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A regularized trace formula for ”weighted” Sturm-Liouville equation with point δ- interaction

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Abstract: In this study, we obtain a formula for the regularized trace formula for ”weighted” Sturm–Liouville equation with point δ- interaction. At the begining, for the correct determination of solutions of analyzed equation at the point of discontinuty, the matching conditions are required. As a result, an equation is derived for the eigenvalues of the differential operator given in this study.

Key words: ”weighted” Sturm–Liouville equation, regularized trace formula, point δ-interaction

1. Introduction

The theory of regularized trace of ordinary differential operators has a long history. Gelfand and Levitan [6] first obtained the trace formula for the Sturm–Liouville differential equation. After this study, several mathematicians were interested in developing trace formulas for different differential operators. The current situation of this subject and studies related to it are presented in the comprehensive survey paper [13]. The regularized trace for differential equations are found in [4, 5, 7–9]. However, there is a small number of words on the regularized trace for Sturm–Liouville operators with singular potentials (see [14–16]). Note that, the trace formulas have applications in the approximate calculation of the eigenvalues of the related operator [4, 13].

We consider the boundary value problem (BVP) for the differential equation

$$ly := -y'' + q(x)\,y = \lambda y, \qquad x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \tag{1.1}$$

with the boundary conditions

$$U(y) := y'(0) = 0, \qquad V(y) := y(\pi) = 0, \tag{1.2}$$

and conditions at the point $x = \frac{\pi}{2}$,

$$I(y) := \begin{cases} y\left(\frac{\pi}{2}+0\right) = y\left(\frac{\pi}{2}-0\right) \equiv y\left(\frac{\pi}{2}\right), \\ y'\left(\frac{\pi}{2}+0\right) - y'\left(\frac{\pi}{2}-0\right) = -\alpha\lambda y\left(\frac{\pi}{2}\right), \end{cases} \tag{1.3}$$

where $q(x)$ is real-valued function in $W_2^1(0,\pi)$ and $\alpha > 0$; λ is spectral parameter.

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Notice that, we can understand problem (1.1),(1.3) as studying the equation

$$-y'' + q(x)\,y = \lambda\rho(x)\,y, \qquad x \in (0,\pi), \tag{1.4}$$

when $\rho(x) = 1 + \alpha\delta\left(x - \frac{\pi}{2}\right)$, where $\delta(x)$ is the Dirac function (see [1]). In this aspect, various inverse spectral problems for the equation (1.4) have been investigated in [12].

In the present paper, after construction of the Hilbert space related to (1.4), we obtain the formula of the first order regularized trace for "weighted" Sturm–Liouville equation with point δ- interaction.

2. Construction of the Hilbert space related to the problem and some properties of its spectral characteristics

In the Hilbert space $\aleph := L_2[0,\pi] \oplus \mathbb{C}$ of two component vectors, we define an inner product by

$$\langle f, g\rangle_{\aleph} := \int_0^{\pi} f_1(x)\,\overline{g}_1(x)\,dx + \frac{1}{\alpha} f_2\overline{g}_2 \tag{2.1}$$

for

$$f = \begin{pmatrix} f_1(x) \\ f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1(x) \\ g_2 \end{pmatrix},$$

where $f_1(x), g_1(x) \in L_2(0,\pi)$, $f_2, g_2 \in \mathbb{C}$. In the Hilbert space $\aleph$ we define the operator L

$$L : \aleph \to \aleph$$

with domain

$$D(L) : \left\{ f \in \aleph \;\middle|\; \begin{array}{c} f_1, f_1' \in AC\left(\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)\right),\; lf_1 \in L_2\left[(0,\pi) \setminus \left\{\frac{\pi}{2}\right\}\right], \\ f_2 = \alpha f_1\left(\frac{\pi}{2}\right), \quad U(f_1) = V(f_1) = 0 \end{array} \right\}$$

and operator rule

$$L(f) := \begin{pmatrix} lf_1 \\ f_1'\left(\frac{\pi}{2} - 0\right) - f_1'\left(\frac{\pi}{2} + 0\right) \end{pmatrix}.$$

Here $AC(.)$ denotes the set of all absolutely continuous functions on related interval. In particular, those functions will have limits at the point $\frac{\pi}{2}$.

It is clear that the eigenvalues of the operator L and the BVP (1.1)-(1.3) are same and eigenfunctions of (1.1)-(1.3) coincide with the first component of corresponding eigenelements of the operator L.

Theorem 2.1 *The operator L is symmetric.*

Proof Let $f, g \in D(L)$. From the inner product defined in (2.1), we obtain

$$\begin{aligned} \langle Lf, g\rangle_{\aleph} - \langle f, Lg\rangle_{\aleph} &= \left[W\left(f, \overline{g}; \frac{\pi}{2} - 0\right) - W(f, \overline{g}; 0)\right] \\ &+ \left[W(f, \overline{g}; \pi) - W\left(f, \overline{g}; \frac{\pi}{2} + 0\right)\right] \\ &+ \frac{1}{\alpha}\left[\alpha W\left(f, \overline{g}; \frac{\pi}{2} + 0\right) - \alpha W\left(f, \overline{g}; \frac{\pi}{2} - 0\right)\right], \end{aligned}$$

where $W(f,g;x)=f(x)g'(x)-f'(x)g(x)$ is the wronskian of the functions f and g. Since f and $\overline{g}$ satisfy the same boundary conditions (1.2) and from the conditions at the point $x=\frac{\pi}{2}$, we obtain $\langle Lf,g\rangle_{\aleph}=\langle f,Lg\rangle_{\aleph}$ for $f,g\in D(L)$. So L is symmetric. □

Corollary 2.2 *The function $W(f,g;x)$ is continuous on $(0,\pi)$.*

Corollary 2.3 *All eigenvalues of the problem (1.1)-(1.3) are real and if λ_1 and λ_2 are two different eigenvalues of the problem (1.1)-(1.3), the corresponding eigenfunctions $y_1(x)$ and $y_2(x)$ are orthogonal in the sense of*

$$\int_0^\pi y_1(x)\,y_2(x)\,dx+\frac{1}{\alpha}y_1\left(\frac{\pi}{2}\right)y_2\left(\frac{\pi}{2}\right)=0.$$

Let $\varphi(x,\lambda),\psi(x,\lambda),C(x,\lambda),S(x,\lambda)$ be solutions of (1.1) under the initial conditions

$$\begin{aligned}C(0,\lambda)&=S'(0,\lambda)=\varphi(0,\lambda)=\psi'(0,\lambda)=1,\\ C'(0,\lambda)&=S(0,\lambda)=\varphi'(0,\lambda)=\psi(\pi,\lambda)=0,\end{aligned}$$

and under the conditions (1.3).

For each fixed x, the functions $\varphi(x,\lambda),\psi(x,\lambda),C(x,\lambda),S(x,\lambda)$ are entire in λ. Clearly,

$$U(\varphi)=\varphi'(0,\lambda)=0,\qquad V(\psi)=\psi(\pi,\lambda)=0.$$

Denote

$$\Delta(\lambda)=W(\varphi,\psi;x).\tag{2.2}$$

By virtue of Corollary 2.2 and the Ostrogradskii–Liouville theorem (see [3], $\Delta(\lambda)$ does not depend on x. The function $\Delta(\lambda)$ is called *characteristic function* of L. Substituting $x=0$ and $x=\pi$ into (2.2), we get

$$\Delta(\lambda)=V(\varphi)=U(\psi).\tag{2.3}$$

The function $\Delta(\lambda)$ is entire in λ, and it has an at most countable set of zeros $\{\lambda_n\}_{n=1,2,3,\ldots}$.

Now, consider the solution $\varphi(x,\lambda)$. Let $C_0(x,\lambda)$ and $S_0(x,\lambda)$ be smooth solutions of (1.1) on the interval $(0,\pi)$ under the initial conditions $C_0(0,\lambda)=S_0'(0,\lambda)=1$, $C_0'(0,\lambda)=S_0(0,\lambda)=0$. Then,

$$C(x,\lambda)=C_0(x,\lambda),\ S(x,\lambda)=S_0(x,\lambda),\qquad x<\frac{\pi}{2},\tag{2.4}$$

$$\begin{array}{l}C(x,\lambda)=A_1C_0(x,\lambda)+B_1S_0(x,\lambda),\\ S(x,\lambda)=A_2C_0(x,\lambda)+B_2S_0(x,\lambda),\end{array}\qquad x>\frac{\pi}{2},\tag{2.5}$$

where

$$\begin{array}{ll}A_1=1+\alpha\lambda C_0\left(\frac{\pi}{2},\lambda\right)S_0\left(\frac{\pi}{2},\lambda\right), & B_1=-\alpha\lambda\left[C_0\left(\frac{\pi}{2},\lambda\right)\right]^2,\\ A_2=\alpha\lambda\left[S_0\left(\frac{\pi}{2},\lambda\right)\right]^2, & B_2=1-\alpha\lambda C_0\left(\frac{\pi}{2},\lambda\right)S_0\left(\frac{\pi}{2},\lambda\right).\end{array}\tag{2.6}$$

Let $\lambda=k^2$. It is easy to verify that the function $C_0(x,\lambda)$ satisfies the following integral equation:

$$C_0(x,\lambda)=\cos kx+\int_0^x\frac{\sin k(x-t)}{k}q(t)\,C_0(t,\lambda)\,dt.\tag{2.7}$$

Solving the equation (2.7) by the method of successive approximations, we obtain

$$C_0(x,\lambda)=\cos kx+\tfrac{\sin kx}{2k}\int_0^x q(t)\,dt+\tfrac{\cos kx}{4k^2}\{q(x)-q(0) \\ -\tfrac{1}{2}\left[\int_0^x q(t)\,dt\right]^2\}+O\left(\tfrac{1}{k^3}\exp(|Imk|\,x)\right), \tag{2.8}$$

Analogously,

$$S_0(x,\lambda)=\tfrac{\sin kx}{k}-\tfrac{\cos kx}{2k^2}\int_0^x q(t)\,dt+\tfrac{\sin kx}{4k^2}\{q(x)+q(0) \\ -\tfrac{1}{2}\left[\int_0^x q(t)\,dt\right]^2\}+O\left(\tfrac{1}{k^4}\exp(|Imk|\,x)\right), \tag{2.9}$$

By virtue of (2.6), (2.8) and (2.9),

$$A_1=\tfrac{\alpha}{2}k\sin k\pi+1-\tfrac{\alpha}{2}\cos k\pi\int_0^{\frac{\pi}{2}} q(t)\,dt \\ +\tfrac{\alpha}{4k}\sin k\pi\left\{q\left(\tfrac{\pi}{2}\right)-\left[\int_0^{\frac{\pi}{2}} q(t)\,dt\right]^2\right\}+O\left(\tfrac{1}{k^2}\right),$$

$$B_1=-\tfrac{\alpha}{2}k^2(1+\cos k\pi)-\tfrac{\alpha}{2}k\sin k\pi\int_0^{\frac{\pi}{2}} q(t)\,dt-\tfrac{\alpha}{4}\left[q\left(\tfrac{\pi}{2}\right)+q(0)\right] \\ -\tfrac{\alpha}{4}\cos k\pi\left\{q\left(\tfrac{\pi}{2}\right)-q(0)-\left[\int_0^{\frac{\pi}{2}} q(t)\,dt\right]^2\right\}+O\left(\tfrac{1}{k}\right),$$

$$A_2=\tfrac{\alpha}{2}(1-\cos k\pi)-\tfrac{\alpha}{2k}\sin k\pi\int_0^{\frac{\pi}{2}} q(t)\,dt+O\left(\tfrac{1}{k^2}\right),$$

$$B_2=-\tfrac{\alpha}{2}k\sin k\pi+1+\tfrac{\alpha}{2}\cos k\pi\int_0^{\frac{\pi}{2}} q(t)\,dt+O\left(\tfrac{1}{k}\right).$$

Since $\varphi(x,\lambda)=C(x,\lambda)$, we calculate using (2.4)–(2.9)

$$\varphi(x,\lambda)=\cos kx+\tfrac{\sin kx}{2k}\int_0^x q(t)\,dt+\tfrac{\cos kx}{4k^2}\{q(x)-q(0) \\ -\tfrac{1}{2}\left[\int_0^x q(t)\,dt\right]^2\}+O\left(\tfrac{1}{k^3}\exp(|Imk|\,x)\right),\qquad x<\tfrac{\pi}{2}, \tag{2.10}$$

$$\varphi(x,\lambda)=-\tfrac{\alpha}{2}k\sin kx+\tfrac{\alpha}{2}k\sin k(\pi-x)+\cos kx\left[1+\tfrac{\alpha}{4}\int_0^x q(t)\,dt\right] \\ +\tfrac{\alpha}{4}\cos k(\pi-x)\left[\int_{\frac{\pi}{2}}^x q(t)\,dt-\int_0^{\frac{\pi}{2}} q(t)\,dt\right]-\tfrac{\alpha}{8k}\sin kx\{q(x)+2q\left(\tfrac{\pi}{2}\right) \\ -q(0)-\tfrac{1}{2}\left[\int_0^x q(t)\,dt\right]^2\}-\tfrac{\alpha}{8k}\sin k(\pi-x)\{q(x)+2q\left(\tfrac{\pi}{2}\right) \\ -q(0)-\tfrac{1}{2}\left[\int_{\frac{\pi}{2}}^x q(t)\,dt-\int_0^{\frac{\pi}{2}} q(t)\,dt\right]^2\}+O\left(\tfrac{1}{k^2}\exp(|\tau|\,x)\right),\qquad x>\tfrac{\pi}{2}. \tag{2.11}$$

It follows from (2.3) and (2.11) that

$$\Delta(\lambda) = -\frac{\alpha}{2}\left(k \sin k\pi - w_1 \cos k\pi + w_2 - w_3 \frac{\sin k\pi}{k}\right) + O\left(\frac{1}{|k|^2}\right), \tag{2.12}$$

where

$$\begin{gathered} w_1 = \tfrac{2}{\alpha} + \tfrac{1}{2}\int\limits_0^{\pi} q(t)\,dt, \quad w_2 = \tfrac{1}{2}\left[\int\limits_0^{\frac{\pi}{2}} q(t)\,dt - \int\limits_{\frac{\pi}{2}}^{\pi} q(t)\,dt\right], \\ w_3 = -\tfrac{1}{4}\left\{q(\pi) + 2q\left(\tfrac{\pi}{2}\right) - q(0) - \tfrac{1}{2}\left[\int\limits_0^{\pi} q(t)\,dt\right]^2\right\}. \end{gathered} \tag{2.13}$$

Using (2.12) and Rouche's theorem, by the well-known method (see [2]) for $n \to \infty$,

$$k_n = n + o(1).$$

Analogously, by using Rouche's theorem, one can prove that, for sufficiently large values of n, every circle $\sigma_n(\delta) = \{k : |k - n| \le \delta\}$ contains exactly one zero of $\Delta(\lambda)$. Since $\delta > 0$ is arbitrary, we must have

$$k_n = n + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \to \infty. \tag{2.14}$$

Since k_n are zeros of $\Delta(\lambda)$, from (2.12), we get

$$n.\sin \varepsilon_n \pi - \left(w_1 \cos \varepsilon_n \pi + (-1)^{n-1} w_2\right) + \upsilon_n = 0, \tag{2.15}$$

where $\upsilon_n = n.\sin \varepsilon_n \pi + o(\exp(|\tau_n|))$, $\tau_n = Imk_n$. Hence, $\sin \varepsilon_n \pi = o\left(\frac{1}{n}\right)$, that is $\varepsilon_n = o\left(\frac{1}{n}\right)$. Using (2.15), we get more precisely

$$\varepsilon_n = \frac{1}{\pi n}\left(w_1 + (-1)^{n-1} w_2\right) + o\left(\frac{1}{n}\right). \tag{2.16}$$

Substituting (2.16) into (2.14), we get

$$k_n = n + \frac{1}{\pi n}\left(w_1 + (-1)^{n-1} w_2\right) + \frac{\xi_n}{n}, \quad (\xi_n) \in l_2. \tag{2.17}$$

3. Trace of the problem

The series

$$s_\lambda := \sum_{n=1}^{\infty}\left[\lambda_n - n^2 - \frac{2}{\pi}\left(w_1 + (-1)^{n-1} w_2\right)\right] \tag{3.1}$$

converges and is called *the regularized trace of first order* for the problem (1.1)-(1.3). The goal of this paper is to find its sum.

Theorem 3.1 *Suppose that $q(x) \in W_2^1(0,\pi)$, then the following first order regularized trace formula holds*

$$s_\lambda = w_3 - \frac{2w_1}{\pi} - \frac{w_1^2}{2},$$

where $w_i\ (i = 1, 3)$ satisfy the relations (2.13).

Proof Since $\Delta(\lambda)$ is an entire function of order $\frac{1}{2}$, from Hadamard's theorem (see, [10], Section 4.2), using (2.12) we obtain:

$$\Delta(\lambda) = A\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right), \tag{3.2}$$

where A is a certain constant to be determined below.

Let $\lambda = -\mu^2$. We calculate the sum s_λ of the series (3.1) by comparing the asymptotic expressions from formulas (2.12) and (3.2) as $\mu \to \infty$. From the (3.2), we have:

$$\Delta\left(-\mu^2\right) = \frac{\left(\lambda_1+\mu^2\right)\sinh\mu\pi}{\mu\pi}C\Phi(\mu), \tag{3.3}$$

where

$$C = \frac{A}{\lambda_1}\prod_{n=2}^{\infty}\frac{n^2}{\lambda_n}, \quad \Phi(\mu) = \prod_{n=2}^{\infty}\left(1-\frac{n^2-\lambda_n}{\mu^2+n^2}\right).$$

We study the asymptotic behaviour of the function $\Phi(\mu)$ for large positive μ. For this, we need the following formulas (see, [11]):

$$\sum_{j=2}^{\infty}\frac{1}{j}\sum_{n=1}^{\infty}\frac{\left|n^2-\lambda_n\right|^j}{\left(\mu^2+n^2\right)^j} = O\left(\frac{1}{\mu^3}\right), \tag{3.4}$$

$$\sum_{n=1}^{\infty}\frac{1}{\mu^2+n^2} = \frac{\pi\coth\pi\mu}{2\mu} - \frac{1}{2\mu^2} = \frac{\pi}{2\mu} - \frac{1}{2\mu^2} + O\left(\exp\left(-2\pi\mu\right)\right), \tag{3.5}$$

and since $\sup\limits_n\left|\lambda_n - n^2 - \frac{2}{\pi}\left(w_1 + (-1)^{n-1}w_2\right)\right| n^2 < \infty$,

$$\frac{1}{\mu^2}\sum_{n=1}^{\infty}\left(\lambda_n - n^2 - \frac{2}{\pi}\left(w_1+(-1)^{n-1}w_2\right)\right)\frac{n^2}{\mu^2+n^2} \le \frac{1}{\mu^2}\left(\sum_{n=1}^{\infty}\zeta_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\frac{1}{\mu^2+n^2}\right)^{\frac{1}{2}} = O\left(\frac{1}{\mu^3}\right). \tag{3.6}$$

From (3.4), (3.5) and (3.6) we calculate:

$$\begin{aligned}\ln\Phi(\mu) &= \sum_{n=2}^{\infty}\ln\left(1-\tfrac{n^2-\lambda_n}{\mu^2+n^2}\right) = -\sum_{n=2}^{\infty}\sum_{j=1}^{\infty}\tfrac{1}{j}\left(\tfrac{n^2-\lambda_n}{\mu^2+n^2}\right)^j \\ &= \tfrac{w_1}{\mu} + \tfrac{1}{\mu^2}\left(s_\lambda + \tfrac{2w_1}{\pi}\right) + O\left(\tfrac{1}{\mu^3}\right).\end{aligned}$$

Therefore, we get

$$\Phi(\mu) = 1 + \frac{w_1}{\mu} + \frac{1}{\mu^2}\left(s_\lambda + \frac{2w_1}{\pi} + \frac{w_1^2}{2}\right) + O\left(\frac{1}{\mu^3}\right),$$

and from (3.3)

$$\Delta\left(-\mu^2\right) = \frac{1}{2}ce^{\mu\pi}\left\{\mu + w_1 + \frac{1}{\mu}\left(s_\lambda + \frac{2w_1}{\pi} + \frac{w_1^2}{2}\right)\right\} + O\left(\frac{1}{\mu^2}\right). \tag{3.7}$$

We now study the asymptotic behaviour of the function $\Delta\left(-\mu^2\right) = \varphi\left(\pi, -\mu^2\right)$ for large negative $\lambda = -\mu^2$. Then, according to formula (2.11), we have

$$\Delta\left(-\mu^2\right) = \frac{1}{4}\alpha e^{\mu\pi}\left\{\mu + w_1 + \frac{1}{\mu}w_3\right\} + O\left(\frac{1}{\mu^2}\right). \tag{3.8}$$

It follows from the equalitites (3.3), (3.7), (3.8) and comparing the coefficients of μ, we obtain

$$c = \frac{\alpha}{2}, \qquad s_\lambda = w_3 - \frac{2w_1}{\pi} - \frac{w_1^2}{2},$$

completing the proof of Theorem 3.1. $\square$

4. Numerical example

The boundary value problem

$$-y'' = \lambda\left(1 + \alpha\delta\left(x - \frac{\pi}{2}\right)\right), \qquad x \in (0, \pi), \tag{4.1}$$

with the boundary conditions

$$y'(0) = y(\pi) = 0, \tag{4.2}$$

is a special case of and conditions at the point $x = \frac{\pi}{2}$,

$$\left\{\begin{array}{c} y\left(\frac{\pi}{2}+0\right) = y\left(\frac{\pi}{2}-0\right) \equiv y\left(\frac{\pi}{2}\right), \\ y'\left(\frac{\pi}{2}+0\right) - y'\left(\frac{\pi}{2}-0\right) = -\alpha\lambda y\left(\frac{\pi}{2}\right), \end{array}\right. \tag{4.3}$$

is a special case of problem (1.1)-(1.3) when $q(x) = 0$.

The first order regularized trace formula for the problem (4.1), (4.2) may be written by

$$s_\lambda := \sum_{n=1}^{\infty}\left(\lambda_n - n^2 - \frac{4}{\pi\alpha}\right) = -\frac{4}{\pi\alpha} - \frac{2}{\alpha^2}.$$

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