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Improved inequalities related to the A-numerical radius for commutators of operators

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Abstract: Let A be a positive bounded linear operator on a complex Hilbert space $\mathcal{H}$ and $\mathbb{B}_A(\mathcal{H})$ be the subspace of all operators which admit A-adjoints operators. In this paper, we establish some inequalities involving the commutator and the anticommutator of operators in semi-Hilbert spaces, i.e. spaces generated by positive semidefinite sesquilinear forms. Mainly, among other inequalities, we prove that for $T, S \in \mathbb{B}_A(\mathcal{H})$ we have

$$\omega_A(TS \pm ST) \leq 2\sqrt{2}\min\Big\{f_A(T,S), f_A(S,T)\Big\},$$

where

$$f_A(X,Y) = \|Y\|_A\sqrt{\omega_A^2(X) - \frac{\left|\left\|\frac{X+X^{\sharp_A}}{2}\right\|_A^2 - \left\|\frac{X-X^{\sharp_A}}{2i}\right\|_A^2\right|}{2}}.$$

This covers and improves the well-known inequalities of Fong and Holbrook. Here $\omega_A(\cdot)$ and $\|\cdot\|_A$ are the A-numerical radius and the A-operator seminorm of semi-Hilbert space operators, respectively and $X^{\sharp_A}$ denotes a distinguished A-adjoint operator of X.

Key words: Positive operator, semiinner product, numerical radius, commutator, anticommutator

1. Introduction

Let $\big(\mathcal{H}, \langle\cdot,\cdot\rangle\big)$ be a complex Hilbert space endowed with the norm $\|\cdot\|$. Let $\mathbb{B}(\mathcal{H})$ stand for the C^*-algebra of all bounded linear operators on $\mathcal{H}$ and I denote the identity operator on $\mathcal{H}$. For every operator $T \in \mathbb{B}(\mathcal{H})$ its range is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$, and its adjoint by T^*. An operator $T \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and we then write $T \geq 0$. The square root of every positive operator T is denoted by $T^{1/2}$. If $T \geq 0$, then the absolute value of T is given by $|T| := (T^*T)^{1/2}$. If $\mathcal{S}$ is a given linear subspace of $\mathcal{H}$, then $\overline{\mathcal{S}}$ stands for its closure in the norm topology of $\mathcal{H}$. Moreover, the orthogonal projection onto a closed linear subspace $\mathcal{S}$ of $\mathcal{H}$ is denoted by $P_{\mathcal{S}}$. Throughout this article, we suppose that $A \in \mathbb{B}(\mathcal{H})$ is a positive operator, which induces the following semiinner product

$$\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C},\ (x,y) \longmapsto \langle x,y\rangle_A := \langle Ax, y\rangle = \langle A^{1/2}x, A^{1/2}y\rangle.$$

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The seminorm induced by $\langle\cdot,\cdot\rangle_A$ is given by $\|x\|_A = \sqrt{\langle x,x\rangle_A}$ for every $x \in \mathcal{H}$. Clearly $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$ which implies that $\|\cdot\|_A$ is a norm on $\mathcal{H}$ if and only if A is an injective operator. Further, it can be seen that the semi-Hilbert space $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. In recent years, many authors investigated several properties of bounded linear operators acting on $(\mathcal{H}, \|\cdot\|_A)$ (see, e.g., [2–7, 9, 12] and their references). In this article we continue the line of research begun in [18–20]. Notice that the inspiration for our investigation comes from the works of Kittaneh et al. [1, 22].

The semiinner product $\langle\cdot,\cdot\rangle_A$ induces an inner product on the quotient space $\mathcal{H}/\mathcal{N}(A)$ which is not complete unless $\mathcal{R}(A)$ is closed subspace in $\mathcal{H}$. However, a canonical construction due to L. de Branges and J. Rovnyak in [13] (see also [15]) shows that the completion of $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathcal{R}(A^{1/2})$ with the inner product

$$\langle A^{1/2}x, A^{1/2}y\rangle_{\mathbf{R}(A^{1/2})} := \langle P_{\overline{\mathcal{R}(A)}}x, P_{\overline{\mathcal{R}(A)}}y\rangle,\ \forall\, x,y \in \mathcal{H}.$$

For the sequel, the Hilbert space $\big(\mathcal{R}(A^{1/2}), \langle\cdot,\cdot\rangle_{\mathbf{R}(A^{1/2})}\big)$ will be denoted by $\mathbf{R}(A^{1/2})$. For more details concerning the Hilbert space $\mathbf{R}(A^{1/2})$, we refer the reader to [4] and the references therein.

For $T \in \mathbb{B}(\mathcal{H})$, an operator $S \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint operator of T if for every $x, y \in \mathcal{H}$, we have $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$, that is, S is solution of the operator equation $AX = T^*A$. This kind of equations can be studied by using the next theorem due to Douglas (for its proof see [14]).

Theorem A *If $T, S \in \mathbb{B}(\mathcal{H})$, then the following statements are equivalent:*

(i) $\mathcal{R}(S) \subseteq \mathcal{R}(T)$.

(ii) $TD = S$ *for some* $D \in \mathbb{B}(\mathcal{H})$.

(iii) *There exists* $\lambda > 0$ *such that* $\|S^*x\| \leq \lambda\|T^*x\|$ *for all* $x \in \mathcal{H}$.

If one of these conditions holds, then there exists a unique solution of the operator equation $TX = S$, denoted by Q, such that $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(T^*)}$. Such Q is called the reduced solution of $TX = S$.

If we denote by $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ the sets of all operators that admit A-adjoints and $A^{1/2}$-adjoints, respectively, then an application of Theorem A gives

$$\mathbb{B}_A(\mathcal{H}) = \big\{T \in \mathbb{B}(\mathcal{H});\ \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\big\},$$

and

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \big\{T \in \mathbb{B}(\mathcal{H});\ \exists\, c > 0;\ \|Tx\|_A \leq c\|x\|_A,\ \forall x \in \mathcal{H}\big\}.$$

Operators in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are called A-bounded. Notice that $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathbb{B}(\mathcal{H})$ which are, in general, neither closed nor dense in $\mathbb{B}(\mathcal{H})$ (see [2]). Moreover, the following inclusions $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$ hold and are in general proper (see [16]). The seminorm of an operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ is given by

$$\|T\|_A := \sup_{\substack{x\in\overline{\mathcal{R}(A)},\\ x\neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup\big\{\|Tx\|_A\,;\ x \in \mathcal{H}, \|x\|_A = 1\big\} < +\infty,$$

(see [16] and the references therein). It may happen that $\|T\|_A = +\infty$ for some $T \in \mathbb{B}(\mathcal{H})$ (see [16, Example 2]). It is not difficult to verify that, for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have $\|Tx\|_A \leq \|T\|_A \|x\|_A$ for all $x \in \mathcal{H}$. This immediately yields that $\|TS\|_A \leq \|T\|_A \|S\|_A$ for every $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$.

Recently, several extensions for the concept of the numerical radius of Hilbert space operators have been investigated. One of these extensions is the notion A-numerical radius of an operator $T \in \mathbb{B}(\mathcal{H})$ which was firstly introduced by Saddi in the study in [26] as

$$\omega_A(T) = \sup\Big\{\big|\langle Tx, x\rangle_A\big|;\ \ x \in \mathcal{H},\ \|x\|_A = 1\Big\}.$$

If $A = I$, we get the classical numerical radius which received considerable attention in the literature (e.g., see [1, 11], and their references). It should be mentioned here that it may happen that $\omega_A(T) = +\infty$ for some $T \in \mathbb{B}(\mathcal{H})$ (see [18]). However, $\omega_A(\cdot)$ defines a seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ which is equivalent to the A-operator seminorm $\|\cdot\|_A$. More precisely, for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A. \tag{1.1}$$

For an account of the results related to the A-numerical radius of A-bounded operators, the reader is referred to [5–10, 16, 17]. The A-Crawford number of an operator T is defined by

$$c_A(T) = \inf\big\{|\langle Tx, x\rangle_A|;\ x \in \mathcal{H},\ \|x\|_A = 1\big\}.$$

An operator $T \in \mathbb{B}(\mathcal{H})$ is called A-selfadjoint if AT is selfadjoint. Further, an operator $T \in \mathbb{B}(\mathcal{H})$ is said A-positive if AT is positive and we note $T \geq_A 0$. Trivially, an A-positive operator is always A-selfadjoint since $\mathcal{H}$ is a complex Hilbert space. Moreover, it was shown in [16] that if T is A-self-adjoint, then

$$\|T\|_A = \omega_A(T). \tag{1.2}$$

If $T \in \mathbb{B}_A(\mathcal{H})$, then the reduced solution to the operator equation $AX = T^*A$ is denoted by $T^{\sharp_A}$. Notice that $T^{\sharp_A} = A^{\dagger}T^*A$, where $A^{\dagger}$ is the Moore–Penrose inverse of A, which is the unique linear mapping from $\mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp}$ into $\mathcal{H}$ satisfying the following Moore–Penrose equations:

$$AXA = A,\ XAX = X,\ XA = P_{\overline{\mathcal{R}(A)}}\ \text{ and }\ AX = P_{\overline{\mathcal{R}(A)}}|_{\mathcal{R}(A)\oplus\mathcal{R}(A)^{\perp}}.$$

We mention here that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathbb{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$ and $\big((T^{\sharp_A})^{\sharp_A}\big)^{\sharp_A} = T^{\sharp_A}$. Moreover, if $T, S \in \mathbb{B}_A(\mathcal{H})$, then $TS \in \mathbb{B}_A(\mathcal{H})$ and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$. Notice that for $T \in \mathbb{B}_A(\mathcal{H})$, we have $T^{\sharp_A}T \geq_A 0$ and $TT^{\sharp_A} \geq_A 0$. Moreover, by using (1.2), we see that

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2. \tag{1.3}$$

For other results covering $T^{\sharp_A}$, we invite the reader to consult [2, 3, 24].

Recently, several improvements of the inequalities in (1.1) have been given (see [8, 18]). For instance, it has been shown in [19], that if $T \in \mathbb{B}_A(\mathcal{H})$, then

$$\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \leq \omega_A^2(T) \leq \frac{1}{2}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A. \tag{1.4}$$

Recently, the present proved in [19] that for every $T, S \in \mathbb{B}_A(\mathcal{H})$, we have

$$\omega_A(TS \pm ST) \leq 2\sqrt{2}\min\Big\{\|T\|_A\omega_A(S), \|S\|_A\omega_A(T)\Big\}. \tag{1.5}$$

Of course, if $A = I$, we get the well-known inequalities of Fong and Holbrook (see [21]). One main target of this paper is to generalize (1.5). Also, a considerable refinement of (1.5) is established.

2. Results

In this section, we prove our results. In order to prove our first result in this section, we need the following lemmas.

Lemma 2.1 *([19, Theorem 2.16.]) Let $T_1, T_2, S_1, S_2 \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(T_1S_1 \pm S_2T_2) \leq \sqrt{\left\|T_1T_1^{\sharp_A} + T_2^{\sharp_A}T_2\right\|_A}\sqrt{\left\|S_1^{\sharp_A}S_1 + S_2S_2^{\sharp_A}\right\|_A}.$$

Lemma 2.2 *([20]) Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ and $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Then, the following assertions hold*

(i) $\omega_{\mathbb{A}}\left[\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}\right] = \max\left\{\omega_A(T), \omega_A(S)\right\}.$

(ii) $\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}\right] = \omega_A(T).$

(iii) $\left\|\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}\right\|_{\mathbb{A}} = \left\|\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}\right\|_{\mathbb{A}} = \max\left\{\|T\|_A, \|S\|_A\right\}.$

Our first result in this paper reads as follows.

Theorem 2.3 *Let $T_1, T_2, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(T_1S \pm ST_2) \leq 4\,\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}\right]\omega_A(S).$$

Proof By letting $S_1 = S_2 = S$ in Lemma 2.1 we obtain

$$\omega_A(T_1S \pm ST_2) \leq \sqrt{\left\|T_1T_1^{\sharp_A} + T_2^{\sharp_A}T_2\right\|_A}\sqrt{\left\|S^{\sharp_A}S + SS^{\sharp_A}\right\|_A}.$$

Moreover, by using the first inequality in (1.4) we get

$$\omega_A(T_1S \pm ST_2) \leq 2\sqrt{\left\|T_1T_1^{\sharp_A} + T_2^{\sharp_A}T_2\right\|_A}\,\omega_A(S). \tag{2.1}$$

Let $\mathbb{T} = \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}$. It can be observed that

$$\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} + \mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} T_1T_1^{\sharp_A} + T_2^{\sharp_A}T_2 & 0 \\ 0 & T_1^{\sharp_A}T_1 + T_2T_2^{\sharp_A} \end{pmatrix}$$

This implies, by Lemma 2.2(iii), that

$$\left\|\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}+\mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}}\right\|_{\mathbb{A}}=\max\left\{\left\|T_1T_1^{\sharp_A}+T_2^{\sharp_A}T_2\right\|_A,\left\|T_1^{\sharp_A}T_1+T_2T_2^{\sharp_A}\right\|_A\right\}.$$

Therefore, by taking into consideration (2.1) we see that

$$\begin{aligned}\omega_A(T_1S\pm ST_2)&\leq 2\sqrt{\left\|T_1T_1^{\sharp_A}+T_2^{\sharp_A}T_2\right\|_A}\,\omega_A(S)\\&\leq 2\sqrt{\left\|\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}+\mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}}\right\|_{\mathbb{A}}}\,\omega_A(S)\\&\leq 4\,\omega_{\mathbb{A}}(\mathbb{T})\,\omega_A(S),\end{aligned}$$

where the last inequality follows from the first inequality in (1.4). Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 2.3 and provides an upper bound for the A-numerical radius of the commutator $TS-ST$ when T and S are in $\mathbb{B}_A(\mathcal{H})$.

Corollary 2.4 *Let $T,S\in\mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS\pm ST)\leq 4\omega_A(T)\,\omega_A(S). \tag{2.2}$$

Moreover, if $TS=ST$, then

$$\omega_A(TS)\leq 2\omega_A(T)\,\omega_A(S). \tag{2.3}$$

Proof By letting $T_1=T_2=T$ in Theorem 2.3 and then using Lemma 2.2(i) we reach (2.2). Further, (2.3) follows immediately from (2.2). □

Now, we aim to prove a generalization of (1.5). In order to achieve this goal, we need to establish the following result.

Proposition 2.5 *Let $T,S\in\mathbb{B}_A(\mathcal{H})$. Then,*

$$\|TT^{\sharp_A}+S^{\sharp_A}S\|_A\leq\max\left\{\|T\|_A^2,\|S\|_A^2\right\}+\|ST\|_A. \tag{2.4}$$

To prove (2.4), we shall require the following lemma.

Lemma B *([4, 24]) Let $T\in\mathbb{B}(\mathcal{H})$. Then $T\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exists a unique $\widetilde{T}\in\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_AT=\widetilde{T}Z_A$. Here, $Z_A:\mathcal{H}\to\mathbf{R}(A^{1/2})$ is defined by $Z_Ax=Ax$. Moreover, we have*

(i) $\|T\|_A=\|\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}$.

(ii) $\widetilde{T^{\sharp_A}}=(\widetilde{T})^*$.

Now, we are ready to prove Proposition 2.5.

Proof of Proposition 2.5 Since, $T,S\in\mathbb{B}_{A^{1/2}}(\mathcal{H})$, then by Lemma B there exists two unique operators $\widetilde{T},\widetilde{S}\in\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_AT=\widetilde{T}Z_A$ and $Z_AS=\widetilde{S}Z_A$. Moreover, clearly $T+S$ and TS are in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. An application of [15, Lemma 2.1.] gives

$$\widetilde{TS}=\widetilde{T}\widetilde{S}\ \text{ and }\ \widetilde{T+S}=\widetilde{T}+\widetilde{S}. \tag{2.5}$$

Now, by using Lemma B together with (2.5), it can be seen that

$$\begin{aligned}\|S^{\sharp_A}S+TT^{\sharp_A}\|_A &= \|\widetilde{S^{\sharp_A}S+TT^{\sharp_A}}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\ &= \|(\widetilde{S})^*\widetilde{S}+\widetilde{T}(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.\end{aligned} \tag{2.6}$$

Moreover, by using basic properties of the spectral radius of Hilbert space operators, we see that

$$\begin{aligned}\|(\widetilde{S})^*\widetilde{S}+\widetilde{T}(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} &= r\left((\widetilde{S})^*\widetilde{S}+\widetilde{T}(\widetilde{T})^*\right)\\ &= r\left[\begin{pmatrix}(\widetilde{S})^*\widetilde{S}+\widetilde{T}(\widetilde{T})^* & 0\\ 0 & 0\end{pmatrix}\right]\\ &= r\left[\begin{pmatrix}|\widetilde{S}| & |(\widetilde{T})^*|\\ 0 & 0\end{pmatrix}\begin{pmatrix}|\widetilde{S}| & 0\\ |(\widetilde{T})^*| & 0\end{pmatrix}\right]\\ &= r\left[\begin{pmatrix}|\widetilde{S}| & 0\\ |(\widetilde{T})^*| & 0\end{pmatrix}\begin{pmatrix}|\widetilde{S}| & |(\widetilde{T})^*|\\ 0 & 0\end{pmatrix}\right]\end{aligned}$$

Hence, we get

$$\|(\widetilde{S})^*\widetilde{S}+\widetilde{T}(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = r\left[\begin{pmatrix}(\widetilde{S})^*\widetilde{S} & |\widetilde{S}|\,|(\widetilde{T})^*|\\ |(\widetilde{T})^*|\,|\widetilde{S}| & \widetilde{T}(\widetilde{T})^*\end{pmatrix}\right]$$

Thus, by using [23, Theorem 1.1.] we obtain

$$\begin{aligned}\|(\widetilde{S})^*\widetilde{S}+\widetilde{T}(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} &\leq r\left[\begin{pmatrix}\|(\widetilde{S})^*\widetilde{S}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} & \|\,|\widetilde{S}|\,|(\widetilde{T})^*|\,\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\ \|\,|(\widetilde{T})^*|\,|\widetilde{S}|\,\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} & \|\widetilde{T}(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\end{pmatrix}\right]\\ &= \left\|\begin{pmatrix}\|\widetilde{S}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))} & \|\widetilde{S}\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\ \|\widetilde{S}\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} & \|\widetilde{T}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\end{pmatrix}\right\|,\end{aligned}$$

where the last equality follows since

$$\|\,|\widetilde{S}|\,|(\widetilde{T})^*|\,\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = \|\,|(\widetilde{T})^*|\,|\widetilde{S}|\,\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} = \|\widetilde{S}\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}.$$

Therefore, we infer that

$$\begin{aligned}\|(\widetilde{S})^*\widetilde{S}+\widetilde{T}(\widetilde{T})^*\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} &\leq \left\|\begin{pmatrix}\|\widetilde{S}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))} & 0\\ 0 & \|\widetilde{T}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\end{pmatrix}\right\| + \left\|\begin{pmatrix}0 & \|\widetilde{S}\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\\ \|\widetilde{S}\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))} & 0\end{pmatrix}\right\|\\ &= \max\left\{\|\widetilde{S}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))}, \|\widetilde{T}\|^2_{\mathbb{B}(\mathbf{R}(A^{1/2}))}\right\} + \|\widetilde{S}\widetilde{T}\|_{\mathbb{B}(\mathbf{R}(A^{1/2}))},\end{aligned}$$

where the last equality follows from Lemma 2.2(iii) by letting $A = I$. Thus, by taking into account (2.6) and then applying Lemma B(i) we get

$$\|TT^{\sharp_A}+S^{\sharp_A}S\|_A \leq \max\left\{\|T\|_A^2, \|S\|_A^2\right\} + \|ST\|_A.$$

This achieves the proof. □

Now, we are in a position to prove the following theorem which generalizes (1.5).

Theorem 2.6 *Let* $T, S, X, Y \in \mathbb{B}_A(\mathcal{H})$ *and* $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. *Then*

$$\begin{aligned}\omega_A(TX \pm YS) &\leq 2\sqrt{\max\left\{\|T\|_A^2, \|S\|_A^2\right\} + \|ST\|_A}\sqrt{\omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\right] - \frac{1}{2}c_A(YX)} \\ &\leq 2\sqrt{2}\max\left\{\|T\|_A, \|S\|_A\right\}\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\right]. \end{aligned} \tag{2.7}$$

Proof Notice first that, it was shown in [19, Theorem 2.8.] that

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\right] \geq \frac{1}{2}\sqrt{\left\|X^{\sharp_A}X + YY^{\sharp_A}\right\|_A + 2c_A(YX)}. \tag{2.8}$$

Now, by applying Lemma 2.1 together with (2.4) we obtain

$$\begin{aligned}\omega_A(TX \pm YS) &\leq \sqrt{\left\|TT^{\sharp_A} + S^{\sharp_A}S\right\|_A}\sqrt{\left\|X^{\sharp_A}X + YY^{\sharp_A}\right\|_A} \\ &\leq \sqrt{\max\left\{\|T\|_A^2, \|S\|_A^2\right\} + \|ST\|_A}\sqrt{\left\|X^{\sharp_A}X + YY^{\sharp_A}\right\|_A} \\ &\leq 2\sqrt{\max\left\{\|T\|_A^2, \|S\|_A^2\right\} + \|ST\|_A}\sqrt{\omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\right] - \frac{1}{2}c_A(YX)}, \end{aligned}$$

where the last inequality follows from (2.8). On the other hand, we see that

$$\|ST\|_A \leq \|T\|_A\|S\|_A \leq \frac{1}{2}\left(\|T\|_A^2 + \|S\|_A^2\right) \leq \max\left\{\|T\|_A^2, \|S\|_A^2\right\}.$$

This immediately proves (2.7). $\square$

Remark 2.7 *By replacing* S *by* T *and* X, Y *by* S *in* (2.7) *and then using Lemma 2.2(ii) we get*

$$\omega_A(TS \pm ST) \leq 2\sqrt{2}\|T\|_A\omega_A(S).$$

Thus, by changing the roles bet ween T *and* S *in the last inequality we reach* (1.5).

For the rest of this paper, for any arbitrary operator $T \in \mathbb{B}_A(\mathcal{H})$, we write

$$\Re_A(T) := \frac{T + T^{\sharp_A}}{2} \quad \text{and} \quad \Im_A(T) := \frac{T - T^{\sharp_A}}{2i}.$$

Our next aim is to improve the inequality (1.5). To do this, we need the following lemma.

Lemma 2.8 *Let* $T \in \mathbb{B}_A(\mathcal{H})$ *be such that* $\omega_A(T) \leq 1$. *Then, for every* $x \in \mathcal{H}$ *with* $\|x\|_A = 1$ *we have*

$$\|Tx\|_A^2 + \|T^{\sharp_A}x\|_A^2 \leq 4\left(1 - \frac{\left|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right|}{2}\right). \tag{2.9}$$

In order to prove Lemma 2.8, we first prove the following result.

Lemma C *Let* $T, S \in \mathbb{B}_A(\mathcal{H})$*. Then,*

$$\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \leq 4\max\left\{\|\Re_A(T)\|_A^2, \|\Im_A(T)\|_A^2\right\} - 2\left|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right|.$$

Proof Notice first that it was shown in [25, Lemma 2.18] that

$$\|X^{\sharp_A}X + Y^{\sharp_A}Y\|_A \leq \max\left\{\|X+Y\|_A^2, \|X-Y\|_A^2\right\} - \frac{\left|\,\|X+Y\|_A^2 - \|X-Y\|_A^2\right|}{2}, \tag{2.10}$$

for every $X, Y \in \mathbb{B}_A(\mathcal{H})$. By replacing X and Y by $(T^{\sharp_A})^{\sharp_A}$ and $T^{\sharp_A}$ in (2.10) respectively we get

$$\begin{aligned}
\|(T^{\sharp_A}T + TT^{\sharp_A})^{\sharp_A}\|_A &\leq \max\left\{\|(T^{\sharp_A}+T)^{\sharp_A}\|_A^2, \|(T^{\sharp_A}-T)^{\sharp_A}\|_A^2\right\} - \frac{\left|\,\|(T^{\sharp_A}+T)^{\sharp_A}\|_A^2 - \|(T^{\sharp_A}-T)^{\sharp_A}\|_A^2\right|}{2} \\
&= \max\left\{\|T^{\sharp_A}+T\|_A^2, \|T^{\sharp_A}-T\|_A^2\right\} - \frac{\left|\,\|T^{\sharp_A}+T\|_A^2 - \|T^{\sharp_A}-T\|_A^2\right|}{2} \\
&= 4\max\left\{\|\Re_A(T)\|_A^2, \|\Im_A(T)\|_A^2\right\} - 2\left|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right|.
\end{aligned}$$

This shows the desired result since $\|S\|_A = \|S^{\sharp_A}\|_A$ for all $S \in \mathbb{B}_A(\mathcal{H})$. □

Now, we are ready to prove Lemma 2.8.

Proof of Lemma 2.8 Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. By using the Cauchy-Schwarz inequality we see that

$$\begin{aligned}
\|Tx\|_A^2 + \|T^{\sharp_A}x\|_A^2 &= \langle (T^{\sharp_A}T + TT^{\sharp_A})x, x\rangle_A \\
&\leq \omega_A\left(T^{\sharp_A}T + TT^{\sharp_A}\right).
\end{aligned}$$

Further, since $T^{\sharp_A}T + TT^{\sharp_A} \geq_A 0$, then (1.2) gives

$$\|Tx\|_A^2 + \|T^{\sharp_A}x\|_A^2 \leq \|T^{\sharp_A}T + TT^{\sharp_A}\|_A. \tag{2.11}$$

On the other hand, since $\Re_A(T)$ and $\Im_A(T)$ are A-selfadjoint operators, then by (1.2) we have

$$\omega_A\Big(\Re_A(T)\Big) = \|\Re_A(T)\|_A \text{ and } \omega_A\Big(\Im_A(T)\Big) = \|\Im_A(T)\|_A.$$

Therefore, by applying (2.11) together with Lemma C, one observes that

$$\begin{aligned}
\|Tx\|_A^2 + \|T^{\sharp_A}x\|_A^2 &\leq 4\max\left\{\|\Re_A(T)\|_A^2, \|\Im_A(T)\|_A^2\right\} - 2\left|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right| \\
&= 4\max\left\{\omega_A^2\Big(\Re_A(T)\Big), \omega_A^2\Big(\Im_A(T)\Big)\right\} - 2\left|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right|.
\end{aligned}$$

On the other hand, it is not difficult to verify that

$$\omega_A\Big(\Re_A(T)\Big) \leq \omega_A(T) \text{ and } \omega_A\Big(\Im_A(T)\Big) \leq \omega_A(T).$$

This implies that

$$\begin{aligned}
\|Tx\|_A^2 + \|T^{\sharp_A}x\|_A^2 &\leq 4\omega_A^2(T) - 2\left|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right| \\
&\leq 4\left(1 - \frac{\left|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right|}{2}\right),
\end{aligned}$$

where the last inequality follows since $\omega_A(T) \leq 1$. $\square$

Now, we are in a position to prove the following theorem.

Theorem 2.9 *Let $T, S, X, Y \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(TXS \pm SYT) \leq 2\sqrt{2}\|S\|_A \max\{\|X\|_A, \|Y\|_A\} \sqrt{\omega_A^2(T) - \frac{|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2|}{2}}. \tag{2.12}$$

Proof Assume first that $\omega_A(T) \leq 1$, $\|X\|_A \leq 1$ and $\|Y\|_A \leq 1$. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. By applying the Cauchy-Schwarz inequality we see that

$$\begin{aligned}
|\langle (TX \pm YT)x, x\rangle_A| &\leq |\langle Xx, T^{\sharp_A}x\rangle_A| + |\langle Tx, Y^{\sharp_A}x\rangle_A| \\
&\leq \|Xx\|_A\|T^{\sharp_A}x\|_A + \|Tx\|_A\|Y^{\sharp_A}x\|_A \\
&\leq \|X\|_A\|T^{\sharp_A}x\|_A + \|Tx\|_A\|Y^{\sharp_A}\|_A \\
&\leq \|T^{\sharp_A}x\|_A + \|Tx\|_A \\
&\leq \sqrt{2}\left(\|T^{\sharp_A}x\|_A^2 + \|Tx\|_A^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, by Lemma 2.8 we get

$$|\langle (TX \pm YT)x, x\rangle_A| \leq 2\sqrt{2}\sqrt{1 - \frac{|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2|}{2}}.$$

Thus, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality we get

$$\omega_A(TX \pm YT) \leq 2\sqrt{2}\sqrt{1 - \frac{|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2|}{2}}. \tag{2.13}$$

Now, let $T, X, Y \in \mathbb{B}_A(\mathcal{H})$ be any operators. If $\max\{\|X\|_A, \|Y\|_A\} = 0$ or $\omega_A(T) = 0$, then obviously the desired result holds. Assume that $\omega_A(T) \neq 0$ and $\max\{\|X\|_A, \|Y\|_A\} \neq 0$. By replacing T, X and Y by $\frac{T}{\omega_A(T)}$, $\frac{X}{\max\{\|X\|_A, \|Y\|_A\}}$ and $\frac{Y}{\max\{\|X\|_A, \|Y\|_A\}}$ respectively in (2.13) we see that

$$\begin{aligned}
\omega_A(TX \pm YT) &\leq 2\sqrt{2}\max\{\|X\|_A, \|Y\|_A\}\,\omega_A(T)\sqrt{1 - \frac{\left|\left\|\Re_A\left(\frac{T}{\omega_A(T)}\right)\right\|_A^2 - \left\|\Im_A\left(\frac{T}{\omega_A(T)}\right)\right\|_A^2\right|}{2}} \\
&= 2\sqrt{2}\max\{\|X\|_A, \|Y\|_A\}\sqrt{\omega_A^2(T) - \frac{\left|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2\right|}{2}}.
\end{aligned} \tag{2.14}$$

By replacing X and Y by XS and SY respectively in the inequality (2.14), we obtain

$$\begin{aligned}
\omega_A(TXS \pm SYT) &\leq 2\sqrt{2}\max\{\|XS\|_A, \|SY\|_A\}\sqrt{\omega_A^2(T) - \frac{|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2|}{2}} \\
&\leq 2\sqrt{2}\|S\|_A \max\{\|X\|_A, \|Y\|_A\}\sqrt{\omega_A^2(T) - \frac{|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2|}{2}}.
\end{aligned}$$

This proves the required result. □

The following result is an immediate consequence of Theorem 2.9 and extends a recent result of Hirzallah and Kittaneh (see [22]). Moreover, the obtained inequality considerably refine the inequality (1.5).

Theorem 2.10 *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(TS \pm ST) \leq 2\sqrt{2}\min\Big\{f_A(T,S), f_A(S,T)\Big\}.$$

where

$$f_A(X,Y) = \|Y\|_A\sqrt{\omega_A^2(X) - \frac{\Big|\,\|\Re_A\,(X)\|_A^2 - \|\Im_A\,(X)\|_A^2\Big|}{2}}.$$

Proof By letting $X = Y = I$ in Theorem 2.9 we get

$$\omega_A(TS \pm ST) \leq 2\sqrt{2}\|S\|_A\sqrt{\omega_A^2(T) - \frac{|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2|}{2}}\,. \tag{2.15}$$

Now, by replacing T and S by S and T respectively in (2.15) we get the desired result. □

As an application of Theorem 2.10, we derive the following two corollaries.

Corollary 2.11 *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_A(T^2) \leq \sqrt{2}\|T\|_A\sqrt{\omega_A^2(T) - \frac{\Big|\,\|\Re_A\,(T)\|_A^2 - \|\Im_A\,(T)\|_A^2\Big|}{2}}\,.$$

Proof Follow immediately by letting $T = S$ in Theorem 2.10. □

Corollary 2.12 *Let $T, S \in \mathbb{B}_A(\mathcal{H})$ be such that $\omega_A(TS \pm ST) = 2\sqrt{2}\|S\|_A\omega_A(T)$ and $AS \neq 0$. Then*

$$\|\Re_A\,(T)\|_A = \|\Im_A\,(T)\|_A\,. \tag{2.16}$$

Proof It follows from Theorem 2.10 that

$$\begin{aligned}\omega_A(TS \pm ST) &\leq 2\sqrt{2}\|S\|_A\sqrt{\omega_A^2(T) - \frac{|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2|}{2}}\\ &\leq 2\sqrt{2}\|S\|_A\omega_A(T).\end{aligned}$$

Therefore, since $\omega_A(TS \pm ST) = 2\sqrt{2}\|S\|_A\omega_A(T)$, then

$$2\sqrt{2}\|S\|_A\sqrt{\omega_A^2(T) - \frac{|\,\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2|}{2}} = 2\sqrt{2}\|S\|_A\omega_A(T).$$

Since $AS \neq 0$, then $\|S\|_A \neq 0$. This immediately proves (2.16) as desired. □

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