Let $(M,\widetilde{\Gamma},\le)$ be simple and $a,b\in M$. Then there exist $x,y\in M$ and $\widetilde{\gamma},\widetilde{\mu}\in\widetilde{\Gamma}$ such that $b\le(x\widetilde{\gamma}a)\widetilde{\mu}y$. Then $b\in(x\widetilde{\gamma}a)\mu y=\{x\widetilde{\gamma}a\}\overline{\mu}\{y\}$ (by [6, Lemma 3.5]). Since $x\widetilde{\gamma}a\in x\gamma a$, we have $\{x\widetilde{\gamma}a\}\subseteq x\gamma a$; then $\{x\widetilde{\gamma}a\}\overline{\mu}\{y\}\subseteq(x\gamma a)\overline{\mu}\{y\}$ (by [6, Lemma 3.5]). Then $\{b\}\preceq(x\gamma a)\overline{\mu}\{y\}$ and so $(M,\Gamma,\le)$ is simple.
Let now $(M,\Gamma,\le)$ be simple and $b\in M$. Then there exist $x,y,t\in M$ and $\gamma,\mu\in\Gamma$ such that $t\in(x\gamma a)\overline{\mu}\{y\}$ and $b\le t$. By [6, Definition 3.2], $t\in u\mu y$ for some $u\in x\gamma a$. Then we have $t\le u\widetilde{\mu}y$ and $u\le x\widetilde{\gamma}a$ and so $t\le x\widetilde{\gamma}a\widetilde{\mu}y$. Then $b\le x\widetilde{\gamma}a\widetilde{\mu}y$ and so $(M,\widetilde{\Gamma},\le)$ is simple. □

The concept of strongly regular ordered semigroups [1] can be extended to ordered Γ-semigroups as follows: An ordered Γ-semigroup $(M,\Gamma,\le)$ is called strongly regular if it is regular and $a\gamma b=b\gamma a$ for every $a,b\in M$ and every $\gamma\in\Gamma$; that is, it is regular and commutative.

The Γ-hypergroupoid M is called commutative if $a\gamma b=b\gamma a$ for every $a,b\in M$ and every $\gamma\in\Gamma$. That is, if $a,b\in M$ and $\gamma\in\Gamma$, then $x\in a\gamma b$ if and only if $x\in b\gamma a$.

Definition 2.23 *An ordered Γ-hypersemigroup $(M,\Gamma,\le)$ is called strongly regular if it is regular and commutative.*

Proposition 2.24 *The ordered Γ-semigroup $(M,\widetilde{\Gamma},\le)$ is strongly regular if and only if the ordered Γ-hypersemigroup $(M,\Gamma,\le)$ is so.*

Proof It is enough to prove that the ordered Γ-groupoid $(M,\widetilde{\Gamma},\leq)$ is commutative if and only if the ordered Γ-hypergroupoid $(M,\Gamma,\leq)$ is commutative.

$\Rightarrow$. Let $(M,\widetilde{\Gamma},\leq)$ be commutative and $a,b\in M$, $\gamma\in\Gamma$. If $x\in a\gamma b$, then $x\leq a\widetilde{\gamma}b=b\widetilde{\gamma}a$, $x\leq b\widetilde{\gamma}a$, and $x\in b\gamma a$. If $x\in b\gamma a$, then $x\leq b\widetilde{\gamma}a=a\widetilde{\gamma}b$, $x\leq a\widetilde{\gamma}b$, and $x\in a\gamma b$. Thus we have $x\in a\gamma b$ if and only if $x\in b\gamma a$ and so $(M,\Gamma,\leq)$ is commutative.

$\Leftarrow$. Let $(M,\Gamma,\leq)$ be commutative and $a,b\in M$, $\widetilde{\gamma}\in\widetilde{\Gamma}$. Since $a\widetilde{\gamma}b\in a\gamma b=b\gamma a$, we have $a\widetilde{\gamma}b\in b\gamma a$ and so $a\widetilde{\gamma}b\leq b\widetilde{\gamma}a$. Since $b\widetilde{\gamma}a\in b\gamma a=a\gamma b$, we have $b\widetilde{\gamma}a\in a\gamma b$ and so $b\widetilde{\gamma}a\leq a\widetilde{\gamma}b$. Thus we have $a\widetilde{\gamma}b=b\widetilde{\gamma}a$ and $(M,\widetilde{\Gamma},\leq)$ is commutative. $\square$

Example 2.25 *Consider the ordered Γ-semigroup $(M,\widetilde{\Gamma},\leq)$ given by Tables 1, 2 and Figure 1 of the Example 2.2. The set $\{a,b,c\}$ is an ideal of $(M,\widetilde{\Gamma},\leq)$. Indeed:*

$$\begin{aligned}\{a,b,c\}\widetilde{\Gamma}M &= \{a,b,c\}\{\widetilde{\gamma},\widetilde{\mu}\}\{a,b,c,d\}\\ &= \{a\widetilde{\gamma}a, a\widetilde{\gamma}b, a\widetilde{\gamma}c, a\widetilde{\gamma}d, b\widetilde{\gamma}a, b\widetilde{\gamma}b, b\widetilde{\gamma}c, b\widetilde{\gamma}d, c\widetilde{\gamma}a, c\widetilde{\gamma}b, c\widetilde{\gamma}c, c\widetilde{\gamma}d, a\widetilde{\mu}a,\\ &\quad a\widetilde{\mu}b, a\widetilde{\mu}c, a\widetilde{\mu}d, b\widetilde{\mu}a, b\widetilde{\mu}b, b\widetilde{\mu}c, b\widetilde{\mu}d, c\widetilde{\mu}a, c\widetilde{\mu}b, c\widetilde{\mu}c, c\widetilde{\mu}d\}\\ &= \{a,b,c\},\end{aligned}$$

we also have $M\widetilde{\Gamma}\{a,b,c\}\subseteq\{a,b,c\}$ and $x\in\{a,b,c\}$ and $M\ni y\leq x$ implies $y\in\{a,b,c\}$.

The set $\{a,b,c\}$ is an ideal of the ordered Γ-hypersemigroup $(M,\Gamma,\leq)$ defined by Tables 3, 4 and Figure 1. Indeed:

$$\begin{aligned}\{a,b,c\}\Gamma M &= \bigcup_{x\in\{a,b,c\},\gamma\in\Gamma,y\in\{a,b,c,d\}} x\gamma y\\ &= a\gamma a\cup a\gamma b\cup a\gamma c\cup a\gamma d\cup b\gamma a\cup b\gamma b\cup b\gamma c\cup b\gamma d\cup c\gamma a\cup c\gamma b\cup c\gamma c\cup c\gamma d\\ &= a\mu a\cup a\mu b\cup a\mu c\cup a\mu d\cup b\mu a\cup b\mu b\cup b\mu c\cup b\mu d\cup c\mu a\cup c\mu b\cup c\mu c\cup c\mu d\\ &= \{a,b,c\};\end{aligned}$$

that is, $\{a,b,c\}$ is a right ideal of $(M,\Gamma,\leq)$. Similarly $M\Gamma\{a,b,c\}\subseteq\{a,b,c\}$ and so the set $\{a,b,c\}$ is an ideal of $(M,\Gamma,\leq)$ (see also Theorem 2.5). In fact, one can check that the set $\{a,b,c\}$ is the only ideal of $(M,\widetilde{\Gamma},\leq)$ and the only ideal of $(M,\Gamma,\leq)$ as well.

Example 2.26 *(see also [5, Example 15]) Consider the ordered Γ-semigroup $(M,\widetilde{\Gamma},\leq)$, where $M=\{a,b,c,d,e\}$ and $\widetilde{\Gamma}=\{\widetilde{\gamma}\}$ defined by Table 11 and Figure 5; this is regular, intra-regular, right regular and left regular.*
The right ideals of $(M,\widetilde{\Gamma},\leq)$ are the sets $\{a,b,d\}$ and M.
The left ideals of $(M,\widetilde{\Gamma},\leq)$ are the sets $\{a\}$, $\{a,c\}$, $\{a,b,d\}$, $\{a,b,c,d\}$, and M.
The bi-ideals and the quasi-ideals of $(M,\widetilde{\Gamma},\leq)$ coincide with the left ideals of $(M,\widetilde{\Gamma},\leq)$.

According to Theorem 2.1, to this ordered Γ-semigroup corresponds the ordered Γ-hypersemigroup given by Table 12 and the same Figure 5.

According to Theorems 2.11, and 2.13 this is also a regular, intra-regular, right regular and left regular ordered Γ-hypersemigroup, having the same right ideals, left ideals, bi-ideals and quasi-ideals as the ordered Γ-semigroup $(M,\widetilde{\Gamma},\leq)$.

Table 11. The ordered Γ-semigroup of the Example 2.26.

$\widetilde{\gamma}$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

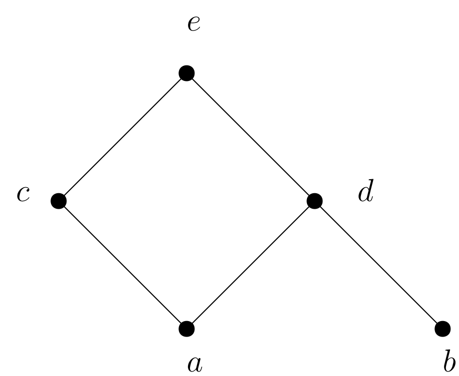

Figure 5. Figure of Example 2.26.

Table 12. The ordered Γ-hypersemigroup of the Example 2.26.

γ	a	b	c	d	e
a	{a}	{a,b,d}	{a}	{a,b,d}	{a,b,d}
b	{a}	{b}	{a}	{a,b,d}	{a,b,d}
c	{a}	{a,b,d}	{a,c}	{a,b,d}	{a,b,c,d,e}
d	{a}	{a,b,d}	{a}	{a,b,d}	{a,b,d}
a	{a}	{a,b,d}	{a,c}	{a,b,d}	{a,b,c,d,e}

Example 2.27 *The ordered Γ-semigroup of the Example 2.2 given by Tables 1, 2 and Figure 1 is an example of regular, intra-regular, right regular, left regular, right quasi-regular, left quasi-regular and semisimple ordered Γ-semigroup. [This is not right simple, not left simple, not simple, not strongly regular]. According to Theorem 2.11, Theorem 2.13, Theorem 2.15 and Theorem 2.17, the ordered Γ-hypersemigroup given by Tables 3, 4 and Figure 1 is an example of a regular, intra-regular, right regular, left regular, right quasi-regular, left quasi-regular and semisimple ordered Γ-hypersemigroup, respectively. According to Theorem 2.5 and Example 2.25, the set $\{a, b, c\}$ is an ideal of the ordered Γ-hypersemigroup. Independently, one can check that the above results are true. It might be mentioned here that for an ordered Γ-semigroup or an ordered Γ-hypersemigroup the ideals are quasi-ideals and the quasi-ideals are bi-ideals. An ordered Γ-semigroup or an ordered Γ-hypersemigroup that is right regular or left regular is intra-regular as well.*

Example 2.28 *The ordered Γ-semigroup of the Example 2.2 given by Tables 1 and 2 and Figure 2, 3 or 4 are also regular, intra-regular, right regular, left regular, right quasi-regular, left quasi-regular and semisimple and so,* *according to Theorem 2.11, Theorem 2.13, Theorem 2.15 and Theorem 2.17, the ordered* Γ*-hypersemigroup given by Tables 5, 6 and Figure 2, the ordered* Γ*-hypersemigroup given by Tables 7, 8 and Figure 3 and the ordered* Γ*-hypersemigroup given by Tables 9, 10 and Figure 4 are also examples of regular, intra-regular, right regular, left regular, right quasi-regular, left quasi-regular and semisimple ordered* Γ*-hypersemigroups. Independently, one can check that the results of the example are true.*

Note The Lemma 11 in [5] should be corrected as follows: Let $(S, \cdot, \leq)$ be an ordered groupoid and $(S, \circ, \leq)$ the ordered hypergroupoid constructed by $(S, \cdot, \leq)$ in the way indicated in [5, Lemma 1]. Let $a, x \in S$. Then $ax = xa$ if and only if $a \circ x = x \circ a$.

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