

# On the spectra of generalized Fibonomial and Jacobsthal-binomial graphs

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**Abstract:** In this work, we first give a more general form of the binomial, Fibonomial and balance-binomial graphs that is called generalized Fibonomial graph. We also argue the spectra of generalized Fibonomial graph. Next, we introduce a new type of graph on Jacobsthal numbers that is called Jacobsthal-binomial graph and denoted by  $JB_n$ . We obtain the adjacency, Laplacian and signless Laplacian characteristic polynomials of  $JB_n$ , respectively. We lastly give inequalities for the adjacency, Laplacian and signless Laplacian energies of  $JB_n$ .

**Key words:** adjacency matrix, Fibonomial graph, balance-binomial graph, spectral characterization, graph energy

## 1. Introduction

A graph  $G$  consists of a vertex set  $V(G) = \{v_1, \dots, v_n\}$  and an edge set  $E(G)$  and it is denoted by  $G = (V(G), E(G))$ . The order of  $G$  is number of vertices in  $G$ , that is  $|V(G)|$  and the size of  $G$  is number of edges in  $G$  that is  $|E(G)|$ . The degree of a vertex  $v_i \in V(G)$  is number of vertices in  $V(G)$  which are adjacent to  $v_i$  and it is denoted by  $deg(v_i)$ . If a vertex in  $G$  is adjacent to itself, then there occurs a loop. If a graph has order  $n \in \mathbb{Z}^+$  and size 0, then it is called a null graph with  $n$  vertices. Also, if all of the vertices of a graph with order  $n$  are adjacent to each other (except itself) then it is called a complete graph and denoted by  $K_n$ . Adjacency matrix of a graph  $G$  is  $A(G) = [a_{ij}]_{n \times n}$  such that

$$a_{ij} = \begin{cases} 1, & (v_i, v_j) \in E(G); \\ 0, & (v_i, v_j) \notin E(G). \end{cases}$$

Degree matrix of a graph  $G$  is the diagonal matrix  $D(G) = diag(deg(v_1), \dots, deg(v_n))$ . Laplacian matrix of a graph  $G$  is  $L(G) = D(G) - A(G)$  and the signless Laplacian matrix of  $G$  is  $Q(G) = D(G) + A(G)$ . For a graph matrix  $M(G)$ , the M-spectrum of  $G$  is the set of the eigenvalues of  $M(G)$ . For more information about spectral graph theory we refer [2]. Energy of a graph is firstly given by Gutman in [6]. This concept recently had great attention by the researchers such as [5, 7, 8, 12]. For a given graph  $G = (V(G), E(G))$ , let  $M(G)$  be the adjacency matrix, Laplacian matrix or signless Laplacian matrix of  $G$  and  $\lambda_1 \geq \dots \geq \lambda_n$  consists of the

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spectrum of  $M(G)$ . Then we compute the energy as follows.

$$\varepsilon(M(G)) = \begin{cases} \sum_{i=1}^n |\lambda_i|, & \text{if } G \text{ is simple;} \\ \sum_{i=1}^n \left| \lambda_i - \frac{\text{trace}(M(G))}{|V_n|} \right|, & \text{otherwise.} \end{cases} \quad (1.1)$$

1 Binomial graphs are first introduced in 1997 by Christopher and Kennedy [4]. For  $0 \leq n \in \mathbb{Z}$ , a binomial graph  
 2  $B_n = (V_n, E_n)$  is defined with  $V_n = \{v_j : j = 0, \dots, 2^n - 1\}$  and  $E_n = \{(v_i, v_j) : \begin{bmatrix} i+j \\ j \end{bmatrix} \equiv 1 \pmod{2}\}$ .  
 3 Adjacency spectrum of  $B_n$  is obtained and closed walks in  $B_n$  are also investigated. Similar to this concept,  
 4 in 2014, Akbulak, Kale and Öteleş [1] introduced a new type of graph called Fibonomial graph. They used  
 5 Fibonomial coefficients instead of binomial coefficients. Spectral properties and energy of Fibonomial graphs  
 6 are investigated in the same paper. Next, balance-binomial graphs are introduced in [9] whose entries are  
 7 dependent on the balancing numbers. Balance-binomial coefficients are used instead of Fibonomial coefficients  
 8 in [9]. But actually, balance-binomial graphs are exactly the same with binomial graphs. This could be seen  
 9 easily by comparing [4] and [9]. By the motivation of these results, in this work we first examine the situation  
 10 for the generalized Fibonacci numbers and so define the generalized Fibonomial graphs with a more general  
 11 perspective. Also, we define the Jacobsthal-binomial graphs by using Jacobsthal binomial coefficients. We  
 12 investigate the spectra and energy of Jacobsthal-binomial graphs according to the adjacency, Laplacian and  
 13 signless Laplacian matrices, respectively. Characteristic polynomials and inequalities for the energies of these  
 14 matrices are obtained here.

15 **2. Generalized Fibonomial Graphs**

The well-known Fibonacci sequence is defined as

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \quad (2.1)$$

where  $F_n$  denotes the  $n$ th Fibonacci number. Similar to the binomial coefficients, Fibonomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{F_n \cdot \dots \cdot F_{n-k+1}}{F_k \cdot \dots \cdot F_1} \quad (2.2)$$

where  $n \geq k \geq 1$ ,  $\begin{bmatrix} n \\ 0 \end{bmatrix}_F = 1$  and  $\begin{bmatrix} n \\ k \end{bmatrix}_F = 0$  for  $n < k$ .

For  $0 \leq n \in \mathbb{Z}$ , a Fibonomial graph  $G_n = (V_n, E_n)$  is defined by Akbulak, Kale and Öteleş [1] with the vertex set  $V_n = \{v_t : t = 0, 1, \dots, 3 \cdot 2^n - 1\}$  and the edge set  $E_n = \{(v_i, v_j) : \begin{bmatrix} i+j \\ j \end{bmatrix}_F \equiv 1 \pmod{2}\}$ . Chen and Sagan [3], examined the fractal nature of the Fibonomial triangle. As a result of this investigation, adjacency matrix of a Fibonomial graph could be written in terms of Kronecker product, that is given in [1] as follows

$$A_n = \left[ \bigotimes_{i=1}^n \mathfrak{B} \right] \otimes A_0 \text{ where } \mathfrak{B} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } A_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (2.3)$$

The balancing numbers are defined by the recursion formulae

$$B_n = 6B_{n-1} - B_{n-2} \text{ with initial values } B_1 = 1 \text{ and } B_2 = 6 \quad (2.4)$$

By using this sequence, a balance-binomial graph  $G'_n = (V'_n, E'_n)$  on  $2^n$  vertices, with vertex set  $V_n = \{v_t : t = 0, 1, \dots, 2^n - 1\}$  and the edge set  $E_n = \{(v_i, v_j) : \begin{bmatrix} i+j \\ j \end{bmatrix}_B \equiv 1(\text{mod } 2)\}$  is given in [9] such that the balance-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_B = \frac{B_n \cdot \dots \cdot B_{n-k+1}}{B_k \cdot \dots \cdot B_1} \tag{2.5}$$

for  $n \geq k \geq 1$  with  $\begin{bmatrix} n \\ 0 \end{bmatrix}_B = 1$  and  $\begin{bmatrix} n \\ k \end{bmatrix}_B = 0$  for  $n < k$ . Similar to the Fibonomial graphs, adjacency matrix of a balance-binomial graph is obtained by using Kronecker product as follows

$$A'_n = \left[ \bigotimes_{i=1}^n \mathfrak{B} \right] \bigotimes A'_0 \text{ where } A'_0 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \mathfrak{B}. \tag{2.6}$$

The generalized Fibonacci sequence is defined as

$$GF_n = pGF_{n-1} + qGF_{n-2} \text{ for } n \geq 2, p, q \in \mathbb{Z}^+ \text{ with } GF_0 = 0 \text{ and } GF_1 = 1. \tag{2.7}$$

Generalized Fibonomial coefficients are similarly defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{GF} = \frac{GF_n \cdot \dots \cdot GF_{n-k+1}}{GF_k \cdot \dots \cdot GF_1} \tag{2.8}$$

- 1 for  $n \geq k \geq 1$  with  $\begin{bmatrix} n \\ 0 \end{bmatrix}_{GF} = 1$  and  $\begin{bmatrix} n \\ k \end{bmatrix}_{GF} = 0$  for  $n < k$ .
- 2 In here, we define the generalized Fibonomial graphs  $GF_n = (V_{GF_n}, E_{GF_n})$  with the vertex set  $V_{GF_n}$  and the
- 3 edge set

$$E_{GF_n} = \{(v_i, v_j) : \begin{bmatrix} i+j \\ j \end{bmatrix}_{GF} \equiv 1(\text{mod } 2) \text{ such that } v_i, v_j \in V_{GF_n}\}.$$

4 All of the situations of the generalized Fibonomial coefficients according to the modulo 2 are explained in [3] by  
 5 Chen and Sagan. Thus, we obtain four kinds of adjacency matrices for the generalized Fibonomial graphs. If  
 6  $p, q$  are both odd, then we obtain the Fibonomial graph with  $3 \cdot 2^n$  vertices. If  $p$  is even and  $q$  is odd, then  
 7 we have the balance-binomial graph with  $2^n$  vertices (actually, this graph could also be easily obtained from  
 8 Pascal's triangle). If  $p$  is odd and  $q$  is even, then we obtain that the adjacency matrix is the all one matrix  
 9  $\mathbf{J}_n$ . If  $p, q$  are both even, then we get that the adjacency matrix is the all zero matrix  $\mathbf{0}_n$ . Hence, we have  
 10 four kind of generalized Fibonomial graphs. These are Fibonomial graphs, balance-binomial graphs (or shortly  
 11 binomial graphs), the graph obtained from a complete graph by adding loop to all vertices and the null graph,  
 12 respectively.

13 So, if we talk about the spectra of the generalized Fibonomial graphs, there are four kinds of the adjacency  
 14 spectrum according to the  $p$  and  $q$ . Adjacency spectrum of the Fibonomial graphs and balance-binomial graphs  
 15 are given in [1] and [4, 9], respectively. Clearly  $\mathbf{J}_n$  has eigenvalues  $\{0^{n-1}, n\}$  and  $\mathbf{0}_n$  has eigenvalues  $\{0^n\}$ .  
 16 Based on these results, in the next section we define a new graph on the Jacobsthal numbers that is different  
 17 from generalized Fibonomial graphs and also we obtain the spectral properties of this graph.

1 **3. Jacobsthal-binomial Graphs**

2 **3.1. Jacobsthal numbers**

The Jacobsthal numbers are given by the following recursion formula

$$J_0 = 0, J_1 = 1, J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2 \tag{3.1}$$

3 where  $J_n$  denotes the  $n$ th Jacobsthal number. Here we give some properties of Jacobsthal numbers that are used in the next sections. The following table for the remainders of Jacobsthal numbers is obtained in [17].

$J_n / m$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	...
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
5	1	2	1	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
11	1	2	3	1	5	4	3	2	1	0	11	11	11	11	11	11	11	11	11
21	1	0	1	1	3	0	5	3	1	10	9	8	7	6	5	4	3	2	1
43	1	1	3	3	1	1	3	7	3	10	7	4	1	13	11	9	7	5	3
85	1	1	1	0	1	1	5	4	5	8	1	7	1	10	5	0	13	9	5
171	1	0	3	1	3	3	3	0	1	6	3	2	3	6	11	1	9	0	5
341	1	2	1	1	5	5	5	8	1	0	5	3	5	11	5	1	17	18	13
683	1	2	3	3	5	4	3	8	3	1	11	7	11	8	11	3	17	18	13
1365	1	0	1	0	3	0	5	6	5	1	9	0	7	0	5	5	15	16	11
2731	1	1	3	1	1	1	3	4	1	3	7	1	1	1	11	11	13	14	9
5461	1	1	1	1	1	1	5	7	1	5	1	1	1	1	5	4	7	8	5
10923	1	0	3	3	3	3	3	6	3	0	3	3	3	3	11	9	15	17	11
21845	1	2	1	0	5	5	5	2	5	10	5	5	5	5	5	0	11	14	9
43691	1	2	3	1	5	4	3	5	1	10	11	11	11	11	11	1	5	10	5
87381	1	0	1	1	3	0	5	0	1	8	9	8	7	6	5	1	9	0	5
174763	1	1	3	3	1	1	3	1	3	6	7	4	1	13	11	3	1	1	5
349525	1	1	1	0	1	1	5	1	5	0	1	7	1	10	5	5	1	1	5
⋮																			

Figure 1. Sets of remainder of Jacobsthal numbers for modulo m

4 **Lemma 3.1.** [10] Let  $n$  be an integer. The well-known Binet-like formula of the Jacobsthal numbers is

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{3.2}$$

5 **Theorem 3.2.** [10] For all  $n, k \in \mathbb{Z}^+$ ,  $J_n \mid J_{kn}$ .

6 **3.2. Construction of Jacobsthal-binomial graphs**

Similar to the binomial coefficients, here we define the Jacobsthal-binomial coefficients for  $n \geq k \geq 1$

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_J = \frac{J_n \cdot \dots \cdot J_{n-k+1}}{J_k \cdot \dots \cdot J_1} \tag{3.3}$$

7 with  $\left[ \begin{matrix} n \\ 0 \end{matrix} \right]_J = 1$  and  $\left[ \begin{matrix} n \\ k \end{matrix} \right]_J = 0$  for  $n < k$ . From Theorem 3.2, one can easily see that  $\left[ \begin{matrix} n \\ k \end{matrix} \right]_J$  always  
 8 takes integer values. So, we define the Jacobsthal-binomial graph  $JB_n = (V_{JB_n}, E_{JB_n})$  with the vertex set  
 9  $V_{JB_n} = \{v_j : j = 0, 1, \dots, 2n\}$  and the edge set

$$E_{JB_n} = \{(v_i, v_j) : \left[ \begin{matrix} i+j \\ j \end{matrix} \right]_J \equiv 1 \pmod{4}\}.$$

1 In this way, we obtain that the adjacency matrix of the Jacobsthal-binomial graph  $A(JB_n) = [a_{ij}]_{m \times m}$  such  
 2 that  $m = 2n + 1$  and

$$a_{ij} = \begin{cases} 0, & \text{if } \begin{bmatrix} i+j \\ j \end{bmatrix}_J \equiv 3 \pmod{4}; \\ 1, & \text{if } \begin{bmatrix} i+j \\ j \end{bmatrix}_J \equiv 1 \pmod{4}. \end{cases}$$

3 From the definition of the Jacobsthal-binomial graph, the first row and the first column of  $A(JB_n)$  are all-one  
 4 vectors. Also we can easily obtain the second row and the second column of  $A(JB_n)$  from the figure 1. Now,  
 5 we find other rows and columns of  $A(JB_n)$  by the following theorem.

6 **Theorem 3.3.** For  $i, j \in \{1, 2, \dots, 2n\}$ , let  $\begin{bmatrix} i+j \\ j \end{bmatrix}_J = s$ . If  $i, j$  are both odd, then  $s \equiv 1 \pmod{4}$ . Otherwise  
 7  $s \equiv 3 \pmod{4}$ .

8 **Proof** We give the proof of the statement by induction on  $j$  in two cases. Assume that the statement holds  
 9 for  $j$ . Then, we need to show that the equality holds for  $j + 2$ .

10 **Case 1:** Let  $j$  be an odd integer. For  $j = 1$ ,

$$\begin{bmatrix} i+1 \\ 1 \end{bmatrix}_J = \frac{J_{i+1}}{J_1} = \frac{J_{i+1}}{1} = J_{i+1}$$

11 From figure 1, it can easily be seen that if  $i$  is odd then  $J_{i+1} \equiv 1 \pmod{4}$ , otherwise  $J_{i+1} \equiv 3 \pmod{4}$ . Now,  
 12 we check that whether the statement holds for  $j + 2$ . Let us consider the situation in two subcases.

13 *Subcase 1:* Suppose that  $i$  is odd. Then,

$$\begin{bmatrix} i+j+2 \\ j+2 \end{bmatrix}_J = \frac{J_{i+j+2} \cdot J_{i+j+1} \cdot \dots \cdot J_{i+1}}{J_{j+2} \cdot J_{j+1} \cdot \dots \cdot J_1}$$

14 Since the statement holds for  $j$ , if  $\begin{bmatrix} i+j \\ j \end{bmatrix}_J = m$  then we have  $m \equiv 1 \pmod{4}$ . Also, by using the Binet-like  
 15 formula (3.2) of the Jacobsthal numbers, we get

$$\frac{J_{i+j+2} \cdot J_{i+j+1}}{J_{j+2} \cdot J_{j+1}} = \frac{(2^{i+j+2} - 1)(2^{i+j+1} + 1)}{(2^{j+2} + 1)(2^{j+1} - 1)} = \frac{2^{2i+2j+3} + 2^{i+j+1} - 1}{2^{2j+3} - 2^{j+1} - 1}.$$

16 If  $\frac{2^{2i+2j+3} + 2^{i+j+1} - 1}{2^{2j+3} - 2^{j+1} - 1} = k$ , then we can see easily that  $k \equiv 1 \pmod{4}$ . Hence, we get  $\begin{bmatrix} i+j+2 \\ j+2 \end{bmatrix}_J \equiv 1 \pmod{4}$   
 17 when  $i$  and  $j$  are both odd.

18 *Subcase 2:* Let  $i$  is even. So we have  $\begin{bmatrix} i+j \\ j \end{bmatrix}_J = m \equiv 3 \pmod{4}$ . Therefore,

$$\begin{bmatrix} i+j+2 \\ j+2 \end{bmatrix}_J = \frac{J_{i+j+2} \cdot J_{i+j+1}}{J_{j+2} \cdot J_{j+1}} \cdot m = \frac{2^{2i+2j+3} - 2^{i+j+1} - 1}{2^{2j+3} - 2^{j+1} - 1} \cdot m$$

19 Thus we get  $\begin{bmatrix} i+j+2 \\ j+2 \end{bmatrix}_J \equiv 3 \pmod{4}$ .

1 **Case 2:** Let  $j$  be even. For  $j = 2$ , we get

$$\left[ \begin{matrix} i+2 \\ 2 \end{matrix} \right]_J = \frac{J_{i+2} \cdot J_{i+1}}{J_2 \cdot J_1} = J_{i+2} \cdot J_{i+1} \equiv 3 \pmod{4}.$$

2 We assume again that statement holds for  $j$  and check for  $j + 2$ . So,  $\left[ \begin{matrix} i+j \\ j \end{matrix} \right]_J = m \equiv 3 \pmod{4}$ . We will  
 3 consider the situation in two subcases.

4 *Subcase 1:* Suppose that  $i$  is even. Then, we have

$$\left[ \begin{matrix} i+j+2 \\ j+2 \end{matrix} \right]_J = \frac{J_{i+j+2} \cdot J_{i+j+1}}{J_{j+2} \cdot J_{j+1}} \cdot m = \frac{2^{2i+2j+3} + 2^{i+j+1} - 1}{2^{2j+3} + 2^{j+1} - 1} \cdot m$$

5 So we get  $\left[ \begin{matrix} i+j+2 \\ j+2 \end{matrix} \right]_J \equiv 3 \pmod{4}$ .

6 *Subcase 2:* Suppose that  $i$  is odd. Thus, we get

$$\left[ \begin{matrix} i+j+2 \\ j+2 \end{matrix} \right]_J = \frac{2^{2i+2j+3} - 2^{i+j+1} - 1}{2^{2j+3} + 2^{j+1} - 1} \cdot m \equiv 3 \pmod{4}$$

7 Consequently, we obtain that if  $i$  and  $j$  are both odd, then  $\left[ \begin{matrix} i+j \\ j \end{matrix} \right]_J \equiv 1 \pmod{4}$ , otherwise  $\left[ \begin{matrix} i+j \\ j \end{matrix} \right]_J \equiv$   
 8  $3 \pmod{4}$ .

9 By the help of the Theorem 3.3, we get  $A(JB_n)$  with  $2n + 1$  vertices in the following form.

$$A(JB_n) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{3.4}$$

10 **Remark 3.4.** For all  $v_i \in V_{JB_n}$ ,

$$\deg(v_i) = \begin{cases} 2n + 2, & \text{if } i=0; \\ n + 2, & \text{if } i \text{ is odd;} \\ 1, & \text{if } i \text{ is even.} \end{cases}$$

11 and  $|E_{JB_n}| = n^2 + 5n + 2$ .

12 The pineapple graph  $K_p^q$  is obtained by appending  $q$  pendant edges to a vertex of a complete graph  $K_p$   
 13 ( $q \geq 1, p \geq 3$ )[14, 15]. From (3.4) and figure 2, we can see easily that  $JB_n$  could also be obtained from a  
 14 pineapple graph  $K_{n+1}^n$  by adding loop to the all of vertices of its maximum clique  $K_{n+1}$ .

**Lemma 3.5.** The characteristic polynomial of the adjacency matrix of Jacobsthal-binomial graph with  $2n+1$  vertices is as follows

$$A(JB_n)(x) = x^{2n-2}(-x^3 + (n+1)x^2 + nx - n^2) \tag{3.5}$$

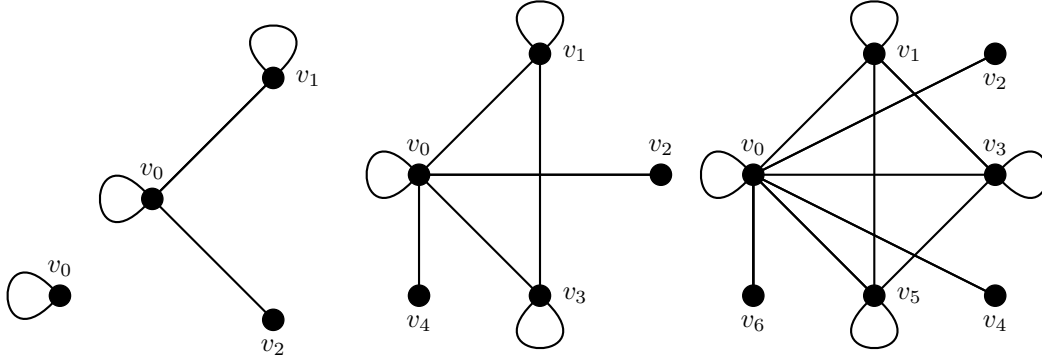


Figure 2. First four Jacobsthal-binomial graphs.

1 **Proof** Let  $\mathbf{1}$  denote the all-one vector,  $\mathbf{J}_n$  denote the all-one  $n \times n$  matrix and  $\mathbf{0}$  denote the all-zero  $n \times n$   
 2 matrix. It is clear that  $A(JB_n)$  could be written in the following form.

$$A(JB_n) = \begin{bmatrix} \mathbf{1} & \mathbf{1}^T & \mathbf{1}^T \\ \mathbf{1} & \mathbf{J}_n & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}_{(2n+1) \times (2n+1)}$$

3 One can easily see that the rank of  $A(JB_n)$  could be equal to at most 3. Hence,  $x^{2n-2}$  is a factor of the  
 4  $A(JB_n)(x)$ . Also, an equitable partition of  $A(JB_n)$  is as follows.

$$P_n = \begin{bmatrix} 1 & n & n \\ 1 & n & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3 \times 3}$$

5 The characteristic polynomial of  $P_n$  is equal to  $P_n(x) = -x^3 + (n+1)x^2 + nx - n^2$ . Since  $P_n(x)$  is a divisor  
 6 of  $A(JB_n)(x)$ , we get that

$$A(JB_n)(x) = x^{2n-2}(-x^3 + (n+1)x^2 + nx - n^2)$$

**Lemma 3.6.** *The Laplacian characteristic polynomial of the Jacobsthal-binomial graph with  $2n+1$  vertices is given below.*

$$L(JB_n)(x) = (x - (n+2))^{n-1}(x-1)^{n-1}(-x^3 + (2n+4)x^2 - (4n+5)x + n+2) \tag{3.6}$$

7 **Proof** Let  $I_n$  denotes the  $n \times n$  identity matrix. Since  $L(JB_n) - I_n$  and  $L(JB_n) - (n+2)I_n$  contains  $n$   
 8 identical rows,  $(x-1)^{n-1}$  and  $(x-(n+2))^{n-1}$  are factors of  $L(JB_n)(x)$ , respectively. Clearly,  $L(JB_n)$  could  
 9 be written in the following form.

$$L(JB_n) = \begin{bmatrix} 2n+1 & -\mathbf{1}^T & -\mathbf{1}^T \\ -\mathbf{1} & (n+2)\mathbf{I}_n - \mathbf{J}_n & \mathbf{0} \\ -\mathbf{1} & \mathbf{0} & \mathbf{I}_n \end{bmatrix}_{(2n+1) \times (2n+1)}$$

10 Hence an equitable partition of  $L(JB_n)$  is as follows.

$$P'_n = \begin{bmatrix} 2n+1 & -n & -n \\ -1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

1 The characteristic polynomial of  $P'_n$  is equal to  $P'_n(x) = -x^3 + (2n+4)x^2 - (4n+5)x + n+2$ . So we obtain  
 2 that,

$$L(JB_n)(x) = (x - (n+2))^{n-1}(x-1)^{n-1}(-x^3 + (2n+4)x^2 - (4n+5)x + n+2)$$

3 **Lemma 3.7.** *The signless Laplacian characteristic polynomial of the Jacobsthal-binomial graph with  $2n+1$   
 4 vertices is in the following form.*

$$Q(JB_n)(x) = (x - (n+2))^{n-1}(x-1)^{n-1}(-x^3 + (4n+6)x^2 - (4n^2 + 12n + 11)x + 2n^2 + 7n + 6) \quad (3.7)$$

5 **Proof**  $Q(JB_n) - I_n$  and  $Q(JB_n) - (n+2)I_n$  has  $n$  identical rows, respectively. Hence  $(x-1)^{n-1}$  and  
 6  $(x - (n+2))^{n-1}$  are factors of  $Q(JB_n)(x)$ . Also  $Q(JB_n)$  could be written as below.

$$Q(JB_n) = \begin{bmatrix} 2n+3 & \mathbf{1}^T & \mathbf{1}^T \\ \mathbf{1} & (n+2)\mathbf{I}_n + \mathbf{J}_n & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{I}_n \end{bmatrix}_{(2n+1) \times (2n+1)}$$

7 So we can obtain an equatable partition of  $Q(JB_n)$  as follows.

$$P''_n = \begin{bmatrix} 2n+3 & n & n \\ 1 & 2n+2 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

8 Hereby the characteristic polynomial of  $P''_n$  is equal to  $P''_n(x) = -x^3 + (4n+6)x^2 - (4n^2 + 12n + 11)x + 2n^2 + 7n + 6$ .

9 So we conclude that

$$Q(JB_n)(x) = (x - (n+2))^{n-1}(x-1)^{n-1}(-x^3 + (4n+6)x^2 - (4n^2 + 12n + 11)x + 2n^2 + 7n + 6)$$

### 10 3.3. Energies of Jacobsthal-binomial graph

11 **Theorem 3.8.** *Let  $\varepsilon(JB_n)$ ,  $\varepsilon_L(JB_n)$  and  $\varepsilon_Q(JB_n)$  denotes the energy, Laplacian energy and signless Lapla-  
 12 cian energy of the Jacobsthal-binomial graph with  $2n+1$  vertices, respectively. Then, the following inequalities  
 13 holds.*

14 (i)  $2n+2 < \varepsilon(JB_n) < 2n+3$

15 (ii)  $n^2 + \frac{5n}{2} - \frac{5}{4} < \varepsilon_L(JB_n) < n^2 + \frac{5n}{2} + \frac{3}{4}$

16 (iii)  $n^2 + \frac{7n}{2} + \frac{23}{4} < \varepsilon_Q(JB_n) < n^2 + \frac{7n}{2} + \frac{31}{4}$



1 **Proof (i)** Let us denote the eigenvalues of  $A(JB_n)$  with  $\lambda_1 \geq \dots \geq \lambda_{2n+1}$ . Then, from (3.5) we have  
 2  $\lambda_1 = n + 1 + \alpha$ ,  $\lambda_2 = 1 + \beta$ ,  $\lambda_3 = \dots = \lambda_{2n} = 0$ ,  $\lambda_{2n+1} = -1 - \alpha - \beta$  such that  $0 < \alpha, \beta < 1$ . We know that  
 3  $\varepsilon(JB_n) = \sum_1^{2n+1} |\lambda_i - \frac{s}{2n+1}|$  where  $s = tr(A(JB_n)) = n + 1$ . Hence,

$$\begin{aligned} \varepsilon(JB_n) &= \sum_{i=1}^2 (\lambda_i - \frac{n+1}{2n+1}) + (\frac{n+1}{2n+1} - \lambda_{2n+1}) + (2n-2)(\frac{n+1}{2n+1}) \\ &= n+3+2(\alpha+\beta) + (2n-3)\frac{n+1}{2n+1} \\ &= n+3+2(\alpha+\beta) + n-1 - \frac{2}{2n+1} \\ &= 2n+2+2(\alpha+\beta) - \frac{2}{2n+1} \\ &= 2n+2+\gamma \end{aligned}$$

4 where  $0 < \gamma = 2(\alpha+\beta) - \frac{2}{2n+1} < 1$ . So we get  $2n+2 < \varepsilon(JB_n) < 2n+3$ .

5 **(ii)** If we denote the eigenvalues of the  $L(JB_n)$  with  $\mu_1 \geq \dots \geq \mu_{2n+1}$ , then from (3.6) we get  $\mu_1 = 2n+1+\alpha_1$ ,  
 6  $\mu_2 = \dots = \mu_n = n+2$ ,  $\mu_{n+1} = 1+\beta_1$ ,  $\mu_{n+2} = \dots = \mu_{2n} = 1$ ,  $\mu_{2n+1} = 2-\alpha_1-\beta_1$  such that  $0 < \alpha_1, \beta_1 < 1$ .  
 7 Also  $s_L = tr(L(JB_n)) = 2n+1+n(n+1)+n = n^2+4n+1$ . Thus, we calculate the Laplacian energy of  $JB_n$   
 8 as follows.

$$\begin{aligned} \varepsilon(L(JB_n)) &= \sum_{i=1}^{2n+1} |\mu_i - \frac{s_L}{2n+1}| \\ &= \sum_{i=1}^{2n+1} |\mu_i - \frac{n^2+4n+1}{2n+1}| \\ &= |2n+1+\alpha_1 - \frac{n^2+4n+1}{2n+1}| + (n-1)|n+2 - \frac{n^2+4n+1}{2n+1}| \\ &+ |1+\beta_1 - \frac{n^2+4n+1}{2n+1}| + (n-1)|1 - \frac{n^2+4n+1}{2n+1}| \\ &+ |2-\alpha_1-\beta_1 - \frac{n^2+4n+1}{2n+1}| \\ &= 2n-2+2\alpha_1 + \frac{n^2+4n+1}{2n+1} + (n-1)(n+1) \\ &= n^2 + \frac{5n}{2} - \frac{5}{4} + 2\alpha_1 + \frac{1}{8n+4} \end{aligned}$$

9 Hence, we get  $n^2 + \frac{5n}{2} - \frac{5}{4} < \varepsilon_L(JB_n) < n^2 + \frac{5n}{2} + \frac{3}{4}$ .

10 **(iii)** We denote the eigenvalues of  $Q(JB_n)$  with  $q_1 \geq \dots \geq q_{2n+1}$ . From (3.7), we obtain that  $q_1 = 2n+a$ ,  
 11  $q_2 = 2n+b$ ,  $q_3 = \dots = q_{n+1} = n+2$ ,  $q_{n+2} = \dots = q_{2n} = 1$ ,  $q_{2n+1} = \alpha_2 = 6-a-b$  and  $0 < \alpha_2 < 1$ . Also,

12  $s_Q = \text{tr}(Q(JB_n)) = n^2 + 6n + 3$ . Hence we may calculate  $\varepsilon_Q(JB_n)$  as below.

$$\begin{aligned}
 \varepsilon(Q(JB_n)) &= \sum_{i=1}^{2n+1} \left| q_i - \frac{s_Q}{2n+1} \right| \\
 &= \sum_{i=1}^{2n+1} \left| q_i - \frac{n^2 + 6n + 3}{2n+1} \right| \\
 &= \left| q_1 - \frac{n^2 + 6n + 3}{2n+1} \right| + \left| q_2 - \frac{n^2 + 6n + 3}{2n+1} \right| \\
 &+ (n-1) \left| n+2 - \frac{n^2 + 6n + 3}{2n+1} \right| \\
 &+ (n-1) \left| 1 - \frac{n^2 + 6n + 3}{2n+1} \right| + \left| q_{2n+1} - \frac{n^2 + 6n + 3}{2n+1} \right| \\
 &= q_1 + q_2 - q_{2n+1} - \frac{n^2 + 6n + 3}{2n+1} + (n-1)(n+1) \\
 &= 4n + 2(a+b) - 6 + n^2 - \frac{n}{2} + \frac{7}{4} - \frac{1}{8n+4} \\
 &= n^2 + \frac{7n}{2} + 2(a+b) - \frac{17}{4} - \frac{1}{8n+4}
 \end{aligned}$$

1 Since  $0 < \alpha_2 < 1$ , we get  $n^2 + \frac{7n}{2} + \frac{23}{4} < \varepsilon_Q(JB_n) < n^2 + \frac{7n}{2} + \frac{31}{4}$ .

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## 4 References

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