Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 638 – 661
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doi:10.3906/mat-2108-34

Multiple positive solutions for nonlinear fractional q-difference equation with p-Laplacian operator

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Received: 07.08.2021 • **Accepted/Published Online:** 04.11.2021 • **Final Version:** 21.01.2022

Abstract: In this paper, we investigate a class of four-point boundary value problems of fractional q-difference equation with p-Laplacian operator which is the first time to be studied and is extended from a bending elastic beam equation. By Avery–Peterson theorem and the method of lower and upper solutions associated with monotone iterative technique, we obtain some sufficient conditions for the existence of multiple positive solutions. As applications, examples are presented to illustrate the main results.

Key words: Fractional q-difference equation, p-Laplacian operator, mixed derivatives, positive solution

1. Introduction

The q-difference calculus or quantum calculus has been of great interest recently. It was initially developed by Jackson [13]. In regard to basic definitions and properties of q-difference calculus, the reader can confirm in the books [11] and [4]. It is well know that the time scale calculus includes q-difference calculus (or quantum calculus) as a special case (i.e. dynamic equations on time scales include related q-difference equations as a special case); see, e.g., the papers [6–8] for more details.

More recently, perhaps due to the explosion in research within the fractional calculus setting, new developments in the theory of fractional q-difference calculus were made. Compared with integer order q-calculus, fractional q-calculus is better and more accurate to describe physical phenomena. Therefore, the theory of fractional q-calculus has been widely used in the fields of mathematical physics, dynamical systems and quantum models and so on[9, 19, 23]. The fractional q-difference calculus had its origin in the works by Al-Salam [3] and Agarwal[1]. Many effective and interesting results can be found in [2, 9, 24] and references therein.

As a matter of fact, p-Laplace equations (equations with p-Laplacian like operators) arise in a variety of real world problems such as in the study of non-Newtonian fluid theory, porous medium problems, chemotaxis models, and so forth; see, e.g., the papers [6–8, 10, 15–18, 26] for more details. Fractional differential equations with p-Laplacian operators have been widely applied in many fields of science and engineering, such as viscoelastic mechanics, non-Newtonian mechanics, electrochemistry, fluid mechanics, combustion theory, materials science, etc. There are some papers dealing with the existence of solutions for fractional differential equations and fractional q-difference equation with p-Laplacian operator, see [12, 14, 20, 21, 25]. For example, very

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2010 *AMS Mathematics Subject Classification:* 26A33, 34A12, 39A27, 39A13

recently, S. Li, Z. Zhang and W. Jiang studied the existence of at least triple positive solutions for four-point boundary value problems of nonlinear fractional differential equations with p-Laplacian operators by using the Avery–Peterson theorem [14].

In [12], Z. Han et al. investigated the following eigenvalue problem of fractional differential equation with generalized p-Laplacian.

$$\begin{cases} D_{0+}^{\beta}(\varphi(D_{0+}^{\alpha}u(t))) = \lambda f(u(t)),\ 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0,\ \varphi(D_{0+}^{\alpha}u(0)) = (\varphi(D_{0+}^{\alpha}u(0)))' = 0, \end{cases}$$

where $2 < \alpha \leq 3,\ 1 < \beta \leq 2$. By using the properties of Green function and Guo-Krasnosel'skii fixed-point theorem in cones, several new existence results of positive solutions in terms of different eigenvalue intervals are obtained.

In [21], X. Li et al. studied the following eigenvalue problems of a class of nonlinear fractional q-difference equations with generalized p-Laplacian

$$\begin{cases} D_q^{\gamma}(\varphi(D_q^{\alpha}u(t))) + \lambda f(u(t)) = 0,\ 0 < t < 1, \\ u(0) = D_q u(0) = 0,\ D_q u(1) = \beta > 0,\ D_q^{\alpha}u(0) = 0, \end{cases}$$

and the second kind is homogeneous boundary conditions

$$\begin{cases} D_q^{\gamma}(\varphi(D_q^{\alpha}v(t))) + \lambda f(v(t)) = 0,\ 0 < t < 1, \\ v(0) = D_q v(0) = 0,\ D_q v(1) = 0,\ D_q^{\alpha}v(0) = 0. \end{cases}$$

By using fixed point theorem in cones, some results for the existence of positive solutions are obtained.

In [25], Q. Yuan and W. Yang considered the fractional q-difference four-point boundary value problem with p-Laplacian operator

$$\begin{cases} D_q^{\beta}(\varphi_p(D_q^{\alpha}u(t))) = f(t, u(t)),\ t \in (0,1), \\ u(0) = 0,\ u(1) = au(\xi),\ D_q^{\alpha}u(0) = 0,\ D_q^{\alpha}u(1) = bD_q^{\alpha}(\eta), \end{cases}$$

where $1 < \alpha \leq 2$, and $0 < a,\ b,\ \xi,\ \eta < 1$. By means of the upper and lower solutions method associated with the Schauder fixed point theorem, some existence results of at least one solution are obtained.

Motivated by the previously mentioned works, we will investigate the following four-point boundary value problems of fractional q-difference equation with p-Laplacian operator

$$D_q^{\beta}(\psi_p({}^cD_q^{\alpha}u(x))) = g(x, u(x), {}^cD_q^{\gamma}u(x)),\ 0 < x < 1, \tag{1.1}$$

subject to the boundary conditions

$$\begin{gathered} {}^cD_q^{\alpha}u(0) = D_q u(0) = 0, \\ u(1) = cu(\lambda),\ {}^cD_q^{\alpha}u(1) = d\,{}^cD_q^{\alpha}u(\zeta), \end{gathered} \tag{1.2}$$

where $0 < q < 1$, $1 < \alpha, \beta \leq 2$, $0 < \gamma \leq \alpha$, $0 < \lambda, \zeta < 1$, and $c, d > 0$. D_q^{β} is the Riemann–Liouville fractional derivative, ${}^cD_q^{\alpha}$ and ${}^cD_q^{\gamma}$ are the Caputo fractional derivative. $\psi_m = \psi_p^{-1}$, ψ_p is the p-Laplacian operator, $\psi_p(t) = |t|^{p-2}t$, $p > 1$, $\frac{1}{p} + \frac{1}{m} = 1$ and $g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$.

The innovation of this paper are as follows:

(i) Compared with [25], we generalize the nonlinear term of the q-difference equation to the case with higher derivative which makes the boundary value problem we study more widely applicable. Especially, when $p=2$, $\alpha=\beta=\gamma=2$ and $c=d=0$, the boundary value problem (1.1)–(1.2) models the deformations of an elastic beam whose two ends are simply supported in equilibrium state, and the ${}^{c}D_{q}^{\gamma}u(x)$ in function g is the bending moment term which represents bending effect.

(ii) Although the solvability of multipoint boundary value problems for fractional q-difference equation has been investigated by some authors, to the best of our knowledge, there are no papers that consider the multiple positive solutions for four-point boundary value problem of fractional q-difference equation with p-Laplacian operator. Inspired by works mentioned above, we aim to fill the gap. Adding the p-Laplacian operator makes this paper posses wider range of potential applications, for instance, compared with the image inpainting method based on total variation model, the image inpainting method based on p-Laplacian operator can effectively improve the image inpainting quality and significantly reduce the operation time. By using the technique of Avery and Peterson and the method of lower and upper solutions, we deduce some sufficient conditions for the existence of multiple positive solutions.

The plan of this paper is as follows. In Section 2, we present necessary definitions, properties and lemmas. In Section 3, we apply the Avery–Peterson theorem to establish the existence criteria of at least triple positive solutions for (1.1)–(1.2) and give rigorous proof. In Section 4, we obtain some new sufficient conditions for the existence of solutions by the method of lower and upper solutions. At the end of this paper, we present two examples to illustrate the effectiveness of the main results.

2. Preliminaries

In this section, we present basic definitions, notations, and lemmas that will be used in this paper. Let $0<q<1$. Define

$$[\alpha]_q := \frac{1-q^{\alpha}}{1-q}, \quad \alpha\in\mathbb{R}, \quad [n]_q! = [n]_q[n-1]_q\cdots[1]_q, \ n\in\mathbb{N}. \tag{2.1}$$

Let $a,b\in\mathbb{R}$. Define the q-analogue of the power function $(a-b)_q^{(n)}$

$$(a-b)_q^{(n)} := \begin{cases} 1, & n=0, \\ \prod\limits_{k=0}^{n-1}(a^k-bq^k), & n\in\mathbb{N}^+. \end{cases}$$

If $\alpha\in\mathbb{R}$, the general form is given by

$$(a-b)_q^{(\alpha)} := a^{\alpha}\prod_{i=0}^{\infty}\Big[\frac{a-bq^i}{a-bq^{\alpha+i}}\Big], \quad a\neq 0.$$

Note that when $b=0$, $(a)_q^{(\alpha)}=a^{\alpha}$. For $0<q<1$, the q-gamma function is defined by

$$\Gamma_q(x) = \begin{cases} \dfrac{(1-q)_q^{(x-1)}}{(1-q)^{x-1}}, & x\in\mathbb{R}\setminus\{0,-1,-2,\cdots\}, \\ [x-1]_q!, & x\in\mathbb{N}, \end{cases}$$

and satisfies $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$. The q-derivative of the function f is defined as

$$D_q f(t) := \frac{f(t) - f(qt)}{(1-q)t}, \ t \neq 0, \quad (D_q f)(0) = \lim_{t\to 0}(D_q f)(t),$$

provided that f is differentiable at 0. And the n order q-derivative $D_q^n f(t)$ is defined by

$$D_q^n f(t) = \begin{cases} f(t), n = 0, \\ D_q D_q^{n-1} f(t), n \in \mathbb{N}^+. \end{cases}$$

The following formulas will be used later, namely

$$(a(t-s))_q^{(\alpha)} = a^\alpha (t-s)_q^{(\alpha)}, \tag{2.2}$$

$${}_tD_q(t-s)_q^{(\alpha)} = [a]_q (t-s)_q^{(\alpha-1)}, \tag{2.3}$$

$${}_sD_q(t-s)_q^{(\alpha)} = -[a]_q (t-qs)_q^{(\alpha-1)}, \tag{2.4}$$

where ${}_tD_q$ or ${}_sD_q$ denotes the derivative with respect to the variable t or s respectively.

Definition 2.1 *[24] Let $0 < q < 1$, f be an arbitrary function. The q-integral of the function f is defined as*

$$\int_0^x f(t)d_qt = (1-q)x\sum_{k=0}^{\infty} q^k f(q^k x), \tag{2.5}$$

provided that the series of right side in (2.5) converges. In this case, f is called q-integrable on $[0, x]$. Denote

$$I_q f(x) = \int_0^x f(t)d_qt.$$

And the q-integral of n order is defied by

$$I_q^0 f(x) = f(x) \text{ and } I_q^n f(x) = I_q(I_q^{n-1} f)(x).$$

Definition 2.2 *[24] Let $0 < q < 1$, f be an arbitrary function, a and b be two real numbers. Then we define*

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Definition 2.3 *[24] Let $\alpha > 0$, $0 < q < 1$. The fractional q-integral is defined by*

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{(\alpha-1)} f(s)d_qs.$$

Definition 2.4 *[22] The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $f : [0, +\infty) \to \mathbb{R}$ is given by*

$$(D_q^\alpha f)(x) = (D_q^N I_q^{N-\alpha} f)(x),$$

and $(D_q^0 f)(x) = f(x)$, where N is the smallest integer greater than or equal to α.

Definition 2.5 *[22] The fractional q-derivative of the Caputo type of order $\alpha > 0$ is defined by*

$$({}^cD_q^\alpha f)(x) = (I_q^{N-\alpha} D_q^N f)(x),$$

and $({}^cD_q^0 f)(x) = f(x)$, where $N-1 < \alpha < N$, $N \in \mathbb{N}$.

Lemma 2.6 *[24] For α, $\beta > 0$, and $0 < q < 1$, q-integral and q-difference operators have the following properties:*

(a) $I_q^\alpha [I_q^\beta f(x)] = I_q^\beta [I_q^\alpha f(x)] = I_q^{\alpha+\beta} f(x)$,

(b) $D_q I_q f(x) = f(x)$, and $I_q D_q f(x) = f(x) - f(0)$.

(c) $D_q^\alpha I_q^\alpha f(x) = f(x)$.

Lemma 2.7 *[22] Let $0 < q < 1$, $\alpha \in (N-1, N]$, $N \in \mathbb{N}$. Then*

$$D_q^\alpha I_q^\alpha f(t) = f(t), \tag{2.6}$$

and

$$I_q^\alpha D_q^\alpha f(t) = f(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{2.7}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$.

Lemma 2.8 *[22] Let $\alpha \in (N-1, N]$, $N \in \mathbb{N}$, and $0 < q < 1$. Then the following is valid*

$${}^cD_q^\alpha I_q^\alpha f(t) = f(t), \tag{2.8}$$

and

$$I_q^{\alpha c} D_q^\alpha f(t) = f(t) + C_1 t^{N-1} + C_2 t^{N-2} + \cdots + C_N, \tag{2.9}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$.

Lemma 2.9 *Assume $1 < \alpha \leq 2$. Then $D_q^\alpha x^{\alpha-1} = 0$, for $x \in \mathbb{R}$.*

Proof By virtue of Definitions 2.1, 2.3 and 2.4, we have

$$\begin{aligned} D_q^\alpha x^{\alpha-1} &= D_q^2 I_q^{2-\alpha} x^{\alpha-1} \\ &= D_q^2 \left[\frac{1}{\Gamma_q(2-\alpha)} \int_0^x (x-qt)_q^{(1-\alpha)} t^{\alpha-1} d_q t \right] \\ &= \frac{1}{\Gamma_q(2-\alpha)} D_q^2 \left[(1-q)x \sum_{k=0}^{\infty} q^k \left(1-q^{k+1}\right)_q^{(1-\alpha)} \left(q^k\right)^{\alpha-1} \right] \\ &= 0. \end{aligned}$$

This completes the proof.

Lemma 2.10 *[4] Let $g \in C_r[0, a]$, $a > 0$, where $g \in C_r[0, a]$ is equivalent that there exists a constant $\gamma < 1$ such that $x^\gamma g \in C[0, a]$. Then*

(1) *$I_q^\alpha g \in C_r[0, a]$.*

(2) *If we additionally assume that $\gamma \leq \alpha$, then $I_q^\alpha g \in C[0, a]$.*

Definition 2.11 *[4] Let* $0<q<1$ *and* $n\in\mathbb{N}^+$. *We define the space* $C_q^n[0,a]$ *to be the space of all continues functions with continuous* q*-derivatives up to order* $n-1$ *on the interval* $[0,a]$.

Lemma 2.12 *[4] Let* $\alpha>0$, $n=\lceil\alpha\rceil$. *If there exists* $\gamma\le\alpha-n+1$, *such that* $f\in C_r[0,a]$, $a>0$, *then* $I_q^\alpha f\in C_q^n[0,a]$.

3. The solvability based on Avery–Peterson theorem

In this section, we shall establish an existence criterion of at least triple positive solutions to the problem (1.1)–(1.2) by the Avery–Peterson theorem. In order to prove our main results, we need the following lemmas.

Lemma 3.1 *Let* $0<q<1$, $1<\alpha\le 2$, $d^{p-1}\zeta^{\beta-1}<1$ *and* $h\in C([0,1],[0,+\infty))$. *Then a function* $u(x)\in C_q^2[0,1]$ *is a solution of the following boundary value problem*

$$\begin{cases} D_q^\beta(\psi_p({}^cD_q^\alpha u(x)))=h(x),\ 0<x<1,\\ {}^cD_q^\alpha u(0)=D_qu(0)=0,\\ u(1)=cu(\lambda),\ {}^cD_q^\alpha u(1)=d^cD_q^\alpha u(\zeta),\end{cases}\tag{3.1}$$

when and only when $u(x)$ *satisfies the integral equation*

$$u(x)=a_1-\frac{1}{\Gamma_q(\alpha)}\int_0^x(x-qs)_q^{(\alpha-1)}k(s)d_qs,\tag{3.2}$$

where

$$a_1=\frac{\int_0^1(1-qs)_q^{(\alpha-1)}k(s)ds-c\int_0^\lambda(\lambda-qs)_q^{(\alpha-1)}k(s)d_qs}{(1-c)\Gamma_q(\alpha)},\tag{3.3}$$

$$k(x)=\psi_m\left[-\left(\frac{1}{\Gamma_q(\beta)}\int_0^x(x-qt)_q^{(\beta-1)}h(t)d_qt+b_1x^{\beta-1}\right)\right],\tag{3.4}$$

and

$$b_1=\frac{d^{p-1}\int_0^\zeta(\zeta-qt)_q^{(\beta-1)}h(t)d_qt-\int_0^1(1-qt)_q^{(\beta-1)}h(t)d_qt}{(1-d^{p-1}\zeta^{\beta-1})\Gamma_q(\beta)}.$$

Proof Assume $u(x)$ is a solution of (3.1). Applying the operator I_q^β on both sides of (3.1), by Lemma 2.7 and Definition 2.3, for $x\in[0,1]$, we have

$$\psi_p({}^cD_q^\alpha u(x))=b_1x^{\beta-1}+b_2x^{\beta-2}+\frac{1}{\Gamma_q(\beta)}\int_0^x(x-qt)_q^{(\beta-1)}h(t)d_qt.$$

According to ${}^cD_q^\alpha u(0)=0$, we have $b_2=0$. By ${}^cD_q^\alpha u(1)=d^cD_q^\alpha u(\zeta)$, it is easy to see

$$b_1=\frac{d^{p-1}\int_0^\zeta(\zeta-qt)_q^{(\beta-1)}h(t)d_qt-\int_0^1(1-qt)_q^{(\beta-1)}h(t)d_qt}{(1-d^{p-1}\zeta^{\beta-1})\Gamma_q(\beta)},\tag{3.5}$$

that is,

$$\psi_p({}^cD_q^\alpha u(x)) = b_1 x^{\beta-1} + \frac{1}{\Gamma_q(\beta)} \int_0^x (x-qt)_q^{(\beta-1)} h(t) d_q t. \tag{3.6}$$

By the above formula, we give the following definition

$${}^cD_q^\alpha u(x) = \psi_m \left(b_1 x^{\beta-1} + \frac{1}{\Gamma_q(\beta)} \int_0^x (x-qt)_q^{(\beta-1)} h(t) d_q t \right) := -k(x). \tag{3.7}$$

Taking operator I_q^α on both sides of (3.7), from Lemma 2.8, we have

$$u(x) = -I_q^\alpha k(x) + a_1 + a_2 x = -\frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qs)_q^{(\alpha-1)} k(s) d_q s + a_1 + a_2 x. \tag{3.8}$$

Differentiating both sides of (3.8), one has

$$D_q u(x) = -\frac{1}{\Gamma_q(\alpha-1)} \int_0^x (x-qs)_q^{(\alpha-2)} k(s) d_q s + a_2.$$

By the boundary condition $D_q u(0) = 0$, we can get $a_2 = 0$. It follows from (3.8) and the boundary condition $u(1) = cu(\lambda)$ that

$$a_1 = \frac{\int_0^1 (1-qs)_q^{(\alpha-1)} k(s) d_q s - c \int_0^\lambda (\lambda-qs)_q^{(\alpha-1)} k(s) d_q s}{(1-c)\Gamma_q(\alpha)}.$$

Hence

$$u(x) = \frac{\int_0^1 (1-qs)_q^{(\alpha-1)} k(s) d_q s - c \int_0^\lambda (\lambda-qs)_q^{(\alpha-1)} k(s) d_q s}{(1-c)\Gamma_q(\alpha)} - \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qs)_q^{(\alpha-1)} k(s) d_q s. \tag{3.9}$$

On the other hand, if $u(x)$ is a solution of (3.2)(i.e.(3.9)), then we have

$$\begin{aligned} u(x) &= -I_q^\alpha k(x) + a_1 \\ &= I_q^\alpha \psi_m [b_1 x^{\beta-1} + I_q^\beta h(x)] + a_1. \end{aligned} \tag{3.10}$$

From the continuity of function h and Lemma 2.10, we have $\psi_m[b_1 x^{\beta-1} + I_q^\beta h(x)] \in C[0,1]$. Then by Lemma 2.12, $I_q^\alpha \psi_m[b_1 x^{\beta-1} + I_q^\beta h(x)] \in C_q^2[0,1]$. Hence $u(x) \in C_q^2[0,1]$. Taking operator ${}^cD_q^\alpha$ on both sides of (3.10), by Lemma 2.8, we can obtain

$${}^cD_q^\alpha u(x) = \psi_m [b_1 x^{\beta-1} + I_q^\beta h(x)],$$

i.e.,

$$\psi_p({}^cD_q^\alpha u(x)) = b_1 x^{\beta-1} + I_q^\beta h(x).$$

Then taking operator D_q^β on both sides of the above equality, by Lemmas 2.7 and 2.9, one has

$$D_q^\beta(\psi_p({}^cD_q^\alpha u(x))) = h(x).$$

In addition, we can easily prove that $u(x)$ satisfies the boundary value conditions in (3.1). This completes the proof.

Lemma 3.2 *If* $1 < \alpha \le 2$, $d^{p-1}\zeta^{\beta-1} < 1$, $h \in C([0,1],[0,+\infty))$ *and the function* $\hbar(x)$ *is defined by*

$$\hbar(x) = a_1 - \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qs)_q^{(\alpha-1)}k(s)d_qs, \tag{3.11}$$

where a_1 *and* $k(s)$ *are given by* (3.3) *and* (3.4), *then* $\hbar(x) \ge 0$.

Proof Since

$$\begin{aligned}
k(s) &= \psi_m\left(-b_1 s^{\beta-1} - \frac{1}{\Gamma_q(\beta)}\int_0^s (s-qt)_q^{(\beta-1)}h(t)d_qt\right)\\
&= \psi_m\left(\frac{-d^{p-1}s^{\beta-1}\int_0^\zeta(\zeta-qt)_q^{(\beta-1)}h(t)d_qt + s^{\beta-1}\int_0^1(1-qt)_q^{(\beta-1)}h(t)d_qt}{(1-d^{p-1}\zeta^{\beta-1})\Gamma_q(\beta)} - \frac{\int_0^s(s-qt)_q^{(\beta-1)}h(t)d_qt}{\Gamma_q(\beta)}\right)\\
&= \psi_m\left(\frac{\int_0^1 s^{\beta-1}(1-qt)_q^{(\beta-1)}h(t)d_qt}{\Gamma_q(\beta)} + \frac{d^{p-1}\zeta^{\beta-1}\int_0^1 s^{\beta-1}(1-qt)_q^{(\beta-1)}h(t)d_qt}{(1-d^{p-1}\zeta^{\beta-1})\Gamma_q(\beta)}\right.\\
&\qquad \left.-\frac{d^{p-1}s^{\beta-1}\int_0^\zeta(\zeta-qt)_q^{(\beta-1)}h(t)d_qt}{(1-d^{p-1}\zeta^{\beta-1})\Gamma_q(\beta)} - \frac{\int_0^s(s-qt)_q^{(\beta-1)}h(t)d_qt}{\Gamma_q(\beta)}\right)\\
&\ge \psi_m\left\{\frac{d^{p-1}s^{\beta-1}\left[\int_0^1 \zeta^{\beta-1}(1-qt)_q^{(\beta-1)}h(t)d_qt - \int_0^\zeta(\zeta-qt)_q^{(\beta-1)}h(t)d_qt\right]}{(1-d^{p-1}\zeta^{\beta-1})\Gamma_q(\beta)} + \frac{\int_s^1 s^{\beta-1}(1-qt)_q^{(\beta-1)}h(t)d_qt}{\Gamma_q(\beta)}\right\}\\
&\ge \psi_m\left(\frac{\int_s^1 s^{\beta-1}(1-qt)_q^{(\beta-1)}h(t)d_qt}{\Gamma_q(\beta)}\right) \ge 0,
\end{aligned} \tag{3.12}$$

it follows from (3.12) that

$$\begin{aligned}
\hbar(x) &= \frac{\int_0^1(1-qs)_q^{(\alpha-1)}k(s)d_qs - c\int_0^\lambda(\lambda-qs)_q^{(\alpha-1)}k(s)d_qs}{(1-c)\Gamma_q(\alpha)} - \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qs)_q^{(\alpha-1)}k(s)d_qs.\\
&= \frac{\int_0^1(1-qs)_q^{(\alpha-1)}k(s)d_qs - \int_0^x(x-qs)_q^{(\alpha-1)}k(s)d_qs}{\Gamma_q(\alpha)}\\
&\quad + \frac{c\left(\int_0^1(1-qs)_q^{(\alpha-1)}k(s)d_qs - \int_0^\lambda(\lambda-qs)_q^{(\alpha-1)}k(s)d_qs\right)}{(1-c)\Gamma_q(\alpha)}\\
&\ge 0.
\end{aligned}$$

Therefore, $\hbar(x)$ is nonnegative. The proof is completed.

Lemma 3.3 *Suppose that* $h \in C([0,1],[0,+\infty))$ *and* $\hbar(x)$ *is defined by* (3.11). *Then there exists a constant* ρ *such that*

$$\max_{x\in[0,1]}|\hbar(x)| \le \rho \max_{x\in[0,1]}|{}^cD_q^\gamma\hbar(x)|.$$

Proof Case 1: $\gamma < \alpha$. From (3.11), Definition 2.3, Property 2.6 and Lemma 2.8, one has

$${}^cD_q^\gamma\hbar(x) = -{}^cD_q^\gamma I_q^\alpha k(x) = -I_q^{\alpha-\gamma}k(x) = -\frac{1}{\Gamma_q(\alpha-\gamma)}\int_0^x (x-qs)_q^{(\alpha-\gamma-1)}k(s)d_qs.$$

Hence

$$\max_{x\in[0,1]} |{}^cD_q^\gamma \hbar(x)| \geq |{}^cD_q^\gamma \hbar(1)| = \frac{1}{\Gamma_q(\alpha-\gamma)} \int_0^1 (1-qs)_q^{(\alpha-\gamma-1)} k(s) d_q s. \tag{3.13}$$

By the definition of $\hbar$ and (3.13), one has

$$\begin{aligned}\max_{x\in[0,1]} |\hbar(x)| &\leq \frac{1}{(1-c)\Gamma_q(\alpha)} \int_0^1 (1-qs)_q^{(\alpha-1)} k(s) d_q s \\ &\leq \frac{\Gamma_q(\alpha-\gamma)}{(1-c)\Gamma_q(\alpha)} \frac{1}{\Gamma_q(\alpha-\gamma)} \int_0^1 (1-qs)_q^{(\alpha-\gamma-1)} k(s) d_q s \\ &\leq \rho \max_{x\in[0,1]} |{}^cD_q^\gamma \hbar(x)|,\end{aligned}$$

where $\rho = \frac{\Gamma_q(\alpha-\gamma)}{(1-c)\Gamma_q(\alpha)}$.

Case 2: $\gamma = \alpha$. By the definition of $\hbar$, Lemmas 2.8 and 3.2, we have

$$\left|{}^cD_q^\gamma \hbar(x)\right| = \left|-{}^cD_q^\gamma I_q^\alpha k(x)\right| = k(x).$$

Therefore,

$$\begin{aligned}\max_{x\in[0,1]} |\hbar(x)| &\leq \frac{1}{(1-c)\Gamma_q(\alpha)} \int_0^1 (1-qs)_q^{(\alpha-1)} k(s) d_q s \\ &\leq \frac{\max_{x\in[0,1]} |{}^cD_q^\gamma \hbar(x)|}{(1-c)\Gamma_q(\alpha)} \int_0^1 (1-qs)_q^{(\alpha-1)} d_q s \\ &= \frac{1}{[\alpha]_q(1-c)\Gamma_q(\alpha)} \max_{x\in[0,1]} |{}^cD_q^\gamma \hbar(x)| \\ &\leq \rho \max_{x\in[0,1]} |{}^cD_q^\gamma \hbar(x)|.\end{aligned}$$

The proof is completed.

Lemma 3.4 *[5](Avery–Peterson theorem) Let P be a cone of a real Banach space B, μ, ν be nonnegative continuous convex functionals on P, ω be a nonnegative continuous concave functional on P, and ϖ be a nonnegative continuous functional on P. For $l, n_1, n_2, r > 0$, define the following sets:*

$$\begin{aligned}&P(\mu, r) = \{x \in P | \mu(x) < r\}, \\ &P(\mu, \omega, n_1, r) = \{x \in P | \omega(x) \geq n_1, \mu(x) < r\}, \\ &P(\mu, \nu, \omega, n_1, n_2, r) = \{x \in P | \omega(x) \geq n_1, \nu(x) \leq n_2, \mu(x) < r\},\end{aligned}$$

and

$$Q(\mu, \varpi, l, r) = \{x \in P | \varpi(x) \geq l, \mu(x) < r\}.$$

Suppose that the functionals μ, ν, ω, ϖ satisfy $\varpi(\varepsilon x) \leq \varepsilon\varpi(x)$, $0 \leq \varepsilon \leq 1$, such that for some R, $r > 0$,

$$\omega(x) \leq \varpi(x), \ ||x|| \leq R\mu(x),$$

for all $x \in \overline{P(\mu,r)}$. *Assume also that* $T:\overline{P(\mu,r)} \to \overline{P(\mu,r)}$ *is completely continuous and there exist* $l,\ n_1,\ n_2 > 0$ *with* $l < m$ *such that*
(S_1*)* $\{x \in P(\mu,\nu,\omega,n_1,n_2,r)|\omega(x) > n_1\} \neq \emptyset$ *and* $\omega(Tx) > n_1$ *for* $x \in P(\mu,\nu,\omega,n_1,n_2,r)$*;*
(S_2*)* $\omega(Tx) > n_1$ *for* $x \in P(\mu,\omega,n_1,r)$ *and* $\nu(Tx) > n_2$*;*
(S_3*)* $0 \notin Q(\mu,\varpi,l,r)$ *and* $\varpi(Tx) < l$ *for* $x \in Q(\mu,\varpi,l,r)$ *with* $\varpi(x) = l$*.*
Then T *has at least three fixed points* $x_1,\ x_2,\ x_3 \in \overline{P(\mu,r)}$ *such that*

$$\mu(x_i) \leq r,\ \ i = 1,2,3;$$

$$m < w(x_1);$$

$$l < \varpi(x_2),\ \ \omega(x_2) < n_1;$$

and

$$\varpi(x_3) < l.$$

Next, we shall consider the existence of multiple positive solutions for the problem (1.1)–(1.2). For convenience, some denotations and hypotheses are presented as follows:

$$M = \left[\frac{(1-c)_q^{(\beta)} - (1-\lambda)_q^{(\beta)}}{\Gamma_q(\beta+1)}\right]^{m-1};$$

$$N_1 = \Gamma_q(\alpha-\gamma+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1};$$

$$N_2 = \frac{[\beta m+2]_q\Gamma_q(\alpha)}{c^{\beta m+2}(1-\lambda)(1-c)_q^{(\alpha-1)}M};$$

$$N_3 = (1-c)\Gamma_q(\alpha+1)\left[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)\right]^{m-1},$$

and

$$(C_1)\ g(x,y,z) \leq (rN_1)^{p-1},\ \ (x,y,z) \in [0,1]\times[0,\rho r]\times[-r,r];$$

$$(C_2)\ g(x,y,z) > (n_1N_2)^{p-1},\ \ (x,y,z) \in [0,1]\times[n_1,n_2]\times[-r,r];$$

$$(C_3)\ g(x,y,z) < (lN_3)^{p-1},\ \ (x,y,z) \in [0,1]\times[0,l]\times[-r,r].$$

Let the Banach space $B = \left\{u|u \in C[0,1],^c D_q^\gamma u(x) \in C[0,1]\right\}$ with the norm

$$||u|| = \max\left\{\max_{x\in[0,1]}|u(x)|,\ \max_{x\in[0,1]}|^cD_q^\gamma u(x)|\right\},$$

and define the cone P by

$$P = \left\{u \in B|u(x) \geq 0,\ \max_{x\in[0,1]}|u(x)| \leq \rho\max_{x\in[0,1]}|^cD_q^\gamma u(x)|,\ x \in [0,1]\right\},$$

where $\rho = \frac{\Gamma_q(\alpha-\gamma)}{(1-c)\Gamma_q(\alpha)}$.

Theorem 3.5 *Let* $g \in C([0,1]\times[0,+\infty)\times\mathbb{R},[0,+\infty))$ *and the operator* $T: P \to B$ *be defined as*

$$Tu(x) = a_1 - \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qs)_q^{(\alpha-1)}k(s)d_qs,$$

where

$$a_1 = \frac{\int_0^1(1-qs)_q^{(\alpha-1)}k(s)ds - c\int_0^\lambda(\lambda-qs)_q^{(\alpha-1)}k(s)d_qs}{(1-c)\Gamma_q(\alpha)},$$

$$k(s) = \psi_m\left[-\left(\frac{1}{\Gamma_q(\beta)}\int_0^s (s-qt)_q^{(\beta-1)}g(t,u(t),{}^cD_q^\gamma u(t))d_qt + b_1 s^{\beta-1}\right)\right],$$

and

$$b_1 = \frac{d^{p-1}\int_0^\zeta(\zeta-qt)_q^{(\beta-1)}g(t,u(t),{}^cD_q^\gamma u(t))d_qt - \int_0^1(1-qt)_q^{(\beta-1)}g(t,u(t),{}^cD_q^\gamma u(t))d_qt}{(1-d^{p-1}\zeta^{\beta-1})\Gamma_q(\beta)},$$

where $d^{p-1}\zeta^{\beta-1} < 1$. *Then* $T: P \to P$ *and is completely continuous.*

Proof Obviously, in view of Lemmas 3.2 and 3.3, we obtain $Tu \geq 0$ and $\max\limits_{x\in[0,1]}|Tu(x)| \leq \rho \max\limits_{x\in[0,1]}|{}^cD_q^\gamma Tu(x)|$ for all $u \in P$. Hence $T(P) \subset P$.

Now assume Ω is a bounded subset in P, which is to say that there exists a positive constant η such that $||u|| \leq \eta$ for all $u \in \Omega$. Let

$$L = \sup_{t\in[0,1],u\in\Omega}|g(t,u(t),{}^cD_q^\gamma u(t))|.$$

Then for all $u \in \Omega$, by the definition of $k(s)$, we have

$$\begin{aligned}
k(s) &\leq \psi_m\left(\frac{\int_0^1 s^{\beta-1}(1-qt)_q^{(\beta-1)}g(t,u(t),{}^cD_q^\gamma u(t))d_qt}{(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta)}\right)\\
&\leq \psi_m\left(\frac{Ls^{\beta-1}\int_0^1(1-qt)_q^{(\beta-1)}d_qt}{(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta)}\right)\\
&\leq \psi_m\left(\frac{L}{(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)}\right)\\
&= \frac{L^{m-1}}{[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}}.
\end{aligned}$$

Hence

$$\begin{aligned}
|Tu(x)| &= a_1 - \frac{1}{\Gamma_q(\alpha)}\int_0^x(x-qs)_q^{(\alpha-1)}k(s)d_qs\\
&\leq \frac{\int_0^1(1-qs)_q^{(\alpha-1)}k(s)d_qs}{(1-c)\Gamma_q(\alpha)}\\
&\leq \frac{L^{m-1}\int_0^1(1-qs)_q^{(\alpha-1)}d_qs}{(1-c)\Gamma_q(\alpha)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}}\\
&\leq \frac{L^{m-1}}{(1-c)\Gamma_q(\alpha+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}}
\end{aligned}$$

and

$$|{}^cD_q^\gamma Tu(x)| = \frac{1}{\Gamma_q(\alpha-\gamma)}\int_0^x (x-qs)_q^{(\alpha-\gamma-1)}k(s)d_qs$$
$$\leq \frac{L^{m-1}\int_0^1(1-qs)_q^{(\alpha-\gamma-1)}d_qs}{\Gamma_q(\alpha-\gamma)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}}$$
$$\leq \frac{L^{m-1}}{\Gamma_q(\alpha-\gamma+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}}.$$

Hence, $T(\Omega)$ is uniformly bounded.

On the other hand, taking any $x_1, x_2 \in [0,1]$ with $x_1 < x_2$, for all $u \in \Omega$, we have

$$|Tu(x_1)-Tu(x_2)| = \frac{1}{\Gamma_q(\alpha)}\left|\int_0^{x_2}(x_2-qs)_q^{(\alpha-1)}k(s)d_qs - \int_0^{x_1}(x_1-qs)_q^{(\alpha-1)}k(s)d_qs\right|$$
$$\leq \frac{1}{\Gamma_q(\alpha)}\left|\int_0^{x_1}\left[(x_2-qs)_q^{(\alpha-1)}-(x_1-qs)_q^{(\alpha-1)}\right]k(s)d_qs + \int_{x_1}^{x_2}(x_2-qs)_q^{(\alpha-1)}k(s)d_qs\right|$$
$$\leq \frac{1}{\Gamma_q(\alpha)}\left|\int_0^{1}\left[(x_2-qs)_q^{(\alpha-1)}-(x_1-qs)_q^{(\alpha-1)}\right]k(s)d_qs + \int_{x_1}^{x_2}k(s)d_qs\right|$$
$$\leq \frac{L^{m-1}\left|\int_0^1\left[(x_2-qs)_q^{(\alpha-1)}-(x_1-qs)_q^{(\alpha-1)}\right]d_qs+(x_2-x_1)\right|}{\Gamma_q(\alpha)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}}, \tag{3.14}$$

Since the function $(x-qs)_q^{(\alpha-1)}$ is continuous with respect to x and s on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. Hence for any $s \in [0,1]$, as $x_1 \to x_2$, we can get

$$(x_2-qs)_q^{(\alpha-1)}-(x_2-qs)_q^{(\alpha-1)} \rightrightarrows 0.$$

It follows that as $x_1 \to x_2$, the right-hand side of the above inequality (3.14) tends to zero. And

$$|{}^cD_q^\gamma Tu(x_1)-{}^cD_q^\gamma Tu(x_2)| = \frac{1}{\Gamma_q(\alpha-\gamma)}\left|\int_0^{x_2}(x_2-qs)_q^{(\alpha-\gamma-1)}k(s)d_qs - \int_0^{x_1}(x_1-qs)_q^{(\alpha-\gamma-1)}k(s)d_qs\right|$$
$$\leq \frac{1}{\Gamma_q(\alpha-\gamma)}\left|\int_0^{x_1}\left[(x_2-qs)_q^{(\alpha-\gamma-1)}-(x_1-qs)_q^{(\alpha-\gamma-1)}\right]k(s)d_qs\right.$$
$$\left.+\int_{x_1}^{x_2}(x_2-qs)_q^{(\alpha-\gamma-1)}k(s)d_qs\right| \tag{3.15}$$
$$\leq \frac{L^{m-1}[x_2^{\alpha-\gamma}-x_1^{\alpha-\gamma}]}{\Gamma_q(\alpha-\gamma)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}}.$$

Similarly, we can get that the right-hand side of the above inequality (3.15) tends to zero with $x_1 \to x_2$.

Therefore, $T(\Omega)$ is equicontinuous on $[0,1]$. We conclude that $T: P \to P$ is relatively compact on basis of Arzela–Ascoli theorem, which completes the proof.

Theorem 3.6 *Assume $g \in C([0,1]\times[0,+\infty)\times\mathbb{R},[0,+\infty))$, $d^{p-1}\zeta^{\beta-1} < 1$, $0 < c < \lambda < 1$ and there exist constants $0 < l < n_1 < n_2 < r$ such that $\frac{n_1}{1-\lambda} < n_2$. Under the assumptions of $(C_1) \sim (C_3)$, the problem (1.1)–(1.2) has at least three positive solutions.*

Proof By Theorem 3.5 and Lemma 3.1, we know that $T : P \to P$ is completely continuous and problem (1.1)–(1.2) has a solution $u = u(x)$ if and only if u satisfies the operator equation $u = Tu$.

Now Let

$$\omega(u) = \min_{x\in[0,\lambda]} |u(x)|,\ \mu(u) = \max_{x\in[0,1]} |{}^cD_q^\gamma u(x)|,\ \nu(u) = \varpi(u) = \max_{x\in[0,1]} |u(x)|.$$

Evidently, $\omega(u) \le \varpi(u)$. By Lemma 3.3, we have $||u|| \le R\mu(u)$, $R = \max\{\rho, 1\}$.

For $u \in \overline{P(\mu, r)}$, by (C_1), one has

$$\begin{aligned}
k(s) &\le \psi_m\left(\frac{\int_0^1 s^{\beta-1}(1-qt)_q^{(\beta-1)} g(t, u(t), {}^c D_q^\gamma u(t)) d_q t}{(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta)}\right)\\
&\le \psi_m\left(\frac{(rN_1)^{p-1}\int_0^1 (1-qt)_q^{(\beta-1)} d_q t}{(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta)}\right)\\
&\le \frac{rN_1}{[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}},
\end{aligned}$$

then

$$\begin{aligned}
\mu(Tu) &= \max_{x\in[0,1]} |{}^cD_q^\gamma Tu(x)|\\
&= \max_{x\in[0,1]} \left| -\frac{1}{\Gamma_q(\alpha-\gamma)} \int_0^x (x-qs)_q^{(\alpha-\gamma-1)} k(s) d_q s \right|\\
&\le \frac{\int_0^1 (1-qs)_q^{(\alpha-\gamma-1)} k(s) d_q s}{\Gamma_q(\alpha-\gamma)}\\
&\le \frac{rN_1 \int_0^1 (1-qs)_q^{(\alpha-\gamma-1)} d_q s}{\Gamma_q(\alpha-\gamma)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}}\\
&= \frac{rN_1}{\Gamma_q(\alpha-\gamma+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1}} = r.
\end{aligned}$$

Hence $T : \overline{P(\mu, r)} \to \overline{P(\mu, r)}$.

Next, we prove that condition $(S_1) \sim (S_3)$ in Lemma 3.4 are true for operator T. Firstly, for constants function $u(x) = \frac{n_1}{1-\lambda} \in P(\mu, \nu, \omega, n_1, n_2, r)$. Since it is easy to see $\mu(u) = 0 \le r$, $\nu(u) = \frac{n_1}{1-\lambda} < n_2$, and $\omega(u) = \frac{n_1}{1-\lambda} > n_1$, $\{u \in P(\mu, \nu, \omega, n_1, n_2, r) : \omega(u) > n_1\} \ne \emptyset$.

If $u \in P(\mu, \nu, \omega, n_1, n_2, r)$, with the help of (3.12) and (C_2), for any $s \in [0, c]$, we have

$$\begin{aligned}
k(s) &\geq \psi_m \left(\frac{\int_s^1 s^{\beta-1}(1-qt)_q^{(\beta-1)} g(t, u(t), {}^cD_q^\gamma u(t)) d_q t}{\Gamma_q(\beta)} \right) \\
&\geq \psi_m \left(\frac{\int_s^\lambda s^{\beta-1}(1-qt)_q^{(\beta-1)} g(t, u(t), {}^cD_q^\gamma u(t)) d_q t}{\Gamma_q(\beta)} \right) \\
&> \psi_m \left(\frac{(n_1 N_2)^{p-1} s^{\beta-1} \int_s^\lambda (1-qt)_q^{(\beta-1)} d_q t}{\Gamma_q(\beta)} \right) \\
&= \frac{s^{\beta m-\beta-m+1}[(1-s)_q^{(\beta)} - (1-\lambda)_q^{(\beta)}]^{m-1}}{[\Gamma_q(\beta+1)]^{m-1}} n_1 N_2 \\
&\geq \frac{s^{\beta m-\beta-m+1}[(1-c)_q^{(\beta)} - (1-\lambda)_q^{(\beta)}]^{m-1}}{[\Gamma_q(\beta+1)]^{m-1}} n_1 N_2 \\
&\geq M n_1 N_2 s^{\beta m+1},
\end{aligned}$$

then

$$\begin{aligned}
\omega(Tu) &= \min_{x \in [0,\lambda]} |Tu(x)| = |Tu(\lambda)| \\
&= \frac{\int_0^1 (1-qs)_q^{(\alpha-1)} k(s) d_q s - c\int_0^\lambda (\lambda-qs)_q^{(\alpha-1)} k(s) d_q s}{(1-c)\Gamma_q(\alpha)} - \frac{1}{\Gamma_q(\alpha)} \int_0^\lambda (\lambda-qs)_q^{(\alpha-1)} k(s) d_q s. \\
&= \frac{\int_0^1 (1-qs)_q^{(\alpha-1)} k(s) d_q s - \int_0^\lambda (\lambda-qs)_q^{(\alpha-1)} k(s) d_q s}{(1-c)\Gamma_q(\alpha)} \\
&= \frac{\int_0^\lambda \left[(1-qs)_q^{(\alpha-1)} - (\lambda-qs)_q^{(\alpha-1)} \right] k(s) d_q s + \int_\lambda^1 (1-qs)_q^{(\alpha-1)} k(s) d_q s}{(1-c)\Gamma_q(\alpha)} \\
&\geq \frac{\int_0^\lambda (1-qs)_q^{(\alpha-1)} (1-\lambda) k(s) d_q s + \int_\lambda^1 (1-qs)_q^{(\alpha-1)} k(s) d_q s}{(1-c)\Gamma_q(\alpha)} \qquad (3.16) \\
&> \frac{\int_0^\lambda (1-qs)_q^{(\alpha-1)} (1-\lambda) k(s) d_q s}{(1-c)\Gamma_q(\alpha)} \\
&> \frac{\int_0^c (1-qs)_q^{(\alpha-1)} (1-\lambda) k(s) d_q s}{(1-c)\Gamma_q(\alpha)} \\
&\geq \frac{(1-\lambda) M n_1 N_2}{(1-c)\Gamma_q(\alpha)} \int_0^c s^{\beta m+1} (1-c)_q^{(\alpha-1)} d_q s \\
&\geq \frac{(1-\lambda)(1-c)_q^{(\alpha-1)} M n_1 N_2 c^{\beta m+2}}{[\beta m+2]_q \Gamma_q(\alpha)} = n_1.
\end{aligned}$$

Hence the condition (S_1) is satisfied.

Secondly, if $u \in P(\mu, \omega, n_1, r)$ and $\nu(Tu) > n_2$, since

$$\nu(Tx) = \max_{x \in [0,1]} |Tu(x)| \leq \frac{\int_0^1 (1-qs)_q^{(\alpha-1)} k(s) d_q s}{(1-c)\Gamma_q(\alpha)},$$

from (3.16), one has

$$\begin{aligned}
\omega(Tu) &= \min_{x\in[0,\lambda]} |Tu(x)| = |Tu(\lambda)| \\
&\geq \frac{\int_0^\lambda (1-qs)_q^{(\alpha-1)}(1-\lambda)k(s)d_qs + \int_\lambda^1 (1-qs)_q^{(\alpha-1)}k(s)d_qs}{(1-c)\Gamma_q(\alpha)} \\
&\geq \frac{(1-\lambda)\int_0^1 (1-qs)_q^{(\alpha-1)}k(s)d_qs}{(1-c)\Gamma_q(\alpha)} \\
&\geq (1-\lambda)\nu(Tu) \\
&> (1-\lambda)n_2 > n_1.
\end{aligned}$$

Hence the condition (S_2) is satisfied.

Finally, if $u \in Q(\mu, \varpi, l, r)$ and $\varpi(u) = l$, since for all $s \in [0,1]$, by the definition of $k(s)$ and assumption (C_3), we have

$$\begin{aligned}
k(s) &\leq \psi_m\left(\frac{\int_0^1 s^{\beta-1}(1-qt)_q^{(\beta-1)} g(t, u(t), {}^cD_q^\gamma u(t))d_qt}{(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta)}\right) \\
&< \psi_m\left[\frac{s^{\beta-1}(lN_3)^{p-1}\int_0^1 (1-qs)_q^{(\beta-1)}d_qs}{(1-\zeta^{p-1}d^{\beta-1})\Gamma_q(\beta)}\right] \\
&\leq \frac{lN_3}{[(1-\zeta^{p-1}d^{\beta-1})\Gamma_q(\beta+1)]^{m-1}}.
\end{aligned}$$

Then by the definition of operator T,

$$\begin{aligned}
\varpi(Tu) = \max_{x\in[0,1]} |Tu(x)| &\leq \frac{\int_0^1 (1-qs)_q^{(\alpha-1)}k(s)d_qs}{(1-c)\Gamma_q(\alpha)} \\
&< \frac{lN_3}{(1-c)\Gamma_q(\alpha+1)[(1-\zeta^{p-1}d^{\beta-1})\Gamma_q(\beta+1)]^{m-1}} = l.
\end{aligned}$$

Furthermore, $0 \notin Q(\mu, \varpi, l, r)$ obviously. So the condition (S_3) also holds. According to the Avery–Peterson theorem, the problem (1.1)–(1.2) has at least three positive solution. The proof is completed.

4. The method of lower and upper solutions

In this section, we shall give a new existence result of multiple positive solutions for (1.1)–(1.2), applying the method of lower and upper solutions based on the monotone iterative technique. In order to prove our main results, we need the following vital lemmas and definition.

Lemma 4.1 *For any given function $h \in C[0,1]$ and $a,\ b \in \mathbb{R}$, $u(x) \in C_q^2[0,1]$ is a solution of the boundary value problem*

$$\begin{cases}
D_q^\beta(\psi_p({}^cD_q^\alpha u(x))) = h(x),\ 0 < x < 1, \\
{}^cD_q^\alpha u(0) = D_q u(0) = 0, \\
u(1) = a,\ {}^cD_q^\alpha u(1) = b,
\end{cases} \tag{4.1}$$

if and only if $u(x)$ satisfies the following integral equation

$$u(x) = a - \int_0^1 G(x,qt)\psi_m\left(\psi_p(b)t^{\beta-1} - \int_0^1 H(t,qs)h(s)d_qs\right)d_qt,$$

where

$$G(x,qt) = \frac{1}{\Gamma_q(\alpha)}\begin{cases}(1-qt)_q^{(\alpha-1)} - (x-qt)_q^{(\alpha-1)}, & 0 \le qt \le x \le 1,\\ (1-qt)_q^{(\alpha-1)}, & 0 \le x \le qt \le 1,\end{cases} \tag{4.2}$$

and

$$H(x,qt) = \frac{1}{\Gamma_q(\beta)}\begin{cases}x^{\beta-1}(1-qt)_q^{(\beta-1)} - (x-qt)_q^{(\beta-1)}, & 0 \le qt \le x \le 1,\\ x^{\beta-1}(1-qt)_q^{(\beta-1)}, & 0 \le x \le qt \le 1.\end{cases} \tag{4.3}$$

Proof Let $v(x) := \varphi_p({}^cD_q^\alpha u(x))$. We can decompose (4.1) into the following coupled boundary value problem

$$\begin{cases}D_q^\beta v(x) = h(x),\ x \in (0,1),\\ v(0) = 0,\ v(1) = \psi_p(b),\end{cases} \tag{4.4}$$

and

$$\begin{cases}{}^cD_q^\alpha u(x) = \psi_m(v(x)),\ x \in (0,1),\\ D_qu(0) = 0,\ u(1) = a.\end{cases} \tag{4.5}$$

Taking operator I_q^β on both sides of (4.4), by Lemma 2.7, we have

$$v(x) = C_1x^{\beta-1} + C_0x^{\beta-2} + I_q^\beta h(x),$$

where $C_i \in \mathbb{R},\ i = 1,2$. By boundary value condition $v(0) = 0$, one has $C_0 = 0$. It follows from $v(1) = \psi_p(b)$ that $C_1 = \psi_p(b) - I_q^\beta h(1)$. Then (4.4) has a unique solution

$$\begin{aligned}v(x) &= \frac{1}{\Gamma_q\beta)}\left(\int_0^x (x-qt)_q^{(\beta-1)}h(t)d_qt - \int_0^1 x^{\beta-1}(1-qt)_q^{(\beta-1)}h(t)d_qt\right) + \psi_p(b)x^{\beta-1}\\ &= \varphi_p(b)x^{\beta-1} - \int_0^1 H(x,qt)h(t)d_qt.\end{aligned} \tag{4.6}$$

Similar to (4.6), the boundary value problem (4.5) has a unique solution, which is given by

$$\begin{aligned}u(x) &= a + \frac{1}{\Gamma_q\alpha)}\left(\int_0^x (x-qt)_q^{(\alpha-1)}\psi_m(v(t))d_qt - \int_0^1 (1-qt)_q^{(\alpha-1)}\psi_m(v(t))d_qt\right)\\ &= a - \int_0^1 G(x,qt)\psi_m(v(t))d_qt\\ &= a - \int_0^1 G(x,qt)\psi_m\left(\varphi_p(b)t^{\beta-1} - \int_0^1 H(t,qs)h(s)d_qs\right)d_qt.\end{aligned} \tag{4.7}$$

On the other hand, assume $u(x)$ satisfies (4.1) (i.e.(4.7)). From (4.7),

$$u(x) = a + I_q^\alpha \psi_m(v(x)) - \int_0^1 (1-qt)_q^{(\alpha-1)} \psi_m(v(t)) d_q t. \tag{4.8}$$

Obviously, $v(t)$ is continuous on $[0,1]$ by the continuity of functions $h(t)$ and $H(t,qs)$. Hence by Lemma 2.12, we get $I_q^\alpha \psi_m(v(x)) \in C^2[0,1]$, i.e. $u(x) \in C^2[0,1]$. Applying the operator ${}^cD_q^\alpha$ on both sides of (4.8), by Lemma 2.8 we can derive

$${}^cD_q^\alpha u(x) = {}^cD_q^\alpha I_q^\alpha \psi_m(v(x)) = \psi_m\left(\psi_p(b)x^{\beta-1} - \int_0^1 H(x,qt)h(t)d_qt\right).$$

i.e.

$$\begin{aligned}\psi_p\left({}^cD_q^\alpha u(x)\right) &= \psi_p(b)x^{\beta-1} - \int_0^1 H(x,qt)h(t)d_qt \\ &= \left[\psi_p(b) - \int_0^1 (1-qs)_q^{(\beta-1)} h(s)d_qs\right] x^{\beta-1} + I_q^\beta h(x).\end{aligned} \tag{4.9}$$

Taking operator D_q^β on both sides of (4.9), by Lemmas 2.7 and 2.9, one has

$$D_q^\beta\left(\psi_p\left({}^cD_q^\alpha u(x)\right)\right) = D_q^\beta I_q^\beta h(x) = h(x).$$

In addition, it is easy to prove that $u(x)$ satisfies the boundary value conditions in (4.1) which completes the proof.

Remark 4.2 *Let $g \in C([0,1]\times\mathbb{R}\times\mathbb{R},[0,+\infty))$. From Lemma 4.1, we conclude that if $u(x) \in C^2[0,1]$ is a solution of (1.1)–(1.2) if and only if $u(x)$ satisfies the following integral equation*

$$u(x) = cu(\lambda) - \int_0^1 G(x,qt)\psi_m\left(\psi_p(d^cD_q^\alpha u(\zeta)t^{\beta-1} - \int_0^1 H(t,qs)g(s,u(s),{}^cD_q^\gamma u(s))d_qs\right)d_qt. \tag{4.10}$$

Lemma 4.3 *[4] Assume the functions $G(x,qt)$ and $H(x,qt)$ are defined by (4.2) and (4.3), respectively. Then $G(x,qt)$ and $H(x,qt)$ satisfy the following conditions:*
(1) $G(x,qt)$ and $H(x,qt)$ are continuous;
(2) $G(x,qt) \geq 0$ and $H(x,qt) \geq 0$ for all $0 \leq x,t \leq 1$.

Definition 4.4 *We say that a function $u(x)$ is a solution of (1.1)–(1.2) if and only if $u(x) \in C[0,1]$ satisfies the problem (1.1)–(1.2) almost everywhere on $[0,1]$.*

Definition 4.5 *Assume $u(x) \in AC^2[0,1]$. We say that $u(x)$ is a lower solution of (1.1)–(1.2), if $u(x)$ satisfies the following inequality*

$$\begin{cases} D_q^\beta(\psi_p({}^cD_q^\alpha u(x))) \leq g(x,u(x),{}^cD_q^\gamma u(x)), \ a.e.\ 0<x<1, \\ {}^cD_q^\alpha u(0) = D_qu(0) = 0, \\ u(1) \leq cu(\lambda), \ {}^cD_q^\alpha u(1) \geq d^cD_q^\alpha u(\zeta). \end{cases} \tag{4.11}$$

Assume $u(x) \in AC^2[0,1]$. *We say that* $u(x)$ *is an upper solution of* (1.1)–(1.2)*, if* $u(x)$ *satisfies the following inequality*

$$\begin{cases} D_q^\beta(\psi_p(^cD_q^\alpha u(x))) \geq g(x,u(x),{}^cD_q^\gamma u(x)), \ a.e.\ 0<x<1, \\ {}^cD_q^\alpha u(0) = D_q u(0) = 0, \\ u(1) \geq cu(\lambda), \ {}^cD_q^\alpha u(1) \leq d^cD_q^\alpha u(\zeta). \end{cases} \tag{4.12}$$

Define $X = \{u : u \in C[0,1], {}^cD_q^\alpha u(x) \in C[0,1], D_q u(0) = 0\}$, with the norm $||u|| = \max\limits_{x\in[0,1]} |u(x)| + \max\limits_{x\in[0,1]} |{}^cD_q^\alpha u(x)|$. Then $(X, ||\cdot||)$ is a Banach space. Define a normal cone P by

$$P = \{u : u \in X, u(x) \geq 0, {}^cD_q^\gamma u(x) \leq 0, x \in [0,1]\}.$$

We define $u \preceq v$ if and only if $v - u \in P$, for $u, v \in X$.

For our purpose, let us present the following assumption:
(H) $g \in C([0,1]\times[0,+\infty)\times(-\infty,0],[0,+\infty))$, and $g(x,y_1,z_1) \leq g(x,y_2,z_2)$, for $0 \leq y_1 < y_2$, $z_2 < z_1 \leq 0$, for any $x \in [0,1]$.

Theorem 4.6 *Suppose the assumption* (H) *holds and* (1.1)–(1.2) *has a nonnegative lower solution* $u_0 \in P$ *and a nonnegative upper solution* $v_0 \in P$ *such that* $u_0 \preceq v_0$. *Then* (1.1)–(1.2) *has the maximal lower solution* u^* *and the minimal upper solution* v^* *on* $[u_0, v_0] \subset P$, *both* u^* *and* v^* *are positive solutions of* (1.1)–(1.2). *Furthermore*

$$0 \leq u_0(x) \leq u^*(x) \leq v^*(x) \leq v_0(x).$$

Proof In the following, our proof process will be divided into three steps.

Step1. We will obtain the lower solution sequence $\{u_k\}$ and the upper solution sequence $\{v_k\}$.

For $u_0 \in P$, we consider the following boundary value problem

$$\begin{cases} D_q^\beta(\psi_p(^cD_q^\alpha u_1(x))) = g(x,u_0(x),{}^cD_q^\gamma u_0(x)), \ 0<x<1, \\ {}^cD_q^\alpha u_1(0) = D_q u_1(0) = 0, \\ u_1(1) = cu_0(\lambda), \ {}^cD_q^\alpha u_1(1) = d^cD_q^\alpha u_0(\zeta). \end{cases} \tag{4.13}$$

By Lemma 4.1, (4.13) has an unique solution $u_1(x)$. Since u_0 is a lower solution of (4.1), we have

$$\begin{cases} D_q^\beta(\psi_p(^cD_q^\alpha u_0(x))) \leq g(x,u_0(x),{}^cD_q^\gamma u_0(x)), \ 0<x<1, \\ {}^cD_q^\alpha u_0(0) = D_q u_0(0) = 0, \\ u_0(1) \leq cu(\lambda), \ {}^cD_q^\alpha u_0(1) \geq d^cD_q^\alpha u_0(\zeta). \end{cases} \tag{4.14}$$

It follows from (4.13) minus (4.14) that

$$\begin{cases} D_q^\beta(\psi_p(^cD_q^\alpha u_1(x)) - \psi_p(^cD_q^\alpha u_0(x))) \geq 0, \ 0<x<1, \\ {}^cD_q^\alpha u_1(0) - {}^cD_q^\alpha u_0(0) = D_q u_1(0) - D_q u_0(0) = 0, \\ u_1(1) - u_0(1) \geq 0, \ {}^cD_q^\alpha u_1(1) - {}^cD_q^\alpha u_0(1) \leq 0. \end{cases} \tag{4.15}$$

Now, let $\omega(x) := \psi_p({}^cD_q^\alpha u_1(x)) - \psi_p({}^cD_q^\alpha u_0(x))$. It is clear that $\omega(0) = 0$ by the boundary value condition ${}^cD_q^\alpha u_0(0) - {}^cD_q^\alpha u_0(0) = 0$. And from ${}^cD_q^\alpha u_1(1) - {}^cD_q^\alpha u_0(1) \leq 0$, we can get $\omega(1) \leq 0$.

Next, let $\varpi(x) := D_q^\beta \left(\psi_p({}^cD_q^\alpha u_1(x)\right) - \psi_p({}^cD_q^\alpha u_0(x)))$ and $\psi_p(b) = \omega(1)$. Then we can obtain the following boundary value problem

$$\begin{cases} D_q^\beta \omega(x) = \varpi(x) \geq 0, \ 0 < x < 1, \\ \omega(0) = 0, \omega(1) = \psi_p(b) \leq 0. \end{cases}$$

By (4.4) and Lemma 4.3, one has

$$\omega(x) = \psi_p({}^cD_q^\alpha u_1(x)) - \psi_p({}^cD_q^\alpha u_0(x)) = \psi_p(b) - \int_0^1 H(x, qt)h(t)d_qt \leq 0.$$

Hence by the monotonicity of p-Laplacian operator φ_p, we have

$$ {}^cD_q^\alpha(u_1(x) - u_0(x)) = {}^cD_q^\alpha u_0(x) - {}^cD_q^\alpha u_1(x) \leq 0, \ x \in [0, 1]. \tag{4.16}$$

Let $\delta(x) := {}^cD_q^\alpha(u_1(x) - u_0(x))$. Then we obtain the following boundary value problem

$$\begin{cases} {}^cD_q^\alpha(u_1(x) - u_0(x)) = \delta(x) \leq 0, \ 0 < x < 1, \\ D_qu_1(0) - D_qu_0(0) = 0, \ u_1(1) - u_0(1) := a \geq 0. \end{cases}$$

Similar to (4.5) and Lemma 4.3, we can get

$$u_1(x) - u_0(x) = a - \int_0^1 G(x, qt)\delta(t)d_qt \geq 0.$$

Besides, from (4.16), we obtain

$${}^cD_q^\gamma(u_1(x) - u_0(x)) = I_q^{\alpha-\gamma} {}^cD_q^\alpha(u_1(x) - u_0(x)) \leq 0, \ x \in [0, 1].$$

To sum up, we can get that $u_0 \preceq u_1$.

From (4.13) and (H), we have

$$\begin{cases} D_q^\beta(\psi_p({}^cD_q^\alpha u_1(x))) = g(x, u_0(x), {}^cD_q^\gamma u_0(x)) \leq g(x, u_1(x), {}^cD_q^\gamma u_1(x)), \ 0 < x < 1, \\ {}^cD_q^\alpha u_1(0) = D_qu_1(0) = 0, \\ u_1(1) = cu_0(\lambda) \leq cu_1(\lambda), \ {}^cD_q^\alpha u_1(1) = d^cD_q^\alpha u_0(\zeta) \geq d^cD_q^\alpha u_1(\zeta). \end{cases}$$

It is obvious that $u_1(x)$ is a lower solution of (1.1)–(1.2) by the Definition 4.5.

Starting from the initial function $u_0(x)$, by the following iterative scheme

$$\begin{cases} D_q^\beta(\psi_p({}^cD_q^\alpha u_k(x))) = g(x, u_{k-1}(x), {}^cD_q^\gamma u_{k-1}(x)), \ 0 < x < 1, \\ {}^cD_q^\alpha u_k(0) = D_qu_k(0) = 0, \\ u_1(1) = cu_{k-1}(\lambda), \ {}^cD_q^\alpha u_k(1) = d^cD_q^\alpha u_{k-1}(\zeta), \end{cases} \tag{4.17}$$

we obtain a sequence $\{u_k\}$, $k \in \mathbb{N}$, where $u = u_k(x)$ are lower solutions of (1.1), and satisfy $u_{k-1} \preceq u_k$, that is to say that $\{u_k\}$ is monotonically increasing.

Similar to the above inference procedure, starting from the given upper solution $v_0 \in P$, by the following iterative scheme

$$\begin{cases} D_q^{\beta}(\psi_p({}^cD_q^{\alpha} v_k(x))) = g(x, v_{k-1}(x), {}^cD_q^{\gamma} v_{k-1}(x)), \ 0 < x < 1, \\ {}^cD_q^{\alpha} v_k(0) = D_q v_k(0) = 0, \\ v_1(1) = cv_{k-1}(\lambda), \ {}^cD_q^{\alpha} v_k(1) = d^cD_q^{\alpha} v_{k-1}(\zeta), \end{cases} \tag{4.18}$$

where $k \in \mathbb{N}$, we can obtain the sequence $\{v_k\}$ which are lower solutions of (1.1)–(1.2) and satisfy $v_{k-1} \succeq v_k$, namely, $\{v_k\}$ is monotonically decreasing.

Step2. We will prove that $u_k \preceq v_k$ by mathematical induction. Suppose $u_{k-1} \preceq v_{k-1}$, $k \in \mathbb{N}$. Then $u_{k-1}(x) \leq v_{k-1}(x)$ and ${}^cD_q^{\gamma} u_{k-1}(x) \geq {}^cD_q^{\gamma} v_{k-1}(x)$. Hence from (H), one has

$$g\left(x, u_{k-1}(x), {}^cD_q^{\gamma} u_{k-1}(x)\right) \leq g\left(x, v_{k-1}(x), {}^cD_q^{\gamma} v_{k-1}(x)\right).$$

By (4.18) and (4.17), it yields that

$$\begin{cases} D_q^{\beta}(\psi_p({}^cD_q^{\alpha} v_k(x)) - \psi_p({}^cD_q^{\alpha} u_k(x))) = g(x, v_{k-1}(x), {}^cD_q^{\gamma} v_{k-1}(x)) - g(x, u_{k-1}(x), {}^cD_q^{\gamma} u_{k-1}(x)) \geq 0, \\ {}^cD_q^{\alpha} v_k(0) - {}^cD_q^{\alpha} u_k(0) = D_q v_k(0) - D_q u_k(0) = 0, \\ v_k(1) - u_k(1) \geq 0, \ {}^cD_q^{\alpha} v_k(1) - {}^cD_q^{\alpha} u_k(1) \leq 0. \end{cases}$$

Similar to (4.15), we can get $u_k \preceq v_k$. Hence

$$u_0 \preceq u_1 \preceq \cdots \preceq u_k \preceq \cdots \preceq \cdots \preceq v_k \preceq \cdots \preceq v_1 \preceq v_0.$$

Since P is a normal cone on X, the $\{u_k\}$ and $\{v_k\}$ are uniformly bounded. And it is easy to see that $\{u_k\}$ and $\{v_k\}$ are equicontinuous by the continuity of functions H, G, φ_p, φ_m and g. Thus, the sequence $\{u_k\}$ and $\{v_k\}$ are relatively compact. Then there exist u^* and v^* such that

$$\lim_{k\to\infty} u_k = u^*, \quad \lim_{k\to\infty} {}^cD_q^{\alpha} u_k = {}^cD_q^{\alpha} u^*, \tag{4.19}$$

and

$$\lim_{k\to\infty} v_k = v^*, \quad \lim_{k\to\infty} {}^cD_q^{\alpha} v_k = {}^cD_q^{\alpha} v^*. \tag{4.20}$$

Further since ${}^cD_q^{\gamma} u_k(x) = I_q^{\alpha-\gamma c} D_q^{\alpha} u_k(x)$ and ${}^cD_q^{\gamma} v_k(x) = I_q^{\alpha-\gamma c} D_q^{\alpha} v_k(x)$ we have

$$\lim_{k\to\infty} {}^cD_q^{\gamma} u_k = {}^cD_q^{\gamma} u^* \ and \ \lim_{k\to\infty} {}^cD_q^{\gamma} v_k = {}^cD_q^{\gamma} v^*, \tag{4.21}$$

that is to say that u^* is the maximum lower solution, v^* is the minimum upper solution of (1.1)–(1.2) in $[u_0, v_0] \subset P$ satisfying $u^* \preceq v^*$.

Step 3. We will prove that u^* and v^* are the solutions of (1.1).

It follows from (4.17) and Lemma 4.1 that

$$u_k(x) = cu_{k-1}(\lambda) - \int_0^1 G(x, qt)\varphi_q\left(\psi_p\left(d^cD_q^{\alpha} u_{k-1}(\zeta)\right)t^{\beta-1} - \int_0^1 H(t, qs) g\left(x, u_{k-1}(x), {}^cD_q^{\gamma} u_{k-1}(x)\right) d_q s\right) d_q t,$$

Let $k \to +\infty$. By (4.19), (4.21) and the continuity of $\varphi_p,\ g,\ G,\ H$, one has

$$u^*(x) = cu_{k-1}(\lambda) - \int_0^1 G(x,qt)\varphi_q\left(\psi_p\left(d^c D_q^\alpha u^*(\zeta)\right)t^{\beta-1} - \int_0^1 H(t,qs)g\left(x,u^*(x),{}^cD_q^\gamma u^*(x)\right)d_qs\right)d_qt,$$

which implies that u^* is a solution of (1.1)–(1.2).

In the same way, we can also prove that v^* is a solution of (1.1)–(1.2). Besides,

$$0 \le u_0(x) \le u^*(x) \le v^*(x) \le v_0(x),$$

The proof is completed.

5. Examples

Example 5.1 *For equation (1.1), let* $q = \frac{1}{2}$, $\alpha = \beta = \frac{3}{2}$, $\gamma = \frac{1}{2}$, $\lambda = \frac{2}{3}$, $\zeta = \frac{1}{4}$, $c = \frac{1}{2}$, $d = 1$, $p = m = 2$ *and*

$$g(x,y,z) = \begin{cases} \dfrac{x}{20} + \dfrac{1}{100}\sin z, & 0 \le x \le 1, 0 \le y \le 1, \\ \dfrac{x}{20} + 3200(y-1) + \dfrac{1}{100}\sin z, & 0 \le x \le 1, 1 \le y \le 2, \\ \dfrac{x}{20} + 3200 + 20(y-2) + \dfrac{1}{100}\sin z, & 0 \le x \le 1, 2 \le y \le 7, \\ \dfrac{x}{20} + 3300 + \dfrac{1}{100}\sin z, & 0 \le x \le 1, y > 7. \end{cases} \tag{5.1}$$

By simple computation, we obtain $\rho = \frac{\Gamma_q(\alpha-\gamma)}{(1-c)\Gamma_q(\alpha)} \approx 1.94544$,

$$M = \left[\frac{(1-c)_q^{(\beta)} - (1-\lambda)_q^{(\beta)}}{\Gamma_q(\beta+1)}\right]^{m-1} \approx 0.10447,$$

$$N_1 = \Gamma_q(\alpha-\gamma+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1} = 0.7710336,$$

$$N_2 = \frac{[\beta m+2]_q\Gamma_q(\alpha)}{c^{\beta m+2}(1-\lambda)(1-c)_q^{(\alpha-1)}M} \approx 1294.25065,$$

$$N_3 = (1-c)\Gamma_q(\alpha+1)[(1-\zeta^{\beta-1}d^{p-1})\Gamma_q(\beta+1)]^{m-1} = 0.59449.$$

In addition, if we take $l = 1$, $n_1 = 2$, $n_2 = 7$ *and* $r = 5000$, *then* $g(x,y,z)$ *satisfies the following conditions:*

$$g(x,y,z) \le (rN_1)^{p-1} = 3855.168,\ (x,y,z) \in [0,1]\times[0,\rho r]\times[-5000,5000],$$

$$g(x,y,z) > (n_1N_2)^{p-1} \approx 2588.50130,\ (x,y,z) \in [0,1]\times[n_1,n_2]\times[-5000,5000],$$

$$g(x,y,z) < (lN_3)^{p-1} = 0.59449,\ (x,y,z) \in [0,1]\times[0,l]\times[-5000,5000].$$

Then all conditions of Theorem 3.6 are satisfied. Thus, the problem (1.1)–(1.2) has at least three fixed point $u_1(x)$, $u_2(x)$ *and* $u_3(x)$.

Example 5.2 *Consider the following four-point boundary value problem of fractional q-difference equation with p-Laplacian operator*

$$\begin{cases} D_q^{\frac{3}{2}}\left(\psi_p\left({}^cD_q^{\frac{3}{2}}u(x)\right)\right) = g\left(x, u(x), {}^cD_q^{\frac{3}{2}}u(x)\right),\ 0 < x < 1, \\ {}^cD_q^{\frac{3}{2}}u(0) = D_qu(0) = 0, \\ u(1) = \frac{1}{5}u\left(\frac{1}{3}\right),\ {}^cD_q^{\frac{3}{2}}u(1) = 2{}^cD_q^{\frac{3}{2}}u\left(\frac{16}{25}\right). \end{cases} \tag{5.2}$$

Let $q = \frac{1}{2}$ and $p = 2$. Then $\psi_p({}^cD_q^{\frac{3}{2}}u(x)) = {}^cD_q^{\frac{3}{2}}u(x)$. Assume that $g(x, y, z) = 2 - e^{-y} - e^z$, which satisfies the assumption (H). We can easily check that $u_0 = u_0(x) \equiv 0$ is a lower solution of (5.2). Let $v_0(x) = 1 + x^3$. It is easy to see

$$D_q(1+x^3) = \frac{1-q^3}{1-q}x = \frac{7}{4}x^2, D_q^2(1+x^3) = D_q(\frac{7}{4}x^2) = \frac{21}{8}x, D_q^3(1+x^3) = D_q(\frac{21}{8}x) = \frac{21}{8},$$

then by the definitions of fractional q-integral and Caputo type q-derivative,

$$\begin{aligned} {}^cD_q^{\frac{3}{2}}v_0(x) = {}^cD_q^{\frac{3}{2}}(1+x^3) = I_q^{0.5}D_q^2(1+x^3) \\ &= \frac{1}{\Gamma_q(0.5)}\int_0^x (x-qt)_q^{(-0.5)}\frac{21}{8}td_qt \\ &= \frac{21}{8}\frac{(1-q)}{\Gamma_q(0.5)}x^{1.5}\sum_{k=0}^{\infty}q^{2k}(1-q^{k+1})_q^{(-0.5)}, \end{aligned}$$

$$D_q^{\frac{3}{2}}\left(\psi_p\left({}^cD_q^{\frac{3}{2}}v_0(x)\right)\right) = D_q^{\frac{3}{2}}\left({}^cD_q^{\frac{3}{2}}v_0(x)\right) = D_q^2I_qD_q^2v_0(x) = \frac{21}{8}.$$

Obviously, $D_q^{\frac{3}{2}}\left(\psi_p\left({}^cD_q^{\frac{3}{2}}v_0(x)\right)\right) \geq g(x, u(x), {}^cD_q^{\gamma}u(x))$. In addition, by simple calculation, we can get

$${}^cD_q^{\frac{3}{2}}v_0(0) = D_qv_0(0) = 0,\ v_0(1) \geq \frac{1}{5}v_0\left(\frac{1}{3}\right),\ \text{and } {}^cD_q^{\frac{3}{2}}v_0(1) \leq 2{}^cD_q^{\frac{3}{2}}v_0\left(\frac{16}{25}\right).$$

Hence $v_0(x)$ is an upper solution of (5.2). According to Theorem 4.6, (5.2) has the maximal lower solution u^* and the minimal upper solution v^*. And both u^* and v^* are solutions of (5.2).

Acknowledgment

This research is supported by Shandong Provincial Natural Science Foundation (ZR2020MA016), also supported by the Natural Science Foundation of China (62073153).

The authors sincerely thank the anonymous reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript.

References

[1] Agarwal RP. Certain fractional q-integrals and q-derivatives. Mathematical Proceedings of the Cambridge Philosophical Society 1969; 66: 365-370.

[2] Ahmad B, Ntouyas SK, Tariboon J, Alsaedi A, Alsulami H. Impulsive fractional q-integro-difference equations with separated boundary conditions. Applied Mathematics and Computation 2016; 281: 199-213.

[3] Al-Salam WA. Some fractional q-integrals and q-derivatives. Proceedings of the Edinburgh Mathematical Society 1966; 15 (2): 135-140.

[4] Annaby MH, Mansour ZS. Fractional q-Difference Equations. Springer Berlin Heidelberg, 2012.

[5] Avery RI, Peterson AC. Three positive fixed points of nonlinear operators on ordered Banach spaces. Computers and Mathematics with Applications 2001; 42: 313-322.

[6] Bohner M, Hassan TS, Li T. Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments. Indagationes Mathematicae 2018; 29 (2): 548-560.

[7] Bohner M, Li T. Kamenev-type criteria for nonlinear damped dynamic equations. Science China Mathematics 2015; 58 (7): 1445-1452.

[8] Bohner M, Li T. Oscillation of second-order p-Laplace dynamic equations with a nonpositive neutral coefficient. Applied Mathematics Letters 2014; 37: 72-76.

[9] Dobrogowska A. The q-deformation of the Morse potential. Applied Mathematics Letters 2013; 26: 769-773.

[10] Džurina J, Grace SR, Jadlovská I, Li T. Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term. Mathematische nachrichten 2020; 293 (5): 910-922.

[11] Ernst T. A Comprehensive Treatment of q-Calculus. Springer Basel, 2012.

[12] Han Z, Lu H, Zhang C. Positive solutions for eigenvalue problems of fractional differential equation with generalized p-Laplacian. Applied Mathematics and Computation 2015; 257: 526-536.

[13] Jackson F. On q-difference equations. American Journal of Mathematics 1910; 32: 305-314.

[14] Li S, Zhang Z, Jiang W. Multiple positive solutions for four-point boundary value problem of fractional delay differential equations with p-Laplacian operator. Applied Numerical Mathematics 2021; 165: 348-356.

[15] Li T, Pintus N, Viglialoro G. Properties of solutions to porous medium problems with different sources and boundary conditions. Zeitschrift für angewandte Mathematik und Physik 2019; 70 (3): 1-18.

[16] Li T, Rogovchenko YV. On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations. Applied Mathematics Letters 2017; 67: 53-59.

[17] Li T, Rogovchenko YV. On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations. Applied Mathematics Letters 2020; 105: 1-7.

[18] Li T, Viglialoro G. Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime. Differential Integral Equations 2021; 34 (5-6): 315-336.

[19] Liu HK. Application of the variational iteration method to strongly nonlinear q-difference equations. Journal of Applied Mathematics 2012; 1: 1-12.

[20] Liu X, Jia M, Xiang X. On the solvability of a fractional differential equation model involving the p-Laplacian operator. Computers and Mathematics with Applications 2012; 64: 3267-3275.

[21] Li X, Han Z, Sun S, Sun L. Eigenvalue problems of fractional q-difference equations with generalized p-Laplacian. Applied Mathematics Letters 2016; 57: 46-53.

[22] Rajkovi PM, Marinkovi SD, Stankovi MS. On q-Analogues of Caputo Derivative and Mittag-Leffler Function. Fractional Calculus and Applied Analysis 2007; 10: 359-373.

[23] Riyasat M, Khan S, Nahid T. q-difference equations for the composite $2D$ q-Appell polynomials and their applications. Cogent Mathematics 2017; 4: 1-23.

[24] Rui AR. Positive solutions for a class of boundary value problems with fractional q-differences. Computers and Mathematics with Applications 2011; 61: 367-373.

[25] Yuan Q, Yang W. Positive solution for q-fractional four-point boundary value problems with p-Laplacian operator. Journal of Inequalities and Applications 2014; 481: 1-14.

[26] Zhang C, Agarwal RP, Li T. Oscillation and asymptotic behavior of higher-order delay differential equations with p-Laplacian like operators. Journal of Mathematical Analysis and Applications 2014; 409 (2): 1093-1106.