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# Decompositions of complete symmetric directed graphs into the oriented heptagons

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**Abstract:** The complete symmetric directed graph of order $v$, denoted by $K_v^*$, is the directed graph on $v$ vertices that contains both arcs $(x,y)$ and $(y,x)$ for each pair of distinct vertices $x$ and $y$. For a given directed graph $D$, the set of all $v$ for which $K_v^*$ admits a $D$-decomposition is called the spectrum of $D$-decomposition. There are 10 nonisomorphic orientations of a 7-cycle (heptagon). In this paper, we completely settled the spectrum problem for each of the oriented heptagons.

**Key words:** Decomposition, directed graph, orientations of a heptagon

## 1. Introduction

For a graph (or directed graph) $D$, we use $V(D)$ and $E(D)$ to denote the vertex set of $D$ and the edge set (or arc set) of $D$, respectively. For a simple graph $G$, we use $G^*$ to denote the symmetric directed graph with vertex set $V(G^*) = V(G)$ and arc set $E(G^*) = \bigcup_{\{x,y\}\in E(G)} \{(x,y),(y,x)\}$. Hence, $K_v^*$ is the complete symmetric directed graph of order $v$. We use $K_{r\times s}$ to denote the complete simple multipartite graph with $r$ parts of size $s$. Also, if $a$ and $b$ are integers with $a \le b$, we let $[a,b]$ denote the set $\{a, a+1, \dots, b\}$.

A decomposition of a directed graph $K$ is a set $\Delta = \{D_1, D_2, \dots, D_t\}$ of subgraphs of $K$ such that each directed edge, or arc, of $K$ appears in exactly one directed graph $D_i \in \Delta$. If each $D_i$ in $\Delta$ is isomorphic to a given directed graph $D$, the decomposition is called a $D$-decomposition of $K$. A $\{G,H\}$-decomposition of $K$ is defined similarly. A $D$-decomposition of $K$ is also known as a $(K,D)$-design. The set of all $v$ for which $K_v^*$ admits a $D$-decomposition is called the spectrum of $D$.

The reverse orientation of $D$, denoted Rev($D$), is the directed graph with vertex set $V(D)$ and arc set $\{(v,u) : (u,v) \in E(D)\}$. We note that the existence of a $D$-decomposition of $K$ necessarily implies the existence of a Rev($D$)-decomposition of Rev($K$). Since $K_v^*$ is its own reverse orientation, we note that the spectrum of $D$ is equal to the spectrum of Rev($D$).

A decomposition of $K_v^*$ into copies of a directed graph $D$ requires the number of edges in $K_v^*$, namely $v(v-1)$ is divisible by the number of the edges in $D$. Moreover, both gcd{outdegree$(x) : x \in V(D)$} and gcd{indegree$(x) : x \in V(D)$} divide $v-1$, which is both indegree and outdegree of every vertex in $K_v^*$. Thus, the necessary conditions for a directed graph $D$ to decompose $K_v^*$ are

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(a) $|V(D)| \leq v$,

(b) $|E(D)|$ divides $v(v-1)$, and

(c) $\gcd\{\text{outdegree}(x) : x \in V(D)\}$ and $\gcd\{\text{indegree}(x) : x \in V(D)\}$ both divide $(v-1)$.

The spectrum problem for certain subgraphs (both bipartite and nonbipartite) of $K_4^*$ has already been studied [6, 9, 10, 13, 14]. There are two nonisomorphic orientations of $K_3$, namely, cyclic and transitive orientations. When $D$ is a cyclic orientation of $K_3$, then a $(K_v^*, D)$-design is known as a Mendelsohn triple system. The spectrum for Mendelsohn triple systems was found independently by Mendelsohn [13] and Bermond [6]. When $D$ is a transitive orientation of $K_3$, then a $(K_v^*, D)$-design is known as a transitive triple system. The spectrum for transitive triple systems was found by Hung and Mendelsohn [10]. There are exactly four orientations of a quadrilateral (i.e. a 4-cycle). It was shown in [14] that if $D$ is a cyclic orientation of a quadrilateral, then a $(K_v^*, D)$-design exists if and only if $v \equiv 0$ or $1 \pmod 4$ and $v \neq 4$. The spectrum problem for the remaining three orientations of a quadrilateral were settled in [9]. In [5], Alspach et al. showed that $K_v^*$ can be decomposed into each of the four orientations of a pentagon (i.e. a 5-cycle) if and only if $v \equiv 0$ or $1 \pmod 5$. In [4], it is shown that for positive integers $m$ and $v$ with $2 \leq m \leq v$ the directed graph $K_v^*$ can be decomposed into directed cycles (i.e. with all the edges being oriented in the same direction) of length $m$ if and only if $m$ divides the number of arcs in $K_v^*$ and $(v, m) \notin \{(4,4), (6,3), (6,6)\}$. Also [2], Adams et al. has recently settled the $\lambda$-fold spectrum problem for all possible orientations of a 6-cycle.

There are ten nonisomorphic orientations of a heptagon (see Figure 1). The spectrum problem was settled for the directed heptagon ($D_{10}$ in Figure 1) in [4].

**Theorem 1.1 ([4])** *For integers $v \geq 7$, there exists a $D_{10}$-decomposition of $K_v^*$ if and only if $v(v-1) \equiv 0 \pmod 7$.*

In this work, we completely settle the spectrum problem for the remaining nine nonisomorphic oriented heptagons (i.e. $D_i$ for $i \in [1, 9]$ as seen in Figure 1).

If $D$ is an oriented heptagon and if there exists a $(K_v^*, D)-$design, then we must have $7 | v(v-1)$. Moreover, if $D$ is an oriented odd cycle, then $\gcd\{\text{outdegree}(x) : x \in V(D)\} = \gcd\{\text{indegree}(x) : x \in V(D)\} = 1$. Thus, from the necessary conditions established above, we have the following.

**Lemma 1.2** *Let $D \in \{D_1, D_2, \ldots, D_9\}$ and let $v \geq 7$ be a positive integer. There exists a $D$-decomposition of $K_v^*$ only if $v(v-1) \equiv 0 \pmod 7$.*

The remainder of this paper is dedicated to showing the existence of the decompositions in question in order to establish sufficiency of these necessary conditions. Henceforth, each of the graphs $D_i$ with $i \in [1, 9]$ in Figure 1, with vertices labeled as in the figure, will be represented by $D_i[v_0, v_1, \ldots, v_6]$. For example, $D_3[v_0, v_1, \ldots, v_6]$ is the graph with vertex set $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$ and arc set $\{(v_1, v_0), (v_1, v_2), (v_3, v_2), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_0)\}$.

The following result of Alspach and Gavlas proves the existence of 7-cycle decompositions of complete graphs.

**Theorem 1.3 ([3])** *Let $v \geq 7$ be an integer. There exists a $C_7$-decomposition of $K_v$ if and only if $v \equiv 1$ or $7 \pmod{14}$.*

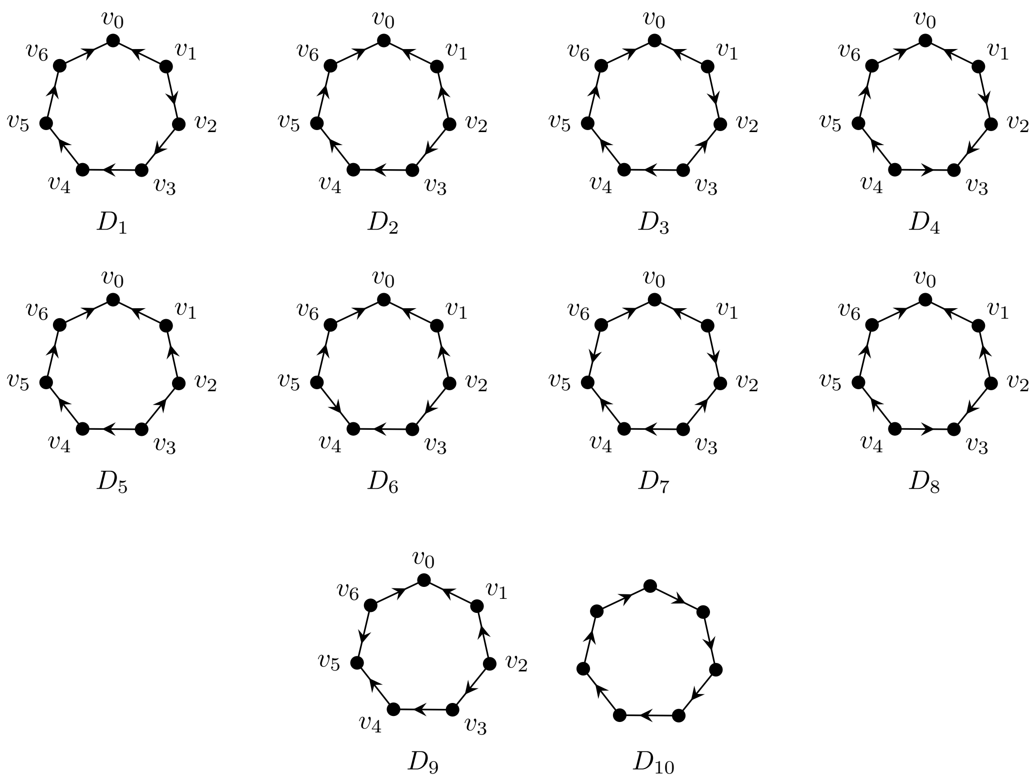


**Figure.** The ten oriented heptagons.

In [11, 12], Liu constructs cycle decompositions of complete equipartite graphs. Even though Liu's result focuses on resolvable decompositions, we will only make use of the following particular instances of his result.

**Proposition 1.4 ([11, 12])** *Let $n \geq 3$ be an odd integer. There exists a $C_7$-decomposition of $K_{n\times 7}$.*

Since each of the oriented 7-cycles $D_i$ is isomorphic to its own reverse for $i = 1, 2, 3, 4, 5, 6,$ and $7$ (see Figure 1), we have the following result.

**Lemma 1.5** *If a graph $G$ has a $C_7$-decomposition, then $G^*$ has a $D_i$-decomposition for $i \in [1, 7]$.*

Theorem 1.3 together with Lemma 1.5 give the following result.

**Corollary 1.6** *If $v \equiv 1$ or $7 \pmod{14}$, then $K_v^*$ has a $D_i$-decomposition for $i \in [1, 7]$.*

Similarly, Proposition 1.4 together with Lemma 1.5 give the following result.

**Corollary 1.7** *For all odd integers $n \geq 3$, there exists a $D_i$-decomposition of $K_{n\times 7}^*$ for $i \in [1, 7]$.*

We also make use of the following results in the next section. All of these results can be found in Handbook of Combinatorial Designs [7] (see [1] and [8]).

**Theorem 1.8** *Let $v \geq 3$ be an odd integer. There exists a $\{K_3, K_5\}$-decomposition of $K_v$.*

**Theorem 1.9** *Let $v \geq 6$ be an even integer. There exists a $\{K_3, K_5\}$-decomposition of $K_v - I$ where $I$ is a $1$-factor of $K_v$.*

## 2. Small designs

We first present several $D_i$-decompositions of various graphs for $i \in [1,8]$. Beyond establishing the existence of necessary base cases, these decompositions are used extensively in the general constructions seen in Section 3.

For given integers $v_0, v_1, \ldots, v_6$ and some $i \in \mathbb{Z}_n$, we define $D[v_0, v_1, \ldots, v_6]+i$ to indicate $D[v_0+i, v_1+i, \ldots, v_6+i]$ where all addition is performed in $\mathbb{Z}_n$. By convention, we define $\infty + i = \infty$.

**Example 2.1** *Let $V\big(K_7^*\big) = \mathbb{Z}_6 \cup \{\infty\}$ and let*

$$\Delta_8 = D_8[0,1,3,4,2,5,\infty] + i : i \in \mathbb{Z}_6. \tag{2.1}$$

*Then $\Delta_8$ is a $D_8$-decomposition of $K_7^*$.*

**Example 2.2** *Let $V\big(K_8^*\big) = \mathbb{Z}_8$ and let*

$$\begin{aligned}
\Delta_1 &= \{D_1[0,2,3,7,4,6,5] + i : i \in \mathbb{Z}_8\},\\
\Delta_2 &= \{D_2[0,5,6,3,1,2,4] + i : i \in \mathbb{Z}_8\},\\
\Delta_3 &= \{D_3[0,2,3,7,4,6,1] + i : i \in \mathbb{Z}_8\},\\
\Delta_4 &= \{D_4[0,1,3,7,4,5,2] + i : i \in \mathbb{Z}_8\},\\
\Delta_5 &= \{D_5[0,1,3,2,5,7,4] + i : i \in \mathbb{Z}_8\},\\
\Delta_6 &= \{D_6[0,1,3,4,7,2,6] + i : i \in \mathbb{Z}_8\},\\
\Delta_7 &= \{D_7[0,1,2,4,7,3,6] + i : i \in \mathbb{Z}_8\},\\
\Delta_8 &= \{D_8[0,1,3,7,2,4,5] + i : i \in \mathbb{Z}_8\}.
\end{aligned}$$

*Then $\Delta_i$ is a $D_i$-decomposition of $K_8^*$ for $i \in [1,8]$.*

**Example 2.3** *Let $V\big(K_{14}^*\big) = \mathbb{Z}_{13} \cup \{\infty\}$ and let*

$$\begin{aligned}
\Delta_1 &= \{\{D_1[0,1,12,2,11,3,9] \cup D_1[0,3,4,12,1,8,\infty]\} + i : i \in \mathbb{Z}_{13}\},\\
\Delta_2 &= \{\{D_2[0,1,12,2,11,3,9] \cup D_2[0,2,5,6,1,8,\infty]\} + i : i \in \mathbb{Z}_{13}\},\\
\Delta_3 &= \{\{D_3[0,1,12,2,11,3,9] \cup D_3[0,5,6,4,7,1,\infty]\} + i : i \in \mathbb{Z}_{13}\},\\
\Delta_4 &= \{\{D_4[0,1,12,2,11,3,5] \cup D_4[0,3,4,10,1,8,\infty]\} + i : i \in \mathbb{Z}_{13}\},\\
\Delta_5 &= \{\{D_5[0,1,12,2,11,3,9] \cup D_5[0,2,1,6,9,3,\infty]\} + i : i \in \mathbb{Z}_{13}\},\\
\Delta_6 &= \{\{D_6[0,1,12,2,11,3,9] \cup D_6[0,2,5,6,11,4,\infty]\} + i : i \in \mathbb{Z}_{13}\},\\
\Delta_7 &= \{\{D_7[0,1,12,2,11,3,9] \cup D_7[0,5,6,\infty,7,1,11]\} + i : i \in \mathbb{Z}_{13}\},\\
\Delta_8 &= \{\{D_8[0,1,12,2,11,3,4] \cup D_8[0,2,5,11,3,10,\infty]\} + i : i \in \mathbb{Z}_{13}\}.
\end{aligned}$$

*Then $\Delta_i$ is a $D_i$-decomposition of $K_{14}^*$ for $i \in [1,8]$.*

**Example 2.4** *Let* $V(K_{15}^{*}) = \mathbb{Z}_{15}$ *and let*

$$\Delta_8 = \{D_8[0,1,3,5,2,14,4] \cup D_8[0,14,10,1,6,13,7]\} + i : i \in \mathbb{Z}_{15}\}.$$

*Then* $\Delta_8$ *is a* $D_8$*-decomposition of* $K_{15}^{*}$.

**Example 2.5** *Let* $V(K_{28}^{*}) = \mathbb{Z}_{27} \cup \{\infty\}$ *and let*

$$\begin{aligned}
\Delta_1 = \{\{&D_1[0,1,26,2,25,3,23] \cup D_1[0,3,25,4,23,5,26] \cup \\
&D_1[0,9,16,6,14,3,13] \cup D_1[0,12,14,26,10,23,\infty]\} + i : i \in \mathbb{Z}_{27}\},\\
\Delta_2 = \{\{&D_2[0,1,26,2,25,3,23] \cup D_2[0,3,19,4,23,5,26] \cup \\
&D_2[0,5,16,6,14,3,13] \cup D_2[0,2,14,20,6,13,\infty]\} + i : i \in \mathbb{Z}_{27}\},\\
\Delta_3 = \{\{&D_3[0,1,26,2,25,3,23] \cup D_3[0,16,19,4,23,5,26] \cup \\
&D_3[0,5,12,6,14,3,13] \cup D_3[0,9,11,21,6,19,\infty]\} + i : i \in \mathbb{Z}_{27}\},\\
\Delta_4 = \{\{&D_4[0,1,26,2,24,3,25] \cup D_4[0,3,23,4,22,5,20] \cup \\
&D_4[0,9,5,16,4,17,10] \cup D_4[0,8,9,13,26,15,\infty]\} + i : i \in \mathbb{Z}_{27}\},\\
\Delta_5 = \{\{&D_5[0,1,26,2,25,3,23] \cup D_5[0,2,14,4,23,5,26] \cup \\
&D_5[0,5,16,26,7,10,21] \cup D_5[0,11,4,17,3,15,\infty]\} + i : i \in \mathbb{Z}_{27}\},\\
\Delta_6 = \{\{&D_6[0,1,26,2,25,3,23] \cup D_6[0,2,5,26,18,1,11] \cup \\
&D_6[0,9,21,26,5,18,19] \cup D_6[0,14,2,13,22,15,\infty]\} + i : i \in \mathbb{Z}_{27}\},\\
\Delta_7 = \{\{&D_7[0,1,26,2,25,3,24] \cup D_7[0,5,26,6,25,16,15],\\
&D_7[0,10,26,11,18,1,14] \cup D_7[0,16,18,9,\infty,23,19]\} + i : i \in \mathbb{Z}_{27}\},\\
\Delta_8 = \{\{&D_8[0,1,26,2,25,3,21] \cup D_8[0,2,5,26,3,25,18],\\
&D_8[0,8,18,7,19,2,16] \cup D_8[0,15,2,3,23,4,\infty]\} + i : i \in \mathbb{Z}_{27}\}.
\end{aligned}$$

*Then* $\Delta_i$ *is a* $D_i$*-decomposition of* $K_{28}^{*}$ *for* $i \in [1,8]$.

**Example 2.6** *Let* $V(K_{29}^{*}) = \mathbb{Z}_{29}$ *and let*

$$\begin{aligned}
\Delta_8 = \{\{&D_8[0,3,21,4,20,5,28] \cup D_8[0,27,5,26,16,25,21],\\
&D_8[0,26,9,27,8,28,23] \cup D_8[0,22,23,21,5,20,24]\} + i : i \in \mathbb{Z}_{29}\}.
\end{aligned}$$

*Then* $\Delta_8$ *is a* $D_8$*-decomposition of* $K_{29}^{*}$.

**Example 2.7** *Let* $V(K_{3\times 7}^{*}) = \mathbb{Z}_{21}$ *with vertex partition* $\{V_i : i \in \mathbb{Z}_3\}$, *where* $V_i = \{j \in \mathbb{Z}_{21} : j \equiv i \pmod 3\}$. *Let*

$$\Delta_8 = \{\{D_8[0,1,5,12,4,17,10] \cup D_8[0,2,7,12,8,18,20]\} + i : i \in \mathbb{Z}_{21}\}.$$

*Then* $\Delta_8$ *is a* $D_8$*-decomposition of* $K_{3\times 7}^{*}$.

**Example 2.8** *Let* $V(K^*_{5\times 7}) = \mathbb{Z}_{35}$ *with vertex partition* $\{V_i : i \in \mathbb{Z}_5\}$, *where* $V_i = \{j \in \mathbb{Z}_{35} : j \equiv i \pmod 5\}$. *Let*

$$\Delta_8 = \{\{D_8[0,1,3,19,2,16,17] \cup D_8[0,3,7,18,6,8,16] \cup D_8[0,6,13,19,12,1,14] \cup D_8[0,8,17,20,16,4,13]\} + i : i \in \mathbb{Z}_{35}\}.$$

*Then* $\Delta_8$ *is a* $D_8$*-decomposition of* $K^*_{5\times 7}$.

## 3. General constructions

For two edge-disjoint graphs (or directed graphs) $G$ and $H$, we use $G \cup H$ to denote the graph (or directed graph) with vertex set $V(G) \cup V(H)$ and edge (or arc) set $E(G) \cup E(H)$. Furthermore, given a positive integer $x$, we use $xG$ to denote the vertex-disjoint union of $x$ copies of $G$.

We now give our constructions for decompositions of $K^*_v$ in Lemmas 3.2, 3.3, 3.5, and 3.6 which cover values of $v$ working modulo 14. The main result is summarized in Theorem 3.7.

**Lemma 3.1** *For every integer* $x \geq 3$ *and each* $i \in [1,8]$, *there exists a* $D_i$*-decomposition of* $K^*_{(2x)\times 7} - xK^*_{7,7}$.

**Proof** By Theorem 1.9, there exists a $\{K^*_3, K^*_5\}$-decomposition of $K^*_{2x} - I^*$. Replacing each vertex of $K^*_{2x}$ by a set of 7 vertices and each edge of $K^*_{2x} - I^*$ by a copy of $K^*_{7,7}$ gives a $\{K^*_{3\times 7}, K^*_{5\times 7}\}$-decomposition of $K^*_{(2x)\times 7} - xK^*_{7,7}$. By Proposition 1.4, there exists a $D_i$-decomposition of $K^*_{3\times 7}$ and $K^*_{5\times 7}$ for $i \in [1,7]$. Also by Examples 2.7 and 2.8, there exist $D_8$-decompositions of $K^*_{3\times 7}$ and $K^*_{5\times 7}$, respectively. Thus, the result now follows. □

**Lemma 3.2** *Let* $D \in \{D_1, D_2, \ldots, D_8\}$. *There exists a* $D$*-decomposition of* $K^*_v$ *for all* $v \equiv 0 \pmod{14}$.

**Proof** Let $D \in \{D_1, D_2, \ldots, D_8\}$ and let $v = 14x$ for some positive integer $x$. For $v = 14$ and $v = 28$, the results follow from Example 2.3 and 2.5, respectively, so we now consider when $x \geq 3$.

Let $K^*_{14x}$ have vertex partition $\{H_j : 1 \leq j \leq 2x\}$, where $|H_j| = 7$ for each $j \in [1,2x]$. For each $j \in [1,x]$, $H_{2j-1} \cup H_{2j}$ induces a $K^*_{14}$, and let $B_j$ be this induced subgraph $K^*_{14x}[H_{2j-1} \cup H_{2j}]$. It is simple to see that $K^*_{14x} - \bigcup_{j=1}^{x} B_j = K^*_{(2x)\times 7} - xK^*_{7,7}$. Thus, we have a decomposition of $K^*_{14x}$ into $x$ copies of $K^*_{14}$ and one copy of $K^*_{(2x)\times 7} - xK^*_{7,7}$. Since both $K^*_{14}$ and $K^*_{(2x)\times 7} - xK^*_{7,7}$ have a $D$-decomposition by Example 2.3 and Lemma 3.1, respectively, we have our desired decomposition of $K^*_v$. □

**Lemma 3.3** *Let* $D \in \{D_1, D_2, \ldots, D_8\}$. *There exists a* $D$*-decomposition of* $K^*_v$ *for all* $v \equiv 1 \pmod{14}$.

**Proof** For $D \in \{D_1, D_2, \ldots, D_7\}$, the result follows from Corollary 1.6. Thus, it remains to prove the lemma for $D = D_8$.

Let $D = D_8$. In the case $v = 15$ and $v = 29$, the results follow from Examples 2.4 and 2.6, respectively. Now, we let $v = 14x + 1$ for some integer $x \geq 3$.

Let $K^*_{14x+1}$ have vertex partition $\{H_j : 1 \leq j \leq 2x\} \cup \{\infty\}$, where $|H_j| = 7$ for each $j \in [1,2x]$. For each $j \in [1,x]$, $H_{2j-1} \cup H_{2j}$ along with the vertex $\infty$ induces a $K^*_{15}$, and let $B_j$ be the induced subgraph $K^*_{14x+1}[H_{2j-1} \cup H_{2j} \cup \{\infty\}]$. Removing union of $B_j$'s from $K^*_{14x+1}$ results in $K^*_{(2x)\times 7} - xK^*_{7,7}$. Thus, $K^*_{14x+1}$

decomposes into $x$ copies of $K_{15}^*$ and one copy of $K_{(2x)\times 7}^* - xK_{7,7}^*$. Since both $K_{15}^*$ and $K_{(2x)\times 7}^* - xK_{7,7}^*$ have a $D$-decomposition by Example 2.4 and Lemma 3.1, respectively, we have our desired decomposition of $K_v^*$. □

**Lemma 3.4** *For every integer $x \geq 1$ and each $i \in [1,8]$, there exists a $D_i$-decomposition of $K_{(2x+1)\times 7}^*$.*

**Proof** By Theorem 1.8, there exists a $\{K_3^*, K_5^*\}$-decomposition of $K_{2x+1}^*$. Replacing each vertex of $K_{2x+1}^*$ by a set of 7 vertices and each edge of $K_{2x+1}^*$ by a copy of $K_{7,7}^*$ gives a $\{K_{3\times 7}^*, K_{5\times 7}^*\}$-decomposition of $K_{(2x+1)\times 7}^*$. By Proposition 1.4, there exists a $D_i$-decomposition of $K_{3\times 7}^*$ and $K_{5\times 7}^*$ for $i \in [1,7]$. Also by Examples 2.7 and 2.8, there exist $D_8$-decompositions of $K_{3\times 7}^*$ and $K_{5\times 7}^*$, respectively. The result then follows. □

**Lemma 3.5** *Let $D \in \{D_1, D_2, \ldots, D_8\}$. There exists a $D$-decomposition of $K_v^*$ for all $v \equiv 7 \pmod{14}$.*

**Proof** By Corollary 1.6, we have the result for $D \in \{D_1, D_2, \ldots, D_7\}$. Thus, it remains to prove the lemma for $D = D_8$.

Let $D = D_8$. When $v$ is 7 the result follows from Example 2.1. Now, we let $v = 14x + 7$ for some integer $x \geq 1$.

Let $H_1, H_2, \ldots, H_{2x+1}$ be disjoint sets of 7 vertices each, and let $K_{(2x+1)\times 7}^*$ have vertex partition $\{H_j : 1 \leq j \leq 2x+1\}$. Now consider $K_{14x+7}^*$ to have vertex set $\bigcup_{j=1}^{2x+1} H_j$ where each $H_j$ induces a $K_7^*$. Thus, $K_{14x+7}^*$ decomposes into copies of $K_7^*$ and one copy of $K_{(2x+1)\times 7}^*$. Since both $K_7^*$ and $K_{(2x+1)\times 7}^*$ have a $D$-decomposition by Example 2.1 and Lemma 3.4, respectively, we have our desired decomposition of $K_v^*$. □

**Lemma 3.6** *Let $D \in \{D_1, D_2, \ldots, D_8\}$. There exists a $D$-decomposition of $K_v^*$ for all $v \equiv 8 \pmod{14}$.*

**Proof** Let $D \in \{D_1, D_2, \ldots, D_8\}$. For $v = 8$, the result follows from Example 2.2. Now, we let $v = 14x + 8$ for some integer $x \geq 1$.

Here we can consider $V(K_{14x+8}^*) = \left(\bigcup_{j=1}^{2x+1} H_j\right) \cup \{\infty\}$, where each $H_j$ is defined as in the proof of Lemma 3.5 with the modification that each $H_j \cup \{\infty\}$ induces a $K_8^*$. Similar to the proof of Lemma 3.5, the desired $D$-decomposition of $K_{14x+8}^*$ can be constructed using $D$-decompositions of $K_8^*$ and $K_{(2x+1)\times 7}^*$, which exist by Example 2.2 and Lemma 3.4, respectively. □

Since $D_9$ is the reverse orientation of $D_8$, existence of a $(K_v^*, D_9)$-design is equivalent to the existence of a $(K_v^*, D_8)$-design. Thus, combining the previous results from Lemmas 3.2, 3.3, 3.5, and 3.6, we obtain the following necessary and sufficient conditions for the existence of a decomposition of $K_v^*$ into the oriented heptagons.

**Theorem 3.7** *Let $D$ be an oriented heptagon. There exists a $D$-decomposition of $K_v^*$ if and only if $v \equiv 0$ or $1 \pmod{7}$.*

## 4. Conclusion and future work

In this article, we considered the spectrum problem for each of the ten nonisomorphic orientations of a heptagon. The necessary condition for such a decomposition is that $v(v-1) \equiv 0 \pmod{7}$. We have shown that this necessary condition is also sufficient.

For future work, it would be interesting to consider a generalization of our problem for the $\lambda$-fold complete symmetric directed graphs.

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