b-Property of sublattices in vector lattices

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Abstract

We study b-property of a sublattice (or an order ideal) F of a vector lattice E. In particular, b-property of E in E^δ, the Dedekind completion of E, b-property of E in E^u, the universal completion of E, and b-property of E in $\hat{E}(\hat{\tau})$, the completion of E.

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1 Introduction and preliminaries

Vector lattices considered here are all real and Archimedean. Vector topologies are assumed to be Hausdorff.

Definition 1. A sublattice F of a vector lattice E is said to have *b-property* in E, if x_α is a net in F^+ and $0 \leq x_\alpha \uparrow\leq e$ for some $e \in E$, then there exists $f \in F$ with $0 \leq x_\alpha \uparrow\leq f$.

Recall that a subset F of E is said to be majorizing in E if, for each $0 < e \in E$, there exists $f \in F$ with $0 \leq e \leq f$.

A subset U of a vector lattice (VL) is called *solid* if $|u| \leq |v|$, $v \in U$, imply $u \in U$. A linear topology τ on a VL E is called *locally solid* if τ has a base of zero consisting of solid sets.

A locally solid VL E (LSVL) satisfies the *Lebesgue property* if $x_\alpha \downarrow 0$ in E implies $x_\alpha \xrightarrow{\tau} 0$.

A LSVL $E(\tau)$ satisfies the *Fatou property* if τ has a base of zero consisting of solid and order closed sets.

A sublattice F in a VL E is *regular* if $\inf A$ is the same as in F and E whenever $A \subset F$ whose infimum exists in F. Ideals are regular in E.

E is called *laterally σ-complete* if the supremum of every disjoint sequence exists in E^+ and *laterally complete* if supremum of every disjoint subset in E^+ exists in E.

A vector lattice E which is both Dedekind (σ-) complete and laterally (σ-) complete is called *universally* (σ-) *complete*.

Example 1. [1, p.198] Let X be a topological space. A function $f : X \to \mathbb{R}$ is called a *step function* if there exists a collection of mutually disjoint subsets $\{V_i\}$ of X such that $\bigcup_i V_i = X$, f is constant on each V_i, and $f \in C^\infty(X)$. Let $S^\infty(X)$ be the space of step functions on an extremally disconnected topological space X. Then $S^\infty(X)$ is a laterally complete VL.

Universal (σ-) completion of a VL E is a laterally (σ-) complete and Dedekind (σ-) complete vector lattice E^u which contains E as an order dense sublattice. Every VL E has a unique universal completion [1, Theorem 7.21]

Lateral completion E^λ of a VL E is defined to be the intersection of all laterally complete vector lattices between E and E^u.

Example 2. Let X be an extremally disconnected topological space. $C^{\infty}(X)$, the space of all extended continuous functions on X with the usual algebraic and lattice operations is a universally complete VL.

A net $(x_\alpha)_{\alpha\in A}$ in a VL E is *order convergent* to $x \in E$ if there exists a net $(x_\beta)_{\beta\in B}$, possibly over a different index set, such that $x_\beta \downarrow 0$ and, for each $\beta \in B$, there exists $\alpha_0 \in A$ with $|x_\alpha - x| \leq x_\beta$ for all $\alpha \geq \alpha_0$. In this case we write $x_\alpha \xrightarrow{o} x$.

A net x_α in E *uo-converges* to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$. In this case we write $x_\alpha \xrightarrow{uo} x$.

Let $E(\tau)$ be a LSVL. A net x_α in E is *$u\tau$-convergent* to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{\tau} 0$ for all $u \in E^+$. A net x_α in E is called *order Cauchy* (*uo-Cauchy*) if the doubly indexed net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha')}$ is order convergent (uo-convergent) to zero. $E(\tau)$ is called *uo-complete* if every uo-Cauchy net is uo-convergent in E.

The b-property of a VL E was defined in [2] as: a VL E has *b-property* if every subset A in E which is order bounded in $(E^{\sim})^{\sim}$, remains to be order bounded in E. We say that a vector sublattice F of a VL E has (countable) b-property in E whenever each (sequence) net f_α in F, with $0 \leq f_\alpha \uparrow \leq e$ for some $e \in E$, is order bounded in F (cf. e.g. [2], [3, p.766]).

Example 3. Every perfect VL, and therefore every order dual, have the b-property. Every reflexive BL and every KB-space have b-property [2, 3, 4, 5]. On the other hand, by considering the basis vectors e_n in c_0, we see that c_0 does not have the b-property in l_∞.

Let us note that Fremlin had considered subsets of a VL E that are order bounded in the universal completion

E^u of E. He proved that if E is a Dedekind σ-complete VL then E is laterally σ-complete iff E has the countable b-property in E^u [1, Theorem 7.38]. That is, each sequence x_n in E with $0 \leq x_n \uparrow\leq e$ for some $e \in E^u$ has an upper bound in E,

Example 4. Each projection band F in a vector lattice E has b-property in E. In particular, every band in a Dedekind complete vector lattice has b-property. An element u in a VL E is called an *atom* if whenever $v \wedge w = 0$, $0 \leq v \leq u$, and $0 \leq w \leq u$ imply either $v = 0$ or $w = 0$. If x is an atom in E, the principal band B_x generated by x is a projection band and therefore has b-property in E.

Example 5. Every majorizing sublattice F has b-property in E. Let $0 \leq x_\alpha \uparrow\leq e$ for some net $x_\alpha \subseteq F$, $e \in E$. As F is majorizing, there exists $f \in F$ with $e \leq f$. Then $0 \leq x_\alpha \leq f$. Since it is well-known that E is majorizing in the Dedekind completion E^δ, the lattice E has b-property in E^δ.

Example 6. Every order ideal F in a vector lattice E with b-property in E is a band of E. Indeed, let x_α be a net in F such that $0 \leq x_\alpha \uparrow e \in E$, then the net x_α is order bounded in F, say $0 \leq x_\alpha \uparrow\leq f \in F$, by the b-property of F in E. Hence $0 \leq e \leq f$ and as F is an ideal, $e \in F$.

Example 7. Let $E \subseteq F$ be a sublattice of F and $I(E)$ be the ideal generated by E in F. Then E has b-property in $I(E)$. Having b-property is transitive: if $E \subseteq F \subseteq G$ are sublattices of a VL X such that E has b-property in F and F has b-property in G, then E has b-property in G. If E has b-property in G, then E has b-property in every sublattice of G containing E as a sublattice.

Example 8. Let F be a norm-closed sublattice of a Banach lattice $(E, \|.\|)$ with order continuous norm. Let x_n be a sequence in F such that $0 \leq x_n \uparrow\leq e$ for some $e \in E$. Then x_n is norm-convergent to some $x \in E$. As F is norm-closed, $x \in F$. Since $x_n \leq x$ for all n, then F has countable b-property in E. Order continuity of the ambient space is essential in this example, if one takes $E = l^\infty$ and $F = c_0$, Then by considering the sequence e_n in c_0, we see that c_0 has no b-property in l^∞.

Example 9. Generalizing Example 8, let $E(\tau)$ be an LSVL with Lebesgue property. Then every τ-closed order ideal F has b-property in $E(\tau)$. This is because every τ-closed ideal is a band and, as $E(\tau)$ is Dedekind complete, it is a projection band.

Example 10. Given a VL E, let us denote by E^λ its lateral completion and E^u its universal completion. Since X is majorizing in X^δ by Example 5, the equality $(E^\lambda)^\delta = (E^\delta)^\lambda = E^u$ (see [1, Exer.10 on p.213]) shows that E^λ is majorizing in E^u and therefore each laterally complete VL E has b-property in its universal completion E^u.

Example 11. If E is a laterally complete VL, then it has the band projection property and every band on E has b-property. Furthermore, a subset $A \subset E^+$ of a laterally complete VL E is order bounded in E^u iff A is order bounded in E by [1, Theorems 7.14 and 7.37].

Let us observe that all Lebesgue topologies on a LSVL $E(\tau)$ induce the same topology on order bounded subsets of E. Therefore, if F is a sublattice of E then on all subsets of F with b-property in E all Lebesgue topologies on E induce the same topology.

Example 12. Let F be an order dense sublattice of a vector lattice E. If F is laterally complete in its own right, then F majorizes E and therefore has b-property in E.

We refer to [1, 10] for all undefined terms.

2 Main results

Lemma 1. *Let F be a sublattice of a LSVL $E(\tau)$. Then each b-bounded in E subset B of F is τ-bounded with respect to induced topology on F.*

Proof. To say that B is b-bounded in E is to say that B is order bounded in E. So, if U is a neighborhood of 0 in τ then $B \subseteq \lambda U$ for some $\lambda > 0$. Then $B \subseteq \lambda U \cap F = \lambda(U \cap F)$. □

Lemma 2. *Let E be a vector lattice and F be an order dense sublattice of E. Then TFAE:*

i) F has b-property in E;

ii) F is majorizing in E.

Proof. $i) \implies ii)$: Let $0 \leq x \in E$ be arbitrary, as F is order dense in E, there exists a net x_α in F such that $0 \leq x_\alpha \uparrow x$. As x_α is b-bounded in E by assumption, there exists $x_0 \in F^+$ with $0 \leq x_\alpha \leq x_0$ for all α, as $x_\alpha \uparrow x$, we have $x \leq x_0$ and F is majorizing.

$ii) \implies i)$: Let x_α be a net in F with $0 \leq x_\alpha \uparrow\leq x$ for some $x \in E$. Since F is assumed to be majorizing E, there exists $y \in F$ with $x \leq y$. Consequently, $0 \leq x_\alpha \uparrow\leq y \in F$ and hence F has b-property in E. □

This yields: E has b-property in E^u iff E is majorizing in E^u. We also have, if $E(\tau)$ is a LSVL where E is an ideal of $\hat{E}(\hat{\tau})$ and $\hat{E}$ is the completion, that E has b-property in $\hat{E}(\hat{\tau})$.

On the other hand, if $E(\tau)$ is a LSVL with Fatou property, then every increasing τ-bounded net of E^+ is order bounded in E^u i.e. every increasing τ-bounded net of E^+ is b-bounded in E^u by [1, Theorem 7.51]

The following property was introduced in [8] and [9].

Definition 2. A locally solid vector lattice $E(\tau)$ is called *boundedly order bounded (BOB)* if every τ-bounded net in E^+ is order bounded in E.

We show BOB is equivalent to b-property if the LSVL $E(\tau)$ has Fatou property.

Lemma 3. *Let $E(\tau)$ be a LSVL with Fatou property. Then E has b-property in E^u iff E is BOB.*

Proof. Suppose E is BOB and x_α be a net in E with $0 \leq x_\alpha \uparrow\leq x_0$ for some $x_0 \in E^u$. Then, by Lemma 1, x_α is τ-bounded in E and, by assumption that E is BOB, $0 \leq x_\alpha \leq x$ for some $x \in E$.

Conversely, suppose x_α is τ-bounded increasing net in E^+, then by [1, Theorem 7.50], x_α is order bounded in E^u. Thus by b-property of E in E^u, there exists $x \in E$ with $0 \leq x_\alpha \leq x$ and $E(\tau)$ is BOB. □

[1, Theorem 7.49] shows that, in a laterally σ-complete LSVL $E(\tau)$, every disjoint sequence in E^+ converges to zero with respect to any LS topology on E. We show a similar result. The proof is similar.

Proposition 1. *Let $E(\tau)$ be a LSVL which has countable b-property in its lateral σ-completion. Then every disjoint* *sequence in E^+ converges to zero with respect to any locally solid topology on E. In particular, every locally solid topology on E has the pre-Lebesgue property.*

Proof. Let x_n be a disjoint sequence in E^+. Then nx_n is also a disjoint sequence in E^+. Then $x = \bigvee_{n=1}^{\infty} nx_n$ exists in the lateral σ-completion, and we have $0 \leq x_n \leq \frac{1}{n}x$ for all n. Countable b-property of E in its lateral completion yields a vector $e \in E$ with $0 \leq x_n \leq \frac{1}{n}e$ for all n. Thus x_n converges to zero with respect to any locally solid topology on E. □

Recall that E has a countable b-property in its lateral completion E^λ if, for each x_n with $0 \leq x_n \uparrow\leq e$ for some $e \in E^\lambda$, there holds $x_n \uparrow\leq x \in E$.

Corollary 1. *Let $E(\tau)$ be an LSVL with Lebesgue property. If E has countable b-property in its lateral σ-completion then the topological completion $\hat{E}$ of $E(\tau)$ is E^u.*

Proof. Under the given conditions, every disjoint sequence in E^+ is τ-convergent to zero by Proposition 1. Thus the corollary follows from [1, Theorem 7.51]. □

Proposition 2. *A laterally complete vector lattice E has b-property in every vector lattice which contains E as an order dense sublattice.*

Proof. In this case, E majorizes the vector lattice that contains it. The result now follows from [1, Theorem 7.15]. □

In [11, Proposition 2.22] it is proved that if $E(\tau)$ is a LSVL with Lebesgue topology, then a sublattice F of E is $u\tau$-closed in E iff it is τ-closed. It was asked in

[11, Question 2.24] whether Lebesgue assumption could be removed. The next result yields an answer utilizing b-property.

Theorem 1. *Let F be an order ideal of an LSVL $E(\tau)$. If F has b-property in E, then F is $u\tau$-closed iff it is τ-closed in E.*

Proof. As $u\tau$ is coarser than τ, the forward implication is clear.

We will show $x \in F$. Suppose F is τ-closed and y_α is a net in F with $y_\alpha \xrightarrow{u\tau} x$ for some $x \in E$. The lattice operations are $u\tau$-continuous, so that $y_\alpha^\pm \xrightarrow{u\tau} x$. Therefore, WLOG we may assume $0 \leq y_\alpha$ for all α. Let $z \in E^+$ be arbitrary, then

$$|y_\alpha \wedge z - x \wedge z| \leq |y_\alpha - x| \wedge z \xrightarrow{\tau} 0.$$

Since $0 \leq y_\alpha \wedge x \leq y_\alpha$ for all α, and F is an order ideal, we have $y_\alpha \wedge x \in F$ for all α and $y_\alpha \wedge x \xrightarrow{\tau} x \wedge x$.

Take $y \in F$, then $y_\alpha \wedge y \xrightarrow{\tau} x \wedge y$, since F is τ-closed we have $x \wedge y \in F$ for each $y \in F^+$. If $z \in F^d$, then $y_\alpha \wedge z = 0$ for all α and we have $x \wedge z = 0$. Thus $x \in F^{dd}$. That is, x is in the band generated by F in E. Hence there exists a net z_β in F^+ such that $0 \leq z_\beta \uparrow |x|$. Therefore z_β is b-bounded in E, by b-property of F in E, $0 \leq z_\beta \leq x_0$ for some $x_0 \in F$ and $|x| \leq x_0$. Hence $x \in F$ as F is an ideal. □

It shown in [1, Theorem 7.39] that a Dedekind complete vector lattice is universally complete iff it is universally σ-complete and has a weak unit. In the next result, we replace universally σ-completeness with countable b-property of E in E^u.

Theorem 2. *Let E be a Dedekind complete vector lattice. Then E has a weak order unit and possesses countable b-property in E^u iff $E = E^u$.*

Proof. If $E = E^u$ then E has b-property in E^u and has a weak order unit (cf. [1, Theorem 7.2]). Now we prove the converse. Let $0 < e$ be a weak order unit for E. Then E is an order ideal in E^u by [1, Theorem 1.40]. Let $0 < u \in E^u$ be arbitrary. Since e is also a weak unit for E^u (E is order dense in E^u), we have $0 < u \wedge ne \uparrow u$. As $u \wedge ne \in E$ for each n, we see that the sequence $u \wedge ne$ is b-bounded in E^u. Therefore the sequence $u \wedge ne$ has an upper bound in E by the assumption. Thus $0 \leq u \wedge ne \leq x$ for some $x \in E$, and hence $0 \leq u \leq x$. As E is an order ideal in E^u, we have $u \in E$. □

It is well known that if $E(\tau)$ is a LSVL with Levi property and τ-complete order intervals, then E is Dedekind complete. In the following we reach to the same conclusion by replacing Levi property with weaker condition that E having b-property in $\hat{E}(\hat{\tau})$.

Proposition 3. *Let $E(\tau)$ be an LSVL with τ-complete order intervals. If $E(\tau)$ has b-property in the τ-completion $\hat{E}$ of $E(\tau)$, then $E(\tau)$ is τ-complete.*

Proof. The assumption on order intervals implies that $E(\tau)$ is an order dense ideal of $\hat{E}$ by [1, Theorem 2.42]. Let $0 < \hat{x} \in \hat{E}$ be arbitrary. Since $E(\tau)$ is order dense in $\hat{E}$, there exists a net x_α such that $0 \leq x_\alpha \uparrow \hat{x}$. By the b-property of $E(\tau)$ in $\hat{E}$, we can find $x_0 \in E$ with $0 \leq x_\alpha \leq x_0$. But then since $x_\alpha \uparrow \hat{x}$, we have $\hat{x} \leq x_0$ and $\hat{x} \in E$ because E is an ideal in $\hat{E}$. Therefore $E(\tau) = \hat{E}$ as reqired. □

Proposition 4. *Let F be a regular sublattice of a Dedekind complete VL E. Then each increasing net of elements of F which is order bounded in E is uo-Cauchy in F.*

Proof. Let x_α be a net in F such that $0 \leq x_\alpha \uparrow \leq e$ for some $e \in E^+$. Since E is Dedekind complete, $x_\alpha \uparrow x$ for some $x \in E^+$. Then x_α is o-Cauchy in E, hence is uo-Cauchy in E. Therefore x_α is uo-Cauchy in F by [7, Theorem 3.2]. $\square$

It was observed in [7, Theorem 3.2] for a net x_α in a regular sublattice F of a vector lattice E, $x_\alpha \xrightarrow{uo} 0$ in F iff $x_\alpha \xrightarrow{uo} 0$ in E. However this may fail for $u\tau$-convergence. $u\tau$-Convergence in a sublattice may not imply $u\tau$-convergence in the entire space. For example, the standard unit vectors e_n in l^∞ is easily seen to be a null sequence in the unbounded norm topology of c_0 but not so in l^∞.

Proposition 5. *Let F be a sublattice of an LSVL $E(\tau)$. Suppose F has b-property in E. For a net x_α in F for which $x_\alpha \xrightarrow{u\tau} 0$ in F, we have $x_\alpha \xrightarrow{u\tau} 0$ in $E(\tau)$.*

Proof. Suppose $x_\alpha \xrightarrow{u\tau} 0$ in F. WLOG we may suppose $0 \leq x_\alpha$ for all α. Then $0 \leq x_\alpha \wedge y \xrightarrow{\tau} 0$ for each $y \in F^+$. On the other hand, for each $x \in E^+$, $0 \leq x_\alpha \wedge x \leq x$ and the net $0 \leq (x_\alpha \wedge x)$ is b-bounded in F, by the hypothesis, there exists $y \in F^+$ such that $0 \leq x_\alpha \wedge x \leq y$ for all α. Then

$$0 \leq x_\alpha \wedge x \leq x_\alpha \wedge y \xrightarrow{\tau} 0$$

from which we obtain, $x_\alpha \wedge x \xrightarrow{\tau} 0$. As x is arbitrary $x_\alpha \xrightarrow{u\tau} 0$ in $E(\tau)$. $\square$

Proposition 6. *Let $E(\tau)$ be a laterally complete vector lattice, then E has b-property in $(E^\sim)^\sim_n$.*

Proof. Recall that E is order dense in $(E^{\sim})^{\sim}_{n}$. Then E is majorizing in $(E^{\sim})^{\sim}_{n}$ by [1, Theorem 7.15]. Therefore E has b-property in $(E^{\sim})^{\sim}_{n}$ $\square$

Theorem 3. *Let $E(\tau)$ be an LSVL with Lebesgue property. Then every order closed sublattice F of $E(\tau)$ has countable b-property in $\hat{E}(\hat{\tau})$.*

Proof. Let $F^{+} \ni x_n \uparrow\leq \hat{x} \in \hat{E}(\hat{\tau})$. Since the topology $\hat{\tau}$ of $\hat{E}(\hat{\tau})$ is also Lebesgue [1, Theorem 3.26] and hence is pre-Lebesgue, the sequence x_n is $\hat{\tau}$-Cauchy in $\hat{E}(\hat{\tau})$, and therefore. $x_n \xrightarrow{\hat{\tau}} z$ for some $z \in \hat{E}(\hat{\tau})$. Since $\hat{\tau}$ is Fatou by [1, Lemma 4.2], and F being order closed is $\hat{\tau}$-closed by [1, Theorem 4.20], thus $z \in F$. As $x_n \uparrow$, $x_n \xrightarrow{\hat{\tau}} z$, hence $z = \sup x_n$ by [1, Theorem 2.21], and F has countable b-property in $\hat{E}(\hat{\tau})$. $\square$

Proposition 7. *Let F be a uo-closed sublattice of a Dedekind complete vector lattice E. Then F has b-property in E.*

Proof. Let x_α be a net in F with $0 \leq x_\alpha \uparrow\leq x$ for some $x \in E$. As E is Dedekind complete, $x_\alpha \uparrow \hat{x}$ for some $\hat{x} \in E$. Then $x_\alpha \xrightarrow{o} \hat{x}$, consequently $x_\alpha \xrightarrow{uo} \hat{x}$ in E as F is *uo*-complete, $\hat{x} \in F$. $\square$

Notice that Theorem 3 follows from Proposition 7 under an additional assumption that $\hat{E}(\hat{\tau})$ is Dedekind complete.

Theorem 4. *Let E be a vector lattice admitting a minimal topology τ. Let x_n be an increasing sequence of elements of E order bounded in E^u. Then x_n is τ-Cauchy in E.*

Proof. Let x_n be such that $0 \leq x_n \uparrow\leq x^u$ for some $x^u \in E^u$. Since E^u is Dedekind complete, x_n being order bounded in E^u, has a supremum in E^u, let it be x. Therefore $x_n \xrightarrow{o} x$, it follows that x_n is *uo*-Cauchy in E^u. Since E is order dense in E^u, and order dense sublattices are regular, E is regular in E^u and by [7, Theorem 3.2], x_n is *uo*-Cauchy in E. As every minimal topology is Lebesgue, τ is Lebesgue and x_n is $u\tau$-Cauchy. As τ is unbounded, it follows that x_n is τ-Cauchy on E. □

Definition 3. A locally solid vector lattice $E(\tau)$ is called *boundedly uo-complete* if every τ-bounded *uo*-Cauchy net in $E(\tau)$ is *uo*-convergent.

Proposition 8. *A boundedly uo-complete LSVL $E(\tau)$ has b-property in E^u.*

Proof. Let $0 \leq x_\alpha \uparrow\leq x^u$, where $x^u \in E^u$, be a net in E. As x_α is a b-bounded subset of E, it is τ-bounded by Lemma 1. We show x_α has an upper bound in E. As E^u is Dedekind complete, $\sup x_\alpha$ exists in E^u. Let this supremum be x. Then $0 \leq x_\alpha \uparrow x$ in E^u. Thus $x_\alpha \xrightarrow{o} x$. It follows that x_α is *uo*-Cauchy in E as E is order dense and a regular sublattice of E^u. Thus, x_α being *uo*-Cauchy and τ-bounded, x_α *uo*-converges to some $x' \in E$. But as $x_\alpha \xrightarrow{o} x$ we have $x = x'$. □

Definition 4. A Banach lattice is *monotonically complete* (*has the Levy property*) if every norm bounded increasing net in E^+ has supremum.

We now show that every boundedly *uo*-complete Banach lattice E has b-property in $(E_n^\sim)_n^\sim$. The proof uses an idea of [6] in that $(E_n^\sim)_n^\sim$ is monotonically complete and the canonical map $J : E \to (E_n^\sim)_n^\sim$ maps a bounded increasing net in E^+ to a net in $(E_n^\sim)_n^\sim$ with similar properties.

Theorem 5. *Let E be a boundedly uo-complete Banach lattice with $E_n^\sim$ separating points of E. If x_α is an increasing net in E^+ which is order bounded in $(E_n^\sim)_n^\sim$, then x_α is order bounded in E.*

Proof. Since the net x_α is order bounded in $(E_n^\sim)_n^\sim$, it is norm bounded in $(E_n^\sim)_n^\sim$ and hence norm bounded in E by Lemma 1.

Let $J : E \to (E_n^\sim)_n^\sim$ be the natural embedding, where $J(x)(f) = f(x)$ for each $x \in E$ and $f \in E_n^\sim$. The map J is a vector lattice isomorphism and the range $J(E)$ in $(E_n^\sim)_n^\sim$ is order dense in $(E_n^\sim)_n^\sim$ by [1, Theorem 1.43]. Therefore, $J(E)$ is a regular sublattice of $(E_n^\sim)_n^\sim$.

By [10, 2.4.19], $(E_n^\sim)_n^\sim$ is a monotonically complete Banach lattice. Thus, the increasing net $J(x_\alpha)$ has a supremum in $(E_n^\sim)_n^\sim$ say x.

So $J(x_\alpha) \uparrow x$ and $J(x_\alpha)$ is order Cauchy in $(E_n^\sim)_n^\sim$. It follows that $J(x_\alpha)$ is *uo*-Cauchy in $(E_n^\sim)_n^\sim$ and in the regular sublattice $J(E)$. As J is 1-1 and onto $J(E)$ is lattice isomorphism, x_α is *uo*-Cauchy in E. Since E is boundedly *uo*-complete, $x_\alpha \xrightarrow{uo} x_1$ for some $x_1 \in E$. On the other hand $0 \leq x_\alpha \uparrow$ implies $x_\alpha \uparrow x_1$ and hence the net x_α is order bounded in E. □

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