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# Some applications of fractional calculus for analytic functions

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**Abstract:** For analytic functions $f(z)$ in the class $A_n$, fractional calculus (fractional integrals and fractional derivatives) $D_z^{\lambda} f(z)$ of order $\lambda$ are introduced. Applying $D_z^{\lambda} f(z)$ for $f(z) \in A_n$, we introduce the interesting subclass $A_n(\alpha_m, \beta, \rho, \lambda)$ of $A_n$. The object of this paper is to discuss some properties of $f(z)$ concerning $D_z^{\lambda} f(z)$.

**Key words:** Analytic function, fractional integral, fractional derivative, Miller and Mocanu lemma

## 1. Introduction

Let $A_n$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \qquad (n \in N = \{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$.
Among various definitions of $f(z) \in A_n$ for fractional calculus given in the literature, we would like to recall here the following definitions for fractional calculus which were used by Owa ([4, 5]) and by Owa and Srivastava [6].

**Definition 1.1** The fractional integral of order $\lambda$ for $f(z) \in A_n$ is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \qquad (\lambda > 0) \tag{1.2}$$

where the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$, and $\Gamma(\lambda)$ is the Gamma function.

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**Definition 1.2** The fractional derivative of order $\lambda$ for $f(z) \in A_n$ is defined by

$$\begin{aligned} D_z^{\lambda} f(z) &= \frac{d}{dz}\left(D_z^{\lambda-1} f(z)\right) \\ &= \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz}\left(\int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt\right), \quad (0 \leq \lambda \leq 1) \end{aligned} \tag{1.3}$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.1 above.

**Definition 1.3** Under the hypothesis of Definition 1.2, the fractional derivative of order $j+\lambda$ for $f(z) \in A_n$ is defined by

$$D_z^{j+\lambda} f(z) = \frac{d^j}{dz^j}\left(D_z^{\lambda} f(z)\right), \qquad (0 \leq \lambda \leq 1) \tag{1.4}$$

where $j = 0, 1, 2, \cdots$.
In view of the above definitions, we know that

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{k=n+1}^{\infty} \frac{k!}{\Gamma(k+1+\lambda)} a_k z^{k+\lambda} \qquad (\lambda > 0), \tag{1.5}$$

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=n+1}^{\infty} \frac{k!}{\Gamma(k+1-\lambda)} a_k z^{k-\lambda} \qquad (0 \leq \lambda \leq 1), \tag{1.6}$$

and

$$D_z^{j+\lambda} f(z) = \frac{1}{\Gamma(2-j-\lambda)} z^{1-j-\lambda} + \sum_{k=n+1}^{\infty} \frac{k!}{\Gamma(k+1-j-\lambda)} a_k z^{k-j-\lambda} \tag{1.7}$$

for $0 \leq \lambda \leq 1$ and $j = 0, 1, 2, \cdots$.
If $\lambda = 1$ in (1.6) and (1.7), then we have $f'(z)$ and $f^{(j+1)}(z)$.
Now, we consider m different boundary points $z_p\,(p = 1, 2, 3, \cdots, m)$ with $|z_p| = 1$ and

$$\alpha_m = \frac{1}{m} \sum_{p=1}^{m} \frac{\Gamma(2-\lambda) D_z^{\lambda} f(z_p)}{z_p^{1-\lambda}}, \qquad (z_j \neq z_k) \tag{1.8}$$

where

$$\alpha_m \in e^{i\beta} \frac{\Gamma(2-\lambda) D_z^{\lambda} f(U)}{z_p^{1-\lambda}}, \tag{1.9}$$

$\alpha_m \neq 1$ and $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$.
If $f(z) \in A_n$ satisfies

$$\left| \frac{e^{i\beta} G_{\lambda}(z) - \alpha_m}{e^{i\beta} - \alpha_m} - 1 \right| < \rho, \qquad (z \in U) \tag{1.10}$$

or some real $\rho > 0$, then we say that $f(z)$ is a member of the class $A_n(\alpha_m, \beta, \rho, \lambda)$, where

$$G_{\lambda}(z) = \frac{\Gamma(2-\lambda) D_z^{\lambda} f(z)}{z^{1-\lambda}} = 1 + \sum_{k=n+1}^{\infty} \frac{k!\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k z^{k-1}, \quad (0 \leq \lambda \leq 1). \tag{1.11}$$

It is clear that $G_{\lambda}(z)$ in (1.11) satisfies

$$|G_{\lambda}(z)-1|<\rho\left|e^{i\beta}-\alpha_m\right|, \qquad (z\in U). \tag{1.12}$$

If we consider a function $f(z)\in A_n$ given by

$$f(z)=z+\frac{\left(e^{i\beta}-\alpha_m\right)\Gamma(n+2-\lambda)}{(n+1)!\Gamma(2-\lambda)}z^{n+1}, \tag{1.13}$$

then $f(z)$ satisfies

$$|G_{\lambda}(z)-1|=\rho\left|e^{i\beta}-\alpha_m\right||z|^n<\rho\left|e^{i\beta}-\alpha_m\right| \qquad (z\in U). \tag{1.14}$$

Therefore, $f(z)$ given by (1.13) belongs to the class $A_n(\alpha_m,\beta,\rho,\lambda)$.
To discuss some interesting properties of $f(z)$ in the class $A_n(\alpha_m,\beta,\rho,\lambda)$, we need the following lemma due to Miller and Mocanu ([2, 3]) (also, due to Jack [1] ).

**Lemma 1.1** Let the function $w(z)$ given by

$$w(z)=a_nz^n+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\cdots \tag{1.15}$$

be analytic in $U$ with $w(0)=0$ and $n\in N$.
If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in U$, then there exists a real number $k\geq n$ such that

$$\frac{z_0w'(z_0)}{w(z_0)}=k \tag{1.16}$$

and

$$\Re\left(1+\frac{z_0w''(z_0)}{w'(z_0)}\right)\geq k. \tag{1.17}$$

## 2. Properties of functions concerning the class $A_n(\alpha_m,\beta,\rho,\lambda)$

Applying Lemma 1.1, we first derive

**Theorem 2.1** If $f(z)\in A_n$ satisfies

$$\left|\frac{zD_z^{1+\lambda}f(z)}{D_z^{\lambda}f(z)}-(1-\lambda)\right|<\frac{\left|e^{i\beta}-\alpha_m\right|n\rho}{1+\left|e^{i\beta}-\alpha_m\right|\rho} \qquad (z\in U) \tag{2.1}$$

for some $\alpha_m$ defined by (1.9) with $\alpha_m\neq 1$ such that $z_p\in\partial U$ $(p=1,2,3,\cdots,m)$, and for some real $\rho>1$, then

$$\left|\frac{\Gamma(2-\lambda)D_z^{\lambda}f(z)}{z^{1-\lambda}}-1\right|<\left|e^{i\beta}-\alpha_m\right|\rho, \qquad (z\in U) \tag{2.2}$$

that is, $f(z)\in A_n(\alpha_m,\beta,\rho,\lambda)$.

**Proof** Let us define a function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{e^{i\beta}G_\lambda(z)-\alpha_m}{e^{i\beta}-\alpha_m}-1 \qquad (2.3)\\ &= \frac{e^{i\beta}}{e^{i\beta}-\alpha_m}\left\{\frac{\Gamma(2-\lambda)(n+1)!}{\Gamma(n+2-\lambda)}a_{n+1}z^n+\frac{\Gamma(2-\lambda)(n+2)!}{\Gamma(n+3-\lambda)}a_{n+2}z^{n+1}+\cdots\right\}, \end{aligned}$$

where $G_\lambda(z)$ is given by $(1.11)$.
It follows that

$$G_\lambda(z)=\frac{\Gamma(2-\lambda)D_z^\lambda f(z)}{z^{1-\lambda}}=1+\left(1-\alpha_m e^{-i\beta}\right)w(z). \qquad (2.4)$$

This implies that

$$\frac{zD_z^{1+\lambda}f(z)}{D_z^\lambda f(z)}-(1-\lambda)=\frac{\left(1-\alpha_m e^{-i\beta}\right)zw'(z)}{1+\left(1-\alpha_m e^{-i\beta}\right)w(z)} \qquad (2.5)$$

and

$$\begin{aligned} \left|\frac{zD_z^{1+\lambda}f(z)}{D_z^\lambda f(z)}-(1-\lambda)\right| &= \left|\frac{\left(1-\alpha_m e^{-i\beta}\right)zw'(z)}{1+\left(1-\alpha_m e^{-i\beta}\right)w(z)}\right| \qquad (2.6)\\ &< \frac{\left|e^{i\beta}-\alpha_m\right|n\rho}{1+\left|e^{i\beta}-\alpha_m\right|\rho} \qquad (z\in U) \end{aligned}$$

with $(2.1)$. We suppose that there exists a point $z_0\in U$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=\rho>1. \qquad (2.7)$$

Then we can write that $w(z_0)=\rho e^{i\theta}$ and $z_0w'(z_0)=kw(z_0)$ with $k\geq n$ by Lemma 1.1. For such a point $z_0\in U$, we have

$$\begin{aligned} \left|\frac{z_0D_z^{1+\lambda}f(z_0)}{D_z^\lambda f(z_0)}-(1-\lambda)\right| &= \left|\frac{\left(1-\alpha_m e^{-i\beta}\right)z_0w'(z_0)}{1+\left(1-\alpha_m e^{-i\beta}\right)w(z_0)}\right| \qquad (2.8)\\ &\geq \frac{\left|1-\alpha_m e^{-i\beta}\right|n\rho}{1+\left|1-\alpha_m e^{-i\beta}\right|\rho}=\frac{\left|e^{i\beta}-\alpha_m\right|n\rho}{1+\left|e^{i\beta}-\alpha_m\right|\rho}. \end{aligned}$$

This contradicts our condition $(2.1)$ for $f(z)\in A_n$.
Therefore, we say that there is no such point $z_0\in U$ such that $|w(z_0)|=\rho>1$. This means that $|w(z)|<\rho$ for all $z\in U$. Therefore, we obtain that

$$\begin{aligned} |w(z)| &= \left|\frac{e^{i\beta}G_\lambda(z)-\alpha_m}{e^{i\beta}-\alpha_m}-1\right| \qquad (2.9)\\ &= \left|\frac{\Gamma(2-\lambda)D_z^\lambda f(z)}{z^{1-\lambda}}-1\right|<\left|e^{i\beta}-\alpha_m\right|\rho, \qquad (z\in U). \end{aligned}$$

This completes the proof of the theorem.

**Example 2.1** We consider a function $f(z) \in A_n$ given by

$$f(z) = z + a_{n+1} z^{n+1} \qquad (z \in U) \tag{2.10}$$

with

$$0 < |a_{n+1}| < \frac{\Gamma(n+2-\lambda)}{2(n+1)!}.$$

Then we see that

$$\frac{zD_z^{1+\lambda} f(z)}{D_z^{\lambda} f(z)} - (1-\lambda) = \frac{\frac{(n+1)! n \Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} a_{n+1} z^n}{1 + \frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} a_{n+1} z^n}. \tag{2.11}$$

This gives us that

$$\left| \frac{zD_z^{1+\lambda} f(z)}{D_z^{\lambda} f(z)} - (1-\lambda) \right| < \frac{\frac{(n+1)! n \Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} |a_{n+1}|}{1 - \frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} |a_{n+1}|}. \tag{2.12}$$

Now, we take five boundary points such that

$$z_1 = e^{-i\frac{\arg(a_{n+1})}{n}}, \tag{2.13}$$

$$z_2 = e^{i\frac{\pi - 6\arg(a_{n+1})}{6n}}, \tag{2.14}$$

$$z_3 = e^{i\frac{\pi - 4\arg(a_{n+1})}{4n}}, \tag{2.15}$$

$$z_4 = e^{i\frac{\pi - 3\arg(a_{n+1})}{3n}}, \tag{2.16}$$

and

$$z_5 = e^{i\frac{\pi - 2\arg(a_{n+1})}{2n}}. \tag{2.17}$$

For the above boundary points, we have that

$$\begin{aligned} \frac{\Gamma(2-\lambda) D_z^{\lambda} f(z_1)}{z_1^{1-\lambda}} &= 1 + \frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} a_{n+1} e^{-i\arg(a_{n+1})} \\ &= 1 + \frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} |a_{n+1}|, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \frac{\Gamma(2-\lambda) D_z^{\lambda} f(z_2)}{z_2^{1-\lambda}} &= 1 + \frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} a_{n+1} e^{i\frac{\pi - 6\arg(a_{n+1})}{6}} \\ &= 1 + \frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} \left( \frac{\sqrt{3}+i}{2} \right) |a_{n+1}|, \end{aligned} \tag{2.19}$$

$$\begin{aligned}\frac{\Gamma(2-\lambda)D_z^\lambda f(z_3)}{z_3^{1-\lambda}} &= 1+\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}a_{n+1}e^{i\frac{\pi-4\arg(a_{n+1})}{4}} \\ &= 1+\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}\left(\frac{1+i}{\sqrt{2}}\right)|a_{n+1}|,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\frac{\Gamma(2-\lambda)D_z^\lambda f(z_4)}{z_4^{1-\lambda}} &= 1+\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}a_{n+1}e^{i\frac{\pi-3\arg(a_{n+1})}{3}} \\ &= 1+\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}\left(\frac{1+\sqrt{3}i}{2}\right)|a_{n+1}|,\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}\frac{\Gamma(2-\lambda)D_z^\lambda f(z_5)}{z_5^{1-\lambda}} &= 1+\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}a_{n+1}e^{i\frac{\pi-2\arg(a_{n+1})}{2}} \\ &= 1+\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}i\,|a_{n+1}|.\end{aligned} \tag{2.22}$$

Therefore, $\alpha_5$ is given by

$$\begin{aligned}\alpha_5 &= \frac{1}{5}\sum_{p=1}^{5}\frac{\Gamma(2-\lambda)D_z^\lambda f(z_p)}{z_p^{1-\lambda}} \\ &= 1+\frac{(n+1)!\Gamma(2-\lambda)\left(3+\sqrt{2}+\sqrt{3}\right)(1+i)}{10\Gamma(n+2-\lambda)}|a_{n+1}|.\end{aligned} \tag{2.23}$$

It follows from $\alpha_5$ that

$$\left|1-\alpha_5 e^{-i\beta}\right| = \frac{\sqrt{2}\left(3+\sqrt{2}+\sqrt{3}\right)(n+1)!\Gamma(2-\lambda)}{10\Gamma(n+2-\lambda)}|a_{n+1}| \tag{2.24}$$

for $\beta=0$. For the above $\alpha_5$ and $\beta=0$, we consider a real $\rho>1$ such that

$$\frac{\frac{(n+1)!n\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}|a_{n+1}|}{1-\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}|a_{n+1}|} \le \frac{\left|e^{i\beta}-\alpha_5\right|n\rho}{1+\left|e^{i\beta}-\alpha_5\right|\rho}. \tag{2.25}$$

Then we have that

$$\rho \ge \frac{10}{\sqrt{2}\left(3+\sqrt{2}+\sqrt{3}\right)} \qquad \left(0<|a_{n+1}|<\frac{\Gamma(n+2-\lambda)}{2(n+1)!}\right). \tag{2.26}$$

Considering $\alpha_5$ in (2.23) and $\rho$ in (2.26), we say that

$$\begin{aligned}\left|\frac{\Gamma(2-\lambda)D_z^\lambda f(z)}{z^{1-\lambda}}-1\right| &< \frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}|a_{n+1}| \\ &\le \frac{\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}|a_{n+1}|}{1-2\frac{(n+1)!}{\Gamma(n+2-\lambda)}|a_{n+1}|} \\ &\le \left|e^{i\beta}-\alpha_5\right|\rho \qquad (\beta=0).\end{aligned} \tag{2.27}$$

Taking $\lambda = 1$ in Theorem 2.1, we see

**Corollary 2.1** If $f(z) \in A_n$ satisfies

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{\left|e^{i\beta} - \alpha_m\right| n\rho}{1 + \left|e^{i\beta} - \alpha_m\right| \rho}, \qquad (z \in U) \tag{2.28}$$

for some $\alpha_m$ defined by (1.9) with $\alpha_m \neq 1$ such that $z_p \in \partial U \quad (p = 1, 2, 3, \cdots, m)$, and for some real $\rho > 1$, then

$$\left|f'(z) - 1\right| < \left|e^{i\beta} - \alpha_m\right| \rho \qquad (z \in U). \tag{2.29}$$

Our next result is as follows:

**Theorem 2.2** Let $G_\lambda(z)$ be defined by (1.11) for $f(z) \in A_n$. If $G_\lambda(z)$ satisfies

$$\left|zG'_\lambda(z) - \frac{zG'_\lambda(z)}{G_\lambda(z)}\right| < \frac{\left|e^{i\beta} - \alpha_m\right|^2 n\rho^2}{1 + \left|e^{i\beta} - \alpha_m\right| \rho} \qquad (z \in U) \tag{2.30}$$

for some $\alpha_m$ defined by (1.9) with $\alpha_m \neq 1$, and for some real $\rho > 1$, then

$$\left|\frac{\Gamma(2-\lambda) D_z^\lambda f(z)}{z^{1-\lambda}} - 1\right| < \left|e^{i\beta} - \alpha_m\right| \rho \qquad (z \in U), \tag{2.31}$$

that is, $f(z) \in A_n(\alpha_m, \beta, \rho, \lambda)$, where $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$.

**Proof** We define the function $w(z)$ by (2.3). Then, it follows from (2.3) that

$$zG'_\lambda(z) - \frac{zG'_\lambda(z)}{G_\lambda(z)} = \frac{\left(1 - \alpha_m e^{-i\beta}\right)^2 zw(z) w'(z)}{1 + \left(1 - \alpha_m e^{-i\beta}\right) w(z)}. \tag{2.32}$$

Supposing that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho > 1, \tag{2.33}$$

Lemma 1.1 gives us $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0) \quad (k \geq n)$.
For such a point $z_0 \in U$, we see that

$$\begin{aligned} \left|z_0 G'_\lambda(z_0) - \frac{z_0 G'_\lambda(z_0)}{G_\lambda(z_0)}\right| &= \left|\frac{\left(1 - \alpha_m e^{-i\beta}\right)^2 z_0 w(z_0) w'(z_0)}{1 + \left(1 - \alpha_m e^{-i\beta}\right) w(z_0)}\right| \\ &= \frac{\left|1 - \alpha_m e^{-i\beta}\right|^2 \rho^2 k}{\left|1 + \left(1 - \alpha_m e^{-i\beta}\right) \rho e^{i\theta}\right|} \\ &\geq \frac{\left|e^{i\beta} - \alpha_m\right|^2 n\rho^2}{1 + \left|e^{i\beta} - \alpha_m\right| \rho}. \end{aligned} \tag{2.34}$$

This contradicts our condition (2.30). Therefore, we say that there is no $z_0 \in U$ such that $|w(z_0)| = \rho > 1$. This shows us that

$$|w(z)| = \left|\frac{e^{i\beta} G_\lambda(z) - \alpha_m}{e^{i\beta} - \alpha_m} - 1\right| < \rho, \qquad (z \in U) \tag{2.35}$$

that is,

$$\left|\frac{\Gamma(2-\lambda)\,D_z^{\lambda}f(z)}{z^{1-\lambda}}-1\right|<\left|e^{i\beta}-\alpha_m\right|\rho \qquad (z\in U). \tag{2.36}$$

**Example 2.2** Let a function $f(z)\in A_n$ be given by

$$f(z)=z+a_{n+1}z^{n+1} \qquad (z\in U) \tag{2.37}$$

with $0<|a_{n+1}|<\frac{\Gamma(n+2-\lambda)}{(n+1)!\Gamma(2-\lambda)}$. Then, $G_\lambda(z)$ satisfies

$$\begin{aligned}\left|zG_\lambda'(z)-\frac{zG_\lambda'(z)}{G_\lambda(z)}\right| &= \left|\frac{n\left(\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}\right)^2 a_{n+1}^2 z^{2n}}{1+\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}a_{n+1}z^n}\right| \\ &< \frac{n\left(\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}\right)^2 |a_{n+1}|^2}{1-\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}|a_{n+1}|} \qquad (z\in U).\end{aligned} \tag{2.38}$$

Considering five boundary points $z_1,z_2,z_3,z_4$ , and $z_5$ in Example 2.1, we have

$$\left|e^{i\beta}-\alpha_5\right|=\frac{\sqrt{2}\left(3+\sqrt{2}+\sqrt{3}\right)(n+1)!\Gamma(2-\lambda)}{10\Gamma(n+2-\lambda)}|a_{n+1}| \tag{2.39}$$

with $\beta=0$. For such $\alpha_5$ and $\beta=0$, we consider a real $\rho>1$ satisfying

$$\frac{n\left(\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}\right)^2 |a_{n+1}|^2}{1-\frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}|a_{n+1}|}\le\frac{\left|e^{i\beta}-\alpha_m\right|^2 n\rho^2}{1+\left|e^{i\beta}-\alpha_5\right|\rho}. \tag{2.40}$$

Then, this $\rho$ satisfies

$$\rho\ge\frac{10}{\sqrt{2}\left(3+\sqrt{2}+\sqrt{3}\right)}>1. \tag{2.41}$$

With such $\alpha_5,\rho$ and $\beta=0$, we know that

$$\begin{aligned}\left|\frac{\Gamma(2-\lambda)\,D_z^{\lambda}f(z)}{z^{1-\lambda}}-1\right| &< \frac{(n+1)!\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}|a_{n+1}| \\ &\le \left|e^{i\beta}-\alpha_5\right|\rho, \qquad (\beta=0).\end{aligned} \tag{2.42}$$

Letting $\lambda=1$ in Theorem 2.2, we have

**Corollary 2.2** If $f(z)\in A_n$ satisfies

$$\left|zf''(z)-\frac{zf''(z)}{f'(z)}\right|<\frac{\left|e^{i\beta}-\alpha_m\right|^2 n\rho^2}{1+\left|e^{i\beta}-\alpha_m\right|\rho} \qquad (z\in U) \tag{2.43}$$

for some $\alpha_m$ defined by

$$\alpha_m=\frac{1}{m}\sum_{p=1}^{m}f'(z_p) \qquad (z_j\ne z_k) \tag{2.44}$$

with $\alpha_m \neq 1,$ and for some real $\rho > 1,$ then

$$\left|f'(z) - 1\right| < \left|e^{i\beta} - \alpha_m\right| \rho, \qquad (z \in U) \tag{2.45}$$

where $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}.$

Finally, we derive

**Theorem 2.3** If $f(z) \in A_n$ satisfies

$$\frac{zD_z^{1+\lambda} f(z) - (1-\lambda) D_z^{\lambda} f(z)}{\Gamma(2-\lambda) D_z^{\lambda} f(z) - z^{1-\lambda}} \neq \frac{k}{\Gamma(2-\lambda)} \qquad (z \in U) \tag{2.46}$$

for some real $k \geq n,$ then

$$\left|\frac{\Gamma(2-\lambda) D_z^{\lambda} f(z)}{z^{1-\lambda}} - 1\right| < \left|e^{i\beta} - \alpha_m\right| \rho, \qquad (z \in U) \tag{2.47}$$

where $\alpha_m$ is defined by $(1.9)$ with $\alpha_m \neq 1, \rho > 1,$ and $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2},$ that is, $f(z) \in A_n(\alpha_m, \beta, \rho, \lambda).$

**Proof** For $f(z) \in A_n$ satisfying $(2.46),$ we consider $w(z)$ given by $(2.3).$ Since

$$G_\lambda(z) = \frac{\Gamma(2-\lambda) D_z^{\lambda} f(z)}{z^{1-\lambda}}, \tag{2.48}$$

we say that

$$\frac{zD_z^{1+\lambda} f(z) - (1-\lambda) D_z^{\lambda} f(z)}{\Gamma(2-\lambda) D_z^{\lambda} f(z) - z^{1-\lambda}} = \frac{zw'(z)}{\Gamma(2-\lambda) w(z)}. \tag{2.49}$$

If we suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho > 1, \tag{2.50}$$

Lemma 1.1 says that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0) \quad (k \geq n).$ From the above fact, we know that

$$\begin{aligned} \frac{z_0 D_z^{1+\lambda} f(z_0) - (1-\lambda) D_z^{\lambda} f(z_0)}{\Gamma(2-\lambda) D_z^{\lambda} f(z_0) - z_0^{1-\lambda}} &= \frac{z_0 w'(z_0)}{\Gamma(2-\lambda) w(z_0)} \\ &= \frac{k}{\Gamma(2-\lambda)}. \end{aligned} \tag{2.51}$$

Since this contradicts our condition $(2.46),$ we have $|w(z)| < \rho$ for all $z \in U.$

Consequently, we obtain that

$$|w(z)| = \left|\frac{\frac{\Gamma(2-\lambda) D_z^{\lambda} f(z)}{z^{1-\lambda}} - 1}{e^{i\beta} - \alpha_m}\right| < \rho, \qquad (z \in U) \tag{2.52}$$

which shows the inequality $(2.47).$

Letting $\lambda = 1$ in Theorem 2.3, we have

**Corollary 2.3** If $f(z) \in A_n$ satisfies

$$\frac{zf''(z)}{f'(z)-1} \neq k \qquad (z \in U), \tag{2.53}$$

for some real $k \geq n$, then $f(z)$ satisfies

$$|f'(z)-1| < \left|e^{i\beta} - \alpha_m\right| \rho \qquad (z \in U), \tag{2.54}$$

where $\alpha_m$ is given by (2.44) with $\alpha_m \neq 1, \rho > 1$, and $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$.

**Remark** In (2.29) of Corollary 2.1, (2.45) of Corollary 2.2, and (2.54) of Corollary 2.3, we say that

$$|f'(z)-1| < \left|e^{i\beta} - \alpha_m\right| \rho, \qquad (z \in U). \tag{2.55}$$

Therefore, if $\left|e^{i\beta} - \alpha_m\right| \rho \leq 1$ for $\alpha_m, \rho$, and $\beta$, then $f(z)$ is closed convex in $U$.

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