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Linear stability of periodic standing waves of the KGZ system

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Abstract: In this work we consider the periodic standing wave solutions for a Klein–Gordon–Zakharov system. We find the conditions on the parameters, for which the periodic waves of dnoidal type are linear stable/unstable.

Key words: Periodic waves, standing waves, linear stability, Klein–Gordon–Zakharov system

1. Introduction

In this paper we consider the following Klein–Gordon–Zakharov (KGZ) system

$$\begin{cases} u_{tt} - u_{xx} + u + uv + \beta|u|^2 u = 0 \\ v_{tt} - v_{xx} = \frac{1}{2}(|u|^2)_{xx}, \end{cases} \tag{1.1}$$

where β is a real parameter, u is a complex valued function and v is a real valued function. The system (1.1) describes the interaction of a Langmuir wave and an ion sound wave in plasma [2, 7].

Theoretically and practically, existence and stability properties of standing wave are a relevant question.

Orbital stability of solitary waves for the KGZ system was studied in [8]. General abstract framework of spectral stability to second order Hamiltonian systems have been developed in [1, 5, 6] recently. In [5, 6], the stability problem of second order in time nonlinear differential equations on the hole line have been studied. The authors applied the abstract results to the Boussinesq, Klein–Gordon, and Klein–Gordon–Zakharov equations.

The question of stability in the periodic case is much more complicated. The spectrum of the linearized operator, which depends on the choice of the function space, is well-known. The spectrum consists of the isolated eigenvalues in the space of periodic functions, while the spectrum is continuous in the space of bounded functions. The orbital stability of periodic standing waves for system (1.1) was considered in [3] in case of $\beta = 0$. Using the theory developed in [5], the linear stability of periodic traveling waves for system (1.1) is studied in [4] when $\beta = 0$.

In this article, we investigate the linear stability of periodic standing waves of dnoidal type. For that purpose, we use the theory developed in [6]. This theory requires some spectral information about the operator of the linearization with the standing waves. The required spectral properties have been achieved by using Floquet theory.

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This article is organised as follows. In Section 2, we construct periodic waves of dnoidal type for system (1.1). In Section 3, we set up the linearized problem and consider the spectral properties of the operator of linearization. In Section 4, we compute the index that give the condition for stability/instability.

2. Periodic standing waves

In this section we construct periodic waves of the form $u(t,x)=e^{i\omega t}\varphi(x)$, $v(t,x)=\psi(x)$ for Klein–Gordon–Zakharov system (1.1). Plugging in (1.1), we get the following system

$$\left\{\begin{array}{l} -\varphi''+\sigma\varphi+\varphi\psi+\beta\varphi^3=0 \\ -\psi''-\frac{1}{2}(\varphi^2)''=0, \end{array}\right. \tag{2.1}$$

where $\sigma=1-\omega^2$. Integrating twice in the second equation of (2.1) and taking the constants of integrations to be zero, we get $\psi=-\frac{1}{2}\varphi^2$. Now, first equation in (2.1) is reduced to the equation

$$-\varphi''+\sigma\varphi-\left(\frac{1}{2}-\beta\right)\varphi^3=0. \tag{2.2}$$

Integrating once the above equation, we get

$$\varphi'^2=\frac{1-2\beta}{4}\left(-\varphi^4+\frac{4}{1-2\beta}\sigma\varphi^2+\frac{4a}{1-2\beta}\right), \tag{2.3}$$

where a is a constant of integration. Hence, the periodic solutions of (2.3) are given by the periodic trajectories $H(\varphi,\varphi')$ of the Hamiltonian vector field $dH=0$, where

$$H(x,y)=y^2+\frac{1-2\beta}{4}x^4-\sigma x^2.$$

The level set $H(x,y)=a$ contains two periodic trajectories if $\sigma>0, 1-2\beta>0, a<0$ and it is unique if $a>0$. Below we consider the case $a<0$ and $\sigma>0$, and $\beta<\frac{1}{2}$. Denoted by $\varphi_0>\varphi_1>0$ the positive solutions of $-\rho^4+\frac{4}{1-2\beta}\sigma\rho^2+a=0$. Then $\varphi_1\le\varphi\le\varphi_0$ and the solution φ is given by

$$\varphi(x)=\varphi_0 dn(\alpha x,\kappa), \tag{2.4}$$

where

$$\varphi_0^2+\varphi_1^2=\frac{4\sigma}{1-2\beta},\quad \alpha=\sqrt{\frac{1}{2}(\frac{1}{2}-\beta)}\varphi_0,\quad \kappa^2=\frac{\varphi_0^2-\varphi_1^2}{\varphi_0^2}=\frac{2\varphi_0^2-\frac{4\sigma}{1-2\beta}}{\varphi_0^2}. \tag{2.5}$$

Since the elliptic function dn has a fundamental period $2K(\kappa)$, then the fundamental period of the solution (2.4) is

$$2T=\frac{2K(\kappa)}{\alpha},\quad T\in I=\left(\frac{\sqrt{2}\pi}{2\sqrt{\sigma}},\infty\right). \tag{2.6}$$

Here and below $K(k)$ and $E(k)$ are, as usual, the complete elliptic integrals of the first and second kinds in a Legendre form.

3. Linearized equation

In this section we study the spectrum of linear operators arising in the linearization of system (1.1). We take the perturbation in the form

$$u(t,x) = e^{i\omega t}(\varphi(x) + p(t,x)), \quad v(t,x) = \psi(x) + q(t,x), \tag{3.1}$$

where $p(t,x)$ is a complex valued function, $q(t,x)$ is a real valued function and which are periodic with the same period as waves. Plugging in the system (1.1), using (2.1), and ignoring all quadratic and higher order terms yield the following linear equation for (p,q)

$$\begin{cases} p_{tt} + 2i\omega p_t + \sigma p - p_{xx} + \varphi q + (-\frac{1}{2}+\beta)\varphi^2 p + 2\beta\varphi^2 Rep = 0 \\ q_{tt} - q_{xx} - \frac{1}{2}(2\varphi Rep)_{xx} = 0. \end{cases} \tag{3.2}$$

To introduce the new function h with mean zero value, so that $q(t,x) = h_x(t,x)$.

$$\begin{cases} p_{tt} + 2i\omega p_t + \sigma p - p_{xx} + \varphi h_x + (-\frac{1}{2}+\beta)\varphi^2 p + 2\beta\varphi^2 Rep = 0 \\ h_{ttx} - h_{xxx} - \frac{1}{2}(2\varphi Rep)_{xx} = 0. \end{cases} \tag{3.3}$$

Integrating by x in second equation and using that h is a function with mean zero value, we get

$$\begin{cases} p_{tt} + 2i\omega p_t + \sigma p - p_{xx} + \varphi h_x + (-\frac{1}{2}+\beta)\varphi^2 p + 2\beta\varphi^2 Rep = 0 \\ h_{tt} - h_{xx} - \varphi' Rep - \varphi Rep_x = 0. \end{cases} \tag{3.4}$$

Splitting real and imaginary parts of complex valued function p as $p = F + iG$ and $h = R$, allows us to rewrite the linearized problem (3.4) as the following system

$$\begin{cases} F_{tt} - 2\omega G_t + \sigma F - F_{xx} + \varphi R_x + (-\frac{1}{2}+\beta)\varphi^2 F + 2\beta\varphi^2 F = 0 \\ G_{tt} + 2\omega F_t + \sigma G - G_{xx} + (-\frac{1}{2}+\beta)\varphi^2 G = 0 \\ R_{tt} - R_{xx} - \varphi' F - \varphi F_x = 0. \end{cases} \tag{3.5}$$

Now we can write the system (3.5) as linearized problem below

$$\vec{\boldsymbol{U}}_{\boldsymbol{tt}} + 2\omega\mathcal{J}\vec{\boldsymbol{U}}_{\boldsymbol{t}} + \mathcal{H}\vec{\boldsymbol{U}} = \boldsymbol{0}, \qquad \vec{\boldsymbol{U}} = \begin{pmatrix} F \\ R \\ G \end{pmatrix}, \tag{3.6}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} H_1 & A & 0 \\ A^* & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}$$

and

$$\begin{aligned} &H_1 = -\partial_x^2 + \sigma + (-\tfrac{1}{2}+3\beta)\varphi^2, \quad H_2 = -\partial_x^2 \\ &H_3 = -\partial_x^2 + \sigma + (-\tfrac{1}{2}+\beta)\varphi^2 \\ &A = \varphi\partial_x, \quad A^* = -\varphi\partial_x - \varphi'. \end{aligned}$$

Note that $\mathcal{H}$ is selfadjoint $\mathcal{H}^* = \mathcal{H}$ and $\mathcal{J}$ is antisymmetric $\mathcal{J}^* = -\mathcal{J}$.

Denote

$$H_0 = \begin{pmatrix} H_1 & A \\ A^* & H_2 \end{pmatrix}.$$

Now, we consider the spectral properties of the operator $\mathcal{H}$ on the $L^2[-T,T]$ with periodic boundary conditions.

Proposition 3.1 *For* $2\beta \neq 1+\frac{M(\kappa)}{K(\kappa)}$ *where* $M(\kappa)=\frac{1}{\kappa^2}\frac{E^2(\kappa)-(1-\kappa^2)K^2(\kappa)}{2(1-\kappa^2)K(\kappa)-(2-\kappa^2)E(\kappa)}<0$, *the selfadjoint operator* H_0 *has an eigenvalue at zero, which is simple. In addition, the unique (up to a multiplicative constant) eigenfunction is given by*

$$\vec{\psi}_1=\begin{pmatrix} \varphi' \\ -\frac{1}{2}\varphi^2+\frac{1}{4T}\int_{-T}^{T}\varphi^2dx \end{pmatrix}.$$

Proof Let $\begin{pmatrix} f \\ g \end{pmatrix}$ be an eigenvector corresponding to a zero eigenvalue, that is $\mathcal{H}\begin{pmatrix} f \\ g \end{pmatrix}=0$. In other words,

$$\begin{aligned} -f''+\sigma f+(-\frac{1}{2}+3\beta)\varphi^2 f+\varphi g'=0 \\ -g''-(\varphi f)'=0. \end{aligned} \tag{3.7}$$

Integrating the second equation in x implies that for some constant c_0, we have

$$g'=-\varphi f+c_0 \tag{3.8}$$

whence the equation for f becomes

$$-f''+\sigma f+(-\frac{3}{2}+3\beta)\varphi^2 f+c_0\varphi=0. \tag{3.9}$$

We show that $c_0=0$ and then $f=d\varphi'$ for some constant d. To that end, recall the defining equation for φ, differentiate it with respect to x, we get

$$-\varphi'''+\sigma\varphi'+(-\frac{3}{2}+3\beta)\varphi^2\varphi'=0. \tag{3.10}$$

We introduce the second order differential operator

$$L=-\partial_x^2+\sigma+(-\frac{3}{2}+3\beta)\varphi^2. \tag{3.11}$$

Using that $\kappa^2sn^2(y)+dn^2(y)=1$ and formulas (2.4) and (2.5), we obtain

$$\begin{aligned} L &= -\partial_x^2+\sigma-3(\beta-\tfrac{1}{2})\varphi_0^2dn^2(\alpha x,\kappa) \\ &= \alpha^2\left[-\partial_y^2+6\kappa^2sn^2(y,\kappa)-(4+\kappa^2)\right], \end{aligned}$$

where $y=\alpha x$.

It is well-known that the first five eigenvalues of $\Lambda_1=-\partial_y^2+6k^2sn^2(y,k)$, with periodic boundary conditions on $[0,4K(k)]$ are simple. These eigenvalues and corresponding eigenfunctions are:

$$\begin{array}{ll} \nu_0=2+2k^2-2\sqrt{1-k^2+k^4}, & \varphi_0(y)=1-(1+k^2-\sqrt{1-k^2+k^4})sn^2(y,k), \\ \nu_1=1+k^2, & \varphi_1(y)=cn(y,k)dn(y,k)=sn'(y,k), \\ \nu_2=1+4k^2, & \varphi_2(y)=sn(y,k)dn(y,k)=-cn'(y,k), \\ \nu_3=4+k^2, & \varphi_3(y)=sn(y,k)cn(y,k)=-k^{-2}dn'(y,k), \\ \nu_4=2+2k^2+2\sqrt{1-k^2+k^4}, & \varphi_4(y)=1-(1+k^2+\sqrt{1-k^2+k^4})sn^2(y,k). \end{array}$$

It follows that the first three eigenvalues of the operator L, equipped with periodic boundary condition on $[-T,T]$ are simple and zero is the second eigenvalue with corresponding eigenfunction φ'.

Now, in that $\varphi \perp \ker L$ form (3.10), we get

$$f = d\varphi' - c_0 L^{-1}\varphi \tag{3.12}$$

and form (3.8), we have

$$g' = -d\varphi\varphi' + c_0(\varphi L^{-1}\varphi + 1). \tag{3.13}$$

Integrating the above equation, we get

$$c_0(\langle L^{-1}\varphi, \varphi\rangle + 2T) = 0. \tag{3.14}$$

It remains to estimate $\langle L^{-1}\varphi, \varphi\rangle$. We do it by constructing of Green function for the operator L.

We have $L\varphi' = 0$. The function

$$\psi(x) = \varphi'(x)\int^x \frac{1}{\varphi'^2(s)}ds, \quad \begin{vmatrix} \varphi' & \psi \\ \varphi'' & \psi' \end{vmatrix} = 1$$

is also solution of $L\psi = 0$. Formally, since φ' has zeros using the identities

$$\frac{1}{cn^2(y,\kappa)} = \frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{sn(x,\kappa)}{cn(y,\kappa)}, \quad \frac{1}{sn^2(y,\kappa)} = -\frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integrating by parts we get

$$\psi(x) = \frac{1}{\alpha^2\kappa^2\varphi_0}\left[\frac{1-2sn^2(\alpha x,\kappa)}{dn(\alpha x,\kappa)} - \alpha\kappa^2 sn(\alpha x,\kappa)cn(\alpha x,\kappa)\int_0^x \frac{1-2sn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)}ds\right].$$

Thus, we may construct Green function

$$L^{-1}f = \varphi'\int_0^x \psi(s)f(s)ds - \psi(s)\int_0^x \varphi'(s)f(s)s + C_f\psi(x),$$

where C_f is chosen such that $L^{-1}f$ is periodic with same period as $\varphi(x)$.

After integrating by parts, we get

$$\langle L^{-1}\varphi, \varphi\rangle = -\langle \varphi^3, \psi\rangle + \frac{\varphi^2(T) + \varphi(0)^2}{2}\langle\varphi, \psi\rangle + C_\varphi\langle\varphi,\psi\rangle. \tag{3.15}$$

We have

$$\begin{aligned} &\langle\varphi,\psi\rangle = \tfrac{1}{\alpha^3\kappa^2}[E(\kappa) - K(\kappa)] \\ &\langle\varphi^3,\psi\rangle = \tfrac{\varphi_0^2}{2\alpha^3\kappa^2}[(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)] \\ &C_\varphi = -\tfrac{\varphi''(T)}{2\psi'(T)}\langle\varphi,\psi\rangle + \tfrac{\varphi^2(T)-\varphi^2(0)}{2}. \end{aligned} \tag{3.16}$$

Finally, we get

$$\langle L^{-1}\varphi, \varphi\rangle = \frac{\varphi_0^2}{2\alpha^3\kappa^2}\frac{E^2(\kappa) - (1-\kappa^2)K^2(\kappa)}{2(1-\kappa^2)K(\kappa) - (2-\kappa^2)E(\kappa)} < 0.$$

Using (2.5), we get

$$\langle L^{-1}\varphi,\varphi\rangle + 2T = \frac{1}{\alpha}\left[\frac{M(\kappa)}{1-2\beta} + K(\kappa)\right].$$

If $2\beta \neq 1 + \frac{M(\kappa)}{K(\kappa)}$, then the right side of the above equality is not zero, whence $c_0 = 0$ and $f = d\varphi'$.

Furthermore, $g' = -\varphi f$ and after integration, we get

$$g = -d\varphi^2 + constant.$$

In the above equation, constant uniquely determined by the fact that g has mean zero value, and whence

$$g = -\frac{1}{2}\varphi^2 + \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx.$$

□

Proposition 3.2 *The operator $\mathcal{H}$ has the following spectral properties*

1. *Has negative simple eigenvalue*
2. *The kernel is two dimensional and spanned by* $\vec{\psi}_1 = \begin{pmatrix} \varphi' \\ -\frac{1}{2}\varphi^2 + \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \\ 0 \end{pmatrix}$ *and* $\vec{\psi}_2 = \frac{1}{||\varphi||}\begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix}$

Proof Not that $\sigma(\mathcal{H}) = \sigma(H_0) \cup \sigma(H_3)$. From Proposition (3.1), we have that H_0 has a one negative eigenvalue, which is simple and kernel is one dimensional.

For the operator H_3 we have

$$H_1 = \alpha^2[-\partial_y^2 + 2k^2 sn^2(y,k) - k^2].$$

The spectrum of $\Lambda_2 = -\partial_y^2 + 2k^2 sn^2(y,k)$ is formed by bands $[k^2,1] \cup [1+k^2,+\infty)$. The first three eigenvalues and the corresponding eigenfunctions with periodic boundary conditions on $[0,4K(k)]$ are simple and

$$\begin{aligned} \epsilon_0 &= k^2, & \theta_0(y) &= dn(y,k), \\ \epsilon_1 &= 1, & \theta_1(y) &= cn(y,k), \\ \epsilon_2 &= 1+k^2, & \theta_2(y) &= sn(y,k). \end{aligned}$$

It follows that zero is an eigenvalue of H_3 and it is the first eigenvalue with corresponding eigenfunction $\varphi(x)$.

Hence, $n(\mathcal{H}) = 1$ and $dimKer\mathcal{H} = 2$, and $\vec{\psi}_1, \vec{\psi}_2 \in Ker\mathcal{H}$. □

4. Stability

Note that we restrict our consideration to the Hilbert space $L^2[-T,T] \times L_0^2[-T,T] \times L^2[-T,T]$, where $L_0^2[-T,T] = \{f \in L^2[-T,T] : \int_{-T}^{T} f dx = 0\}$.

Definition 4.1 *We say that the standing wave solution* $\vec{\varphi} = (e^{i\omega t}\varphi, -\frac{1}{2}\varphi^2)$ *is linear unstable, if there exists a* $2T$ *periodic function* $\vec{\psi} \in D(\mathcal{H})$ *and* $\lambda : \Re\lambda > 0$, *so that*

$$\lambda^2\vec{\psi} + 2z\lambda\mathcal{J}\vec{\psi} + \mathcal{H}\vec{\psi} = 0. \tag{4.1}$$

Otherwise, we say that $\vec{\varphi}$ *is stable.*

Next, we give precise statements of the results in [6]. Let $L^2 = X^+ \oplus X^-$ so that $\mathcal{H}$ acts invariantly on both $X^{\pm}$ and $\mathcal{J} : X^{\pm} \to X^{\mp}$. We have a number of eigenvectors in the kernel of $\mathcal{H}$ and $\vec{\psi}_1 \in X^+$ and $\vec{\psi}_2 \in X^-$. Moreover, we have the following assumptions:

$$\overline{\mathcal{H}u} = \mathcal{H}\overline{u}, \mathcal{H}^* = \mathcal{H}, \tag{4.2}$$

$$\overline{\mathcal{J}u} = \mathcal{J}\overline{u}, \mathcal{J}^* = -\mathcal{J}, \langle\vec{\psi}_1, \mathcal{J}\vec{\psi}_2\rangle = 0, \mathcal{J}(\mathcal{H}+1)^{-1} \in B(L^2). \tag{4.3}$$

In addition to (4.2) and (4.3), we assume the following for the spectrum of $\mathcal{H}$

$$\begin{cases} \mathcal{H}\varphi = -\delta^2\varphi, \mathcal{H}\mid_{\{\varphi\}^\perp} \geq 0 \\ Ker[\mathcal{H}] = span[\vec{\psi}_1, \vec{\psi}_2] \\ \vec{\psi}_1 \in X^+, \vec{\psi}_2 \in X^- \end{cases} \tag{4.4}$$

The following theorem is proved in [6].

Theorem 4.2 *Let* $\mathcal{H}$ *be a selfadjoint operator on a Hilbert space* H. *Assume that it satisfies the assumptions (4.2), (4.3), and (4.4).*

Then, if $\langle\mathcal{H}^{-1}[\mathcal{J}\vec{\psi}_2], \mathcal{J}\vec{\psi}_2\rangle \geq 0$, *one has a solution of to (4.1), that is instability in sense of Definition 4.1. Otherwise, supposing that* $\langle\mathcal{H}^{-1}[\mathcal{J}\vec{\psi}_2], [\mathcal{J}\vec{\psi}_2]\rangle < 0$.

- *the problem (4.1) has solution, if* w *satisfies the inequality*

$$|z| < \frac{1}{2\sqrt{-\langle\mathcal{H}^{-1}[\mathcal{J}\vec{\psi}_2], [\mathcal{J}\vec{\psi}_2]\rangle}} =: z^*(\mathcal{H}) \tag{4.5}$$

- *the problem (4.1) does not has solutions (i.e. stability), if* z *satisfies the reverse inequality*

$$|z| > z^*(\mathcal{H}) \tag{4.6}$$

Theorem 4.3 *If* $2\beta > 1 + \frac{M(\kappa)}{K(\kappa)}$, *then periodic waves* $\vec{\varphi}$ *are unstable. If* $2\beta < 1 + \frac{M(\kappa)}{K(\kappa)}$, *then periodic waves are unstable for* $|w| < \sqrt{\frac{1}{1+4N(\kappa)}}$ *and stable if* $|w| > \sqrt{\frac{1}{1+4N(\kappa)}}$, *where*

$$N(\kappa) = -\frac{2(2-\kappa^2)K(\kappa)}{E(\kappa)} \frac{M(\kappa)}{\left[\frac{M(\kappa)}{1-2\beta} + K(\kappa)\right]}.$$

Proof We have $\mathcal{J}\vec{\psi}_2 = \frac{1}{||\varphi||}\begin{pmatrix} -\varphi \\ 0 \\ 0 \end{pmatrix}$. Thus, $\langle \vec{\psi}_1, J\vec{\psi}_2\rangle = 0$. Obviously, assumptions (4.2) and (4.3) are satisfied. From Propositions (3.1) and (3.2), we have that operator $\mathcal{H}$ satisfies the assumption (4.4). It remains to estimate the index $\langle \mathcal{H}^{-1}[\mathcal{J}\vec{\psi}_2], [\mathcal{J}\vec{\psi}_2]\rangle$.

Let $\mathcal{H}\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -\varphi \\ 0 \\ 0 \end{pmatrix}$. Thus

$$\left| \begin{array}{l} -f'' + \sigma f + \left(-\frac{1}{2} + 3\beta\right)\varphi^2 f + \varphi g' = -\varphi \\ -\varphi f - \varphi' f - g'' = 0 \\ H_1 h = 0. \end{array} \right. \tag{4.7}$$

We need to find only f. From the second equation of (4.7), we have $g'' = -(\varphi f)'$. Integrating, we get

$$g' = -\varphi f + c_1, \tag{4.8}$$

where c_1 is a constant of integration. Plugging in the first equation of (4.7), we get

$$-f'' + \sigma f - \left(\frac{3}{2} - 3\beta\right)\varphi^2 f = -c_1\varphi - \varphi \tag{4.9}$$

or

$$Lf = -(1 + c_1)\varphi. \tag{4.10}$$

Hence

$$f = d\varphi' - (1 + c_1)L^{-1}\varphi. \tag{4.11}$$

Now, replacing in (4.8) and integrating, we get

$$c_1 = -\frac{\langle L^{-1}\varphi, \varphi\rangle}{\langle L^{-1}\varphi, \varphi\rangle + 2T}. \tag{4.12}$$

We have

$$||\varphi||^2 = \int_{-T}^{T} \varphi_0^2 dn^2(\alpha x, \kappa)dx = \frac{2\varphi_0^2}{\alpha}\int_0^{K(\kappa)} dn^2(y, \kappa)dy = \frac{2\varphi_0^2}{\alpha}E(\kappa).$$

Finally, from (4.11) and (4.12), we have

$$\begin{aligned} \langle \mathcal{H}^{-1}[\mathcal{J}\vec{\psi}_2], [\mathcal{J}\vec{\psi}_2]\rangle &= \tfrac{(1+c_1)\langle L^{-1}\varphi,\varphi\rangle}{||\varphi||^2} = \tfrac{2T}{\langle L^{-1}\varphi,\varphi\rangle + 2T}\tfrac{\langle L^{-1}\varphi,\varphi\rangle}{||\varphi||^2} \\ &= \tfrac{1}{\sigma}\tfrac{2(2-\kappa^2)K(\kappa)}{E(\kappa)}\tfrac{M(\kappa)}{\left[\frac{M(\kappa)}{1-2\beta} + K(\kappa)\right]}. \end{aligned} \tag{4.13}$$

If $2\beta > 1 + \frac{M(\kappa)}{K(\kappa)}$, then $\langle \mathcal{H}^{-1}[\mathcal{J}\vec{\psi}_2], [\mathcal{J}\vec{\psi}_2]\rangle > 0$ and hence we have instability.

If $2\beta < 1 + \frac{M(\kappa)}{K(\kappa)}$, then $\langle \mathcal{H}^{-1}[\mathcal{J}\vec{\psi}_2], [\mathcal{J}\vec{\psi}_2]\rangle < 0$ and hence we have instability if $|w| < \sqrt{\frac{1}{1+4N(\kappa)}}$ and stability if $|w| > \sqrt{\frac{1}{1+4N(\kappa)}}$. □

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