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Explicit formulas and recurrence relations for generalized Catalan numbers

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Abstract: In this paper, we present an explicit formula and recurrent relation for generalized Catalan numbers, from which we can give corresponding formulas for Schröder numbers, large and small generalized Catalan numbers for the special cases of our results.

Key words: Catalan number, Schröder number, explicit formula, recursive formula

1. Introduction

For any nonnegative integer n, the Catalan numbers C_n are defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$ and can be generated by

$$\frac{2}{1+\sqrt{1-4t}} = \sum_{n=0}^{\infty} C_n t^n.$$

These numbers are located in a very important position in combinatorial mathematics and a set of exercises of Chapter 6 which describe 66 different interpretations of these numbers can be found in Stanley's book [33]. A number of generalizations of Catalan numbers have naturally appeared from the combinatorial respects and many of properties have been discussed.

In recent years, Qi and his colleagues have investigated plenty of properties, identities, and relations for Catalan numbers and their certain extensions. The authors of [19] discussed some integral representations of Catalan numbers along with their some applications. Also, in [20], Qi et al. analyzed miscellaneous features such as a novel expression, generating function, integral representation, asymptotic expansions, logarithmic convexity, and inequalities for Catalan numbers and its some generalizations, called as Catalan function and Catalan-Qi function. Moreover, congruence properties for Catalan numbers have been considered in [7, 8, 12–14].

For detailed knowledge, one can consult to the monograph [9] and newly other published articles [17, 18, 22, 23].

For any $(c, r) \in \mathbb{Z}^2$, $c, r \neq 0$, a very large class of Catalan numbers with two parameters $d_n^{(c,r)}$, which generalize a kind of generalized Catalan numbers, classical Catalan numbers and Schröder numbers, is introduced

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by

$$d_{c,r}(t) = \frac{1-(c-r)t-\sqrt{1-2(c+r)t+(c-r)^2t^2}}{2rt} = \sum_{n=0}^{\infty} d_n(c,r)\,t^n \tag{1.1}$$

and some significant properties and some combinatorial interpretations are provided in [6]. For brevity, we say these numbers as $(c,r)-$Catalan numbers. Some special cases of these numbers are as follows:

· For $(c,r)=(c,1)$ and $(c,r)=(1,r)$, we have large and small generalized Catalan numbers, respectively, defined in [6].

· For $(c,r)=(1,1)$, we have classical Catalan numbers C_n.

· For $(c,r)=(2,1)$, we get large Schröder numbers S_n, defined by [2, Theorem 8.5.7]

$$\frac{1-t-\sqrt{1-6t+t^2}}{2t} = \sum_{n=0}^{\infty} S_n t^n.$$

· For $(c,r)=(1,2)$, we deduce small Schröder numbers s_n, defined by [2, Theorem 8.5.6]

$$\frac{1+t-\sqrt{1-6t+t^2}}{4t} = \sum_{n=0}^{\infty} s_n t^n.$$

Various number theoretic and analytic aspects of Schröder numbers can be found in [3, 4, 11, 15, 24, 27, 34, 35].

In this paper, the author derives a novel explicit formula for $(c,r)-$Catalan numbers helped by the excellent identity for the Bell polynomials of the second kind, so-called Faà di Bruno formula (See Lemma 2.1, below). Also, we deduce a recursive formula for $(c,r)-$Catalan numbers via analytic methods. Notice that setting some particular cases of our formulas established here yields the counterpart formulas for large and small generalized Catalan numbers, ordinary Catalan numbers, and large (and small) Schröder numbers.

Concretely, we achieve the following conclusions.

Theorem 1.1 *The $(c,r)-$Catalan numbers $d_n(c,r)$ can be computed explicitly as*

$$d_n(c,r) = \frac{1}{2r}\left(\frac{(c-r)^2}{2(c+r)}\right)^{n+1} \sum_{l=0}^{n+1} \frac{2^l\,(2l-3)!!\,(-1)^{n+1-l}}{l!}\left(\frac{c+r}{c-r}\right)^{2l}\binom{l}{n+1-l},$$

where $[-(2n+1)]!!$ denotes the double factorial of negative odd integers $-(2n+1)$, given by

$$[-(2n+1)]!! = \frac{(-1)^n}{(2n-1)!!} = (-1)^n\frac{2^n n!}{(2n)!}, \quad \textit{for } n=0,1,\ldots.$$

Theorem 1.2 *The $(c,r)-$Catalan numbers satisfy the following recursive formula*

$$d_0(c,r)=1$$

and

$$d_{n+1}(c,r) = (c-r)d_n(c,r) + r\sum_{l=0}^{n} d_l(c,r)\,d_{n-l}(c,r), \quad \textit{for } n\geq 0.$$

2. Auxiliary theorems

We recall several lemmas below so as to prove our main results.

Lemma 2.1 *([5, p. 134 and 139]) For $n \geq k \geq 0$, the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, ..., x_{n-k+1})$ is defined by*

$$B_{n,k}(x_1, x_2, ..., x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n,\ l_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n} i l_i = n,\ \sum_{i=1}^{n} l_i = k}}^{\infty} \frac{n!}{\prod_{i=1}^{l-k+1} l_i!} \prod_{i=1}^{l-k+1} \left(\frac{x_i}{i!}\right)^{l_i}.$$

The Faà di Bruno formula can be described as

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) B_{n,k}\left(h'(t), h''(t), ..., h^{(n-k+1)}(t)\right).$$

Lemma 2.2 *([5, p. 135]) Let a and b be any complex numbers and let $n \geq k \geq 0$, then, we have*

$$B_{n,k}\left(abx_1, ab^2x_2, ..., ab^{n-k+1}x_{n-k+1}\right) = a^k b^n B_{n,k}(x_1, x_2, ..., x_{n-k+1}).$$

Lemma 2.3 *([29, Sect. 3])For $n \geq k \geq 0$, we have*

$$B_{n,k}(x, 1, 0, ..., 0) = \frac{(n-k)!}{2^{n-k}} \binom{n}{k} \binom{k}{n-k} x^{2k-n}.$$

Lemma 2.4 *([1, p. 40, Entry 5]) Let $u(x)$ and $v(x)$ be two differentiable functions such that $v(x) \neq 0$, then, we have for $k \geq 0$*

$$\frac{d^k}{dx^k}\left[\frac{u(x)}{v(x)}\right]$$
$$= \frac{(-1)^k}{v^{k+1}} \begin{vmatrix} u & v & 0 & \dots & 0 & 0 \\ u' & v' & v & \dots & 0 & 0 \\ u'' & v'' & \binom{2}{1}v' & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{(k-2)} & v^{(k-2)} & \binom{k-2}{1}v^{(k-3)} & \dots & v & 0 \\ u^{(k-1)} & v^{(k-1)} & \binom{k-1}{1}v^{(k-2)} & \dots & \binom{k-1}{k-2}v' & v \\ u^{(k)} & v^{(k)} & \binom{k}{1}v^{(k-1)} & \dots & \binom{k}{k-2}v'' & \binom{k}{k-1}v' \end{vmatrix}.$$

3. Proofs

In this section, we give the proofs of our theorems.

3.1. Proof of Theorem 1.1

Using Lemmas 2.1, 2.2 and 2.3, one has

$$\left(\sqrt{1-2(c+r)t+(c-r)^2t^2}\right)^{(k+1)}$$

$$= \sum_{l=0}^{k+1} \left(\frac{1}{2}\right)_l \left(1 - 2(c+r)t + (c-r)^2 t^2\right)^{1/2-l}$$

$$\times B_{k+1,l}\left(-2(c+r) + 2(c-r)^2 t, 2(c-r)^2, 0, 0, ..., 0\right)$$

$$\to \sum_{l=0}^{k+1} \left(\frac{1}{2}\right)_l B_{k+1,l}\left(-2(c+r), 2(c-r)^2, 0, 0, ..., 0\right), \quad \text{as } t \to 0$$

$$= \sum_{l=0}^{k+1} \left(\frac{1}{2}\right)_l \left(2(c-r)^2\right)^l B_{k+1,l}\left(\frac{-(c+r)}{(c-r)^2}, 1, 0, 0, ..., 0\right)$$

$$= \sum_{l=0}^{k+1} \left(\frac{1}{2}\right)_l 2^l (c-r)^{2l} \frac{(k+1-l)!}{2^{k+1-l}} \binom{k+1}{l} \binom{l}{k+1-l} \left[\frac{-(c+r)}{(c-r)^2}\right]^{2l-k-1}, \tag{3.1}$$

where $(x)_n$ denotes the falling factorial, defined for $x \in \mathbb{R}$ by

$$(x)_n = \prod_{k=0}^{n-1} (x-k) = \begin{cases} x(x-1)...(x-n+1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$$

On the other hand, if we take $u(t) = 1 - (c-r)t - \sqrt{1 - 2(c+r)t + (c-r)^2 t^2}$ and $v(t) = 2rt$ in Lemma 2.4, then we have

$$\frac{d^n}{dt^n}\left(d_{c,r}(t)\right)$$

$$= \frac{1}{2r}\frac{(-1)^n}{t^{n+1}} \begin{vmatrix} u & t & 0 & \dots & 0 & 0 & 0 \\ u' & 1 & \binom{1}{1}t & \dots & 0 & 0 & 0 \\ u'' & 0 & \binom{2}{1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u^{(n-2)} & 0 & 0 & ... & \binom{n-2}{n-3} & \binom{n-2}{n-2}t & 0 \\ u^{(n-1)} & 0 & 0 & \dots & 0 & \binom{n-1}{n-2} & \binom{n-1}{n-1}t \\ u^{(n)} & 0 & 0 & \dots & 0 & 0 & \binom{n}{n-1} \end{vmatrix}$$

$$= \frac{1}{2r}\frac{(-1)^n}{t^{n+1}} \left| (-1)^n u^{(n)}(t) \begin{vmatrix} t & 0 & \dots & 0 & 0 & 0 \\ 1 & \binom{1}{1}t & \dots & 0 & 0 & 0 \\ 0 & \binom{2}{1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & \binom{n-2}{n-3} & \binom{n-2}{n-2}t & 0 \\ 0 & 0 & \dots & 0 & \binom{n-1}{n-2} & \binom{n-1}{n-1}t \end{vmatrix} \right.$$

$$\left. + \binom{n}{n-1} \begin{vmatrix} u & t & 0 & \dots & 0 & 0 \\ u' & 1 & \binom{1}{1}t & \dots & 0 & 0 \\ u'' & 0 & \binom{2}{1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{(n-2)} & 0 & 0 & ... & \binom{n-2}{n-3} & \binom{n-2}{n-2}t \\ u^{(n-1)} & 0 & 0 & \dots & 0 & \binom{n-1}{n-2} \end{vmatrix} \right|$$

$$= \frac{1}{2r}\frac{u^{(n)}(t)}{t} + \frac{1}{2r}\frac{(-1)^n}{t^{n+1}} n \begin{vmatrix} u & t & 0 & \dots & 0 & 0 \\ u' & 1 & \binom{1}{1}t & \dots & 0 & 0 \\ u'' & 0 & \binom{2}{1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{(n-2)} & 0 & 0 & \dots & \binom{n-2}{n-3} & \binom{n-2}{n-2}t \\ u^{(n-1)} & 0 & 0 & \dots & 0 & \binom{n-1}{n-2} \end{vmatrix}$$

$$= \frac{1}{2r}\frac{u^{(n)}(t)}{t} - \frac{n}{t}\frac{1}{2r}\frac{(-1)^{n-1}}{t^{n}} \begin{vmatrix} u & t & 0 & \dots & 0 & 0 \\ u' & 1 & \binom{1}{1}t & \dots & 0 & 0 \\ u'' & 0 & \binom{2}{1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{(n-2)} & 0 & 0 & \dots & \binom{n-2}{n-3} & \binom{n-2}{n-2}t \\ u^{(n-1)} & 0 & 0 & \dots & 0 & \binom{n-1}{n-2} \end{vmatrix}$$

$$= \frac{1}{2r}\frac{u^{(n)}(t)}{t} - \frac{n}{t}\frac{d^{n-1}}{dt^{n-1}}\left(d_{c,r}(t)\right)$$

$$= \frac{1}{t}\left[\frac{u^{(n)}(t)}{2r} - n\frac{d^{n-1}}{dt^{n-1}}\left(d_{c,r}(t)\right)\right].$$

Now, applying the L'Hospital rule, one can write that

$$\begin{aligned}\lim_{t\to 0}\frac{d^n}{dt^n}\left(d_{c,r}(t)\right) &= \lim_{t\to 0}\frac{1}{t}\left[\frac{u^{(n)}(t)}{2r} - n\frac{d^{n-1}}{dt^{n-1}}\left(d_{c,r}(t)\right)\right] \\ &= \lim_{t\to 0}\left[\frac{u^{(n+1)}(t)}{2r} - n\frac{d^{n}}{dt^{n}}\left(d_{c,r}(t)\right)\right],\end{aligned}$$

from which

$$\lim_{t\to 0}\frac{d^n}{dt^n}\left(d_{c,r}(t)\right) = \frac{1}{n+1}\lim_{t\to 0}\frac{u^{(n+1)}(t)}{2r},$$

which can be reformulated as

$$d_n(c,r) = \frac{1}{n!(n+1)2r}\lim_{t\to 0}u^{(n+1)}(t),$$

from the generating function (1.1). We have already evaluated that the value $\lim_{t\to 0}u^{(n+1)}(t)$ equals to (3.1). Thus, substituting this in (3.1) and simplifying give the desired identity.

Remark 3.1 *In particular* $(c,r) = (c,1)$ *and* $(c,r) = (1,r)$, *we have*

$$d_n(c,1) = \frac{1}{2}\left(\frac{(c-1)^2}{2(c+1)}\right)^{n+1}\sum_{l=0}^{n+1}\frac{2^l(2l-3)!!(-1)^{n+1-l}}{l!}\left(\frac{c+1}{c-1}\right)^{2l}\binom{l}{n+1-l}$$

and

$$d_n(1,r) = \frac{1}{2r}\left(\frac{(1-r)^2}{2(1+r)}\right)^{n+1}\sum_{l=0}^{n+1}\frac{2^l(2l-3)!!(-1)^{n+1-l}}{l!}\left(\frac{1+r}{1-r}\right)^{2l}\binom{l}{n+1-l},$$

which are the explicit formulas for large and small generalized Catalan numbers, respectively. For $(c,r)=(2,1)$ *and* $(c,r)=(1,2)$, *we have the explicit formulas for large and small Schröder numbers as*

$$d_n(2,1)=S_n=2s_{n+1}=\frac{1}{2}\frac{1}{6^{n+1}}\sum_{l=0}^{n+1}\frac{2^l(2l-3)!!(-1)^{n+1-l}}{l!}9^l\binom{l}{n+1-l},$$

which is Theorem 1 of [21].

3.2. Proof of Theorem 1.2

By (1.1), one can write

$$\sqrt{1-2(c+r)t+(c-r)^2t^2}=1-(c-r)t-2rtd_{c,r}(t).$$

If we square on both sides of this equation, then,

$$\begin{aligned}
&1-2(c+r)t+(c-r)^2t^2\\
&=\left(1-(c-r)t-2r\sum_{n=0}^{\infty}d_n(c,r)t^{n+1}\right)^2\\
&=1-2(c-r)t+(c-r)^2t^2+4r^2\left(\sum_{n=0}^{\infty}d_n(c,r)t^{n+1}\right)^2\\
&-4r\sum_{n=0}^{\infty}d_n(c,r)t^{n+1}+4r(c-r)\sum_{n=0}^{\infty}d_n(c,r)t^{n+2}\\
&=1-2(c-r)t+(c-r)^2t^2+4r^2t^2\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}d_l(c,r)d_{n-l}(c,r)\right]t^n\\
&-4r\sum_{n=1}^{\infty}d_{n-1}(c,r)t^n+4r(c-r)\sum_{n=2}^{\infty}d_{n-2}(c,r)t^n\\
&=1-2(c-r)t+(c-r)^2t^2+4r^2\sum_{n=2}^{\infty}\left[\sum_{l=0}^{n-2}d_l(c,r)d_{n-l-2}(c,r)\right]t^n\\
&-4r\sum_{n=1}^{\infty}d_{n-1}(c,r)t^n+4r(c-r)\sum_{n=2}^{\infty}d_{n-2}(c,r)t^n\\
&=1-2(c-r)t+(c-r)^2t^2-4rtd_0(c,r)-4r\sum_{n=2}^{\infty}d_{n-1}(c,r)t^n\\
&+4r(c-r)\sum_{n=2}^{\infty}d_{n-2}(c,r)t^n+4r^2\sum_{n=2}^{\infty}\left[\sum_{l=0}^{n-2}d_l(c,r)d_{n-l-2}(c,r)\right]t^n\\
&=1-2(c-r+2rd_0(c,r))t+(c-r)^2t^2\\
&-4r\sum_{n=2}^{\infty}\left[d_{n-1}(c,r)-(c-r)d_{n-2}(c,r)-r\sum_{l=0}^{n-2}d_l(c,r)d_{n-l-2}(c,r)\right]t^n,
\end{aligned}$$

from which, we conclude that

$$d_0(c,r) = 1$$

and

$$d_{n-1}(c,r) - (c-r)\, d_{n-2}(c,r) - r\sum_{l=0}^{n-2} d_l(c,r)\, d_{n-l-2}(c,r) = 0, \qquad \text{for } n \geq 2.$$

Hence, the proof is completed.

Remark 3.2 *In particular $(c,r) = (c,1)$, we have the counterpart recursive formula for large generalized Catalan numbers as*

$$d_0(c,1) = 1$$

and

$$d_{n+1}(c,1) = (c-1)d_n(c,1) + \sum_{l=0}^{n} d_l(c,1)\, d_{n-l}(c,1), \qquad n \geq 0.$$

Similarly, we can get the recursive formula for small generalized Catalan numbers by taking $(c,r) = (1,r)$.

Furthermore, for $(c,r) = (2,1)$ and $(c,r) = (1,2)$, our formula reduces to the recursive formulas large and small Schröder numbers

$$S_{n+3} = 3S_{n+2} + \sum_{l=0}^{n} S_{l+1}S_{n-l+1}$$

and

$$s_{n+4} = 3s_{n+3} + 2\sum_{l=0}^{n} s_{l+2}S_{n-l+2},$$

given by [25, Eqs. 6 and 7], respectively.

Remark 3.3 *Recently, some significant studies such as [10, 16, 26, 28–32, 36–38] have been exhibited in order to cope with some exhaustive applications of the Bell polynomials of the second kind.*

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