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On Characterization of Tripotent Matrices in Triangular Matrix Rings

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Abstract: Let $\mathfrak{R}$ be a ring with identity 1 whose tripotents are only -1, 0, and 1. It is characterized the structure of tripotents in $\mathcal{T}(\mathfrak{R})$ which is the ring of triangular matrices over $\mathfrak{R}$. In addition, when $\mathfrak{R}$ is finite, it is given number of the tripotents in $\mathcal{T}_n(\mathfrak{R})$ which is the ring of $n\times n$ dimensional triangular matrices over $\mathfrak{R}$ with n being a positive integer.

Key words: Tripotent matrix; triangular matrix; matrix rings.

1. Introduction

An element x of a ring R is called idempotent (or tripotent) if $x^2 = x$ (or $x^3 = x$). Notice that every idempotent element is tripotent. In addition, if x is tripotent, then x^2 is idempotent. In case $x^2 = -x$, the element x is said to be skew-idempotent. And also, every skew-idempotent element is a tripotent element. If $x^2 = e$, then the element x is called involutive, where e is the identity element of R. An element x of a ring R is called an essentially tripotent if $x^3 = x$ and $x^2 \neq \pm x$. So, the set of tripotent elements in a ring covers the sets of idempotent, involutive, skew idempotent, and essentially tripotent. Therefore, studying tripotency is of particular importance.

Special types of matrices such as idempotent, involutive, tripotent, quadratic, triangular, etc., are important concepts in linear algebra, number theory, and matrix theory. In literature, there are many works on the characterization of linear combinations of special type of matrices, see, for instance, [1–7, 10, 15, 16, 18, 19, 21, 23]. Similar works are also studied in ring theory. For example, Hirano and Tominaga proved in [[13], Theorem 1] that a ring R is tripotent if and only if every element of R is a sum of two commuting idempotents. In 2009, Chen et al. worked on rings whose elements can be expressed uniquely as the sum of an idempotent and a unit [8]. In 2016, by Ying et al., the class of these rings was extended in [27] to the class of rings R such that their elements are first the sum of an idempotent and a tripotent that commute, then the sum or difference of commuting two idempotents, and then the sum of two tripotents that commute. In that work, the authors proved that R is tripotent ring if and only if every element x of R is a difference of two commuting idempotents such that $x = \frac{1}{2}(y^2+y) - \frac{1}{2}(y^2-y)$. In 2017, Sheibani and Chen studied on a matrix ring, each element of which is the sum of a tripotent and a nilpotent matrix [22]. In 2018, Zhou first studied on rings

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with each element being the sum of a nilpotent, an idempotent, and a tripotent that commute, and then on rings with each element being the sum of a nilpotent and two tripotent that commute [28]. In 2018, Danchev worked on rings whose elements are the sum of three idempotents or the negative sum of two idempotents that commute [11]. In 2019, Cheraghpour and Ghosseiri calculated the number of idempotents and zero divisors of a matrix ring on a finite field F [9]. In 2019, Tang et al. studied matrices that are the sum of three idempotent and three involutive matrices on a commutative ring [24] inspired by Hirano-Tomigana's work on rings whose each element is the sum of two idempotents (see: [13]) and Seguins Pazzis's work on the decomposition into idempotents of each matrix on an field with positive characteristics (see: [17])

In 2019, Hou, by characterizing the idempotency of triangular matrices over a ring of identity 1 with idempotents only 0 and 1 (trivial idempotents), obtained a result that determines the number of such matrices when the ring is finite [14]. In 2020, Wright characterized the structure of triangular idempotent matrices on more general rings that do not have to be identity element and whose idempotents do not have to be only 0 and 1. [26]. Petik et al. considered a problem similar to that in [14] for involutive matrices in [20].

The motivation of this work comes from the sources: [14] and [20]. Inspired by Hou's work, one may ask: How to characterize tripotent matrices (which contain idempotent and involutive matrices) in triangular matrix rings? And also, how can we count tripotents in finite dimensional matrix rings? Here we address these questions for matrices over a ring with identity 1 whose tripotents are only -1, 0, and 1.

Throughout the work, $\mathfrak{R}$ denotes a ring whose tripotents are only -1, 0, and 1 with identity 1, $|\mathfrak{R}|$ denotes the number of the elements in $\mathfrak{R}$, $\mathcal{T}^U(\mathfrak{R})$ denotes the ring of upper-triangular matrices over the ring $\mathfrak{R}$. Similarly, $\mathcal{T}^L(\mathfrak{R})$ denotes the ring of lower-triangular matrices whose elements are taken from the ring $\mathfrak{R}$. It is clear that $\mathcal{T}(\mathfrak{R}) = \mathcal{T}^U(\mathfrak{R}) \cup \mathcal{T}^L(\mathfrak{R})$. In addition, when we want to emphasize the size of triangular matrices, we will denote the ring of $n \times n$ dimensional upper-triangular matrices over $\mathfrak{R}$ by $\mathcal{T}_n^U(\mathfrak{R})$. Also, the notation $\langle f, g \rangle$ will be used for dot product of the vectors f and g. Finally, $\mathbf{0}$ denotes zero matrix of suitable size.

Moreover, we will also need to use the following concepts in the work: The "super diagonal" of a matrix is the diagonal of entries immediately above the main diagonal. The "sub diagonal" of a matrix is the diagonal of entries immediately below the main diagonal.

2. On Characterization of Tripotent Matrices in Triangular Matrix Rings

In this section, a main result characterizing structure of tripotent matrices in the matrix ring $\mathcal{T}^U(\mathfrak{R})$ is given.

Theorem 2.1 *Let $\mathfrak{R}$ be a ring whose tripotents are only -1, 0, and 1 with identity 1. Then, $X \in \mathcal{T}^U(\mathfrak{R})$ is tripotent if and only if the entries of X have the following structure:*

(i) $x_{ii} \in \{-1, 0, 1\}$ *for all* i.

(ii) For $i < j$, *if* $x_{ii} = x_{jj} = 0$, *then*

$$x_{ij} = \begin{cases} 0, & j = i+1; \\ \sum\limits_{l=i+1}^{j-1} \left(\sum\limits_{m=1}^{l-i} x_{i,i+m} x_{i+m,l} \right) x_{lj}, & j > i+1. \end{cases}$$

(iii) For $i<j$, *if* $x_{ii}=x_{jj}\in\{-1,1\}$, *then*

$$x_{ij}=\begin{cases} 0, & j=i+1;\\ -\frac{1}{2}\left(\sum\limits_{l=i+1}^{j-1}(\sum\limits_{m=0}^{l-i}x_{i,i+m}x_{i+m,l})x_{lj}+\sum\limits_{m=i+1}^{j-1}x_{im}x_{mj}x_{jj}\right), & j>i+1.\end{cases}$$

(iv) For $i<j$, *if* $x_{ii}\neq x_{jj}$, *then* x_{ij} *is arbitrary.*

Proof First, let's prove the necessary part of the theorem.

Let $X\in\mathcal{T}^U(\mathfrak{R})$ be tripotent. Tripotency of the matrix X yields the following system of equations.

$$\begin{aligned}
&x_{ii}^3=x_{ii}\\
&x_{ii}^2x_{i,i+1}+x_{ii}x_{i,i+1}x_{i+1,i+1}+x_{i,i+1}x_{i+1,i+1}^2=x_{i,i+1}\\
&x_{ii}^2x_{i,i+2}+x_{ii}x_{i,i+1}x_{i+1,i+2}+x_{i,i+1}x_{i+1,i+1}x_{i+1,i+2}+x_{ii}x_{i,i+2}x_{i+2,i+2}+\\
&x_{i,i+1}x_{i+1,i+2}x_{i+2,i+2}+x_{i,i+2}x_{i+2,i+2}^2=x_{i,i+2}\\
&\quad\vdots\\
&\sum_{l=0}^{s}\left(\sum_{m=0}^{l}(x_{i,i+m}x_{i+m,i+l})x_{i+l,i+s}\right)=x_{i,i+s}\\
&\quad\vdots
\end{aligned}$$

Let us denote $\rho_{i,i+q}:=x_{ii}^2+x_{ii}x_{i+q,i+q}+x_{i+q,i+q}^2-1$ for each q. So, the system above is equivalent to the following system of equations:

$$\begin{aligned}
&x_{ii}^3=x_{ii}\\
&\rho_{i,i+1}x_{i,i+1}=0\\
&\rho_{i,i+2}x_{i,i+2}+(x_{ii}+x_{i+1,i+1}+x_{i+2,i+2})\,x_{i,i+1}x_{i+1,i+2}=0\\
&\quad\vdots\\
&\rho_{i,i+s}x_{i,i+s}+\sum_{l=1}^{s-1}\left(\sum_{m=0}^{l}x_{i,i+m}x_{i+m,i+l}\right)x_{i+l,i+s}+\sum_{m=1}^{s-1}x_{i,i+m}x_{i+m,i+s}x_{i+s,i+s}=0\\
&\quad\vdots
\end{aligned}\tag{2.1}$$

It is easy to see that $x_{ii}\in\{-1,0,1\}$ from the first equality of the system of equations (2.1) since $\mathfrak{R}$ is a ring whose tripotents are only -1, 0, and 1.

Now, we consider two entries on main diagonal of the matrix X, say x_{ii} and x_{jj} with $i<j$. So, either $j-i=1$ or $j-i>1$.

Firstly, suppose that $j-i=1$. From the second equality of (2.1), we know that

$$(x_{ii}^2+x_{ii}x_{i+1,i+1}+x_{i+1,i+1}^2-1)x_{i,i+1}=0.\tag{2.2}$$

If $x_{ii}=x_{i+1,i+1}\in\{-1,0,1\}$, then from the equality (2.2), we get $x_{i,i+1}=0$. If $x_{ii}\neq x_{i+1,i+1}$, then it seen that $x_{i,i+1}$ is arbitrary from the equality (2.2) since $(x_{ii},x_{i+1,i+1})\in\{(0,1),(0,-1),(-1,0),(-1,1),(1,0),(1,-1)\}$. So, the item (i) is obvious.

Now, we consider the case $j-i>1$, and denote it as $j-i=s$. Note that s can be as many as the super diagonal number of the matrix X at most. We know that $(i,i+s)$. entries of both sides of the equality

1 $X^3 = X$ give $(s+1)$. equality of the system of equations (2.1):

$$\rho_{i,i+s}x_{i,i+s} + \sum_{l=1}^{s-1}\left(\sum_{m=0}^{l} x_{i,i+m}x_{i+m,i+l}\right)x_{i+l,i+s} + \sum_{m=1}^{s-1} x_{i,i+m}x_{i+m,i+s}x_{i+s,i+s} = 0 \tag{2.3}$$

Since $(x_{ii}, x_{i+s,i+s}) \in \{(1,1), (-1,-1), (0,0)\}$ when $x_{ii} = x_{i+s,i+s}$, it is obtained that

$$\rho_{i,i+s} = x_{ii}^2 + x_{ii}x_{i+s,i+s} + x_{i+s,i+s}^2 - 1 = \begin{cases} 2, & x_{ii} \in \{-1,1\}; \\ -1, & x_{ii} = 0. \end{cases}$$

For $x_{ii} = 0$, premultiplying the equality (2.3) by $\left(x_{ii}^2 + x_{ii}x_{i+s,i+s} + x_{i+s,i+s}^2 - 1\right)^{-1}$ leads to

$$x_{i,i+s} + \rho_{i,i+s}\left[\sum_{l=1}^{s-1}\left(\sum_{m=0}^{l} x_{i,i+m}x_{i+m,i+l}\right)x_{i+l,i+s} + \sum_{m=1}^{s-1} x_{i,i+m}x_{i+m,i+s}x_{i+s,i+s}\right] = 0.$$

From this, one can easily obtain the equality

$$x_{i,i+s} = \sum_{l=1}^{s-1}\left(\sum_{m=1}^{l} x_{i,i+m}x_{i+m,i+l}\right)x_{i+l,i+s}.$$

2 Thus, it is obtained the desired result in item (ii) of the theorem.

Next, let us handle the case $x_{ii} \in \{-1,1\}$. In this case, we have the equality $x_{ii}^2 + x_{ii}x_{i+s,i+s} + x_{i+s,i+s}^2 - 1 = 2$. Since $\mathfrak{R}$ is a ring with identity 1, from the last equality, we get $\left(x_{ii}^2 + x_{ii}x_{i+s,i+s} + x_{i+s,i+s}^2 - 1\right)^{-1} = \frac{1}{2}$. If the equality (2.3) is premultiplied by $\left(x_{ii}^2 + x_{ii}x_{i+s,i+s} + x_{i+s,i+s}^2 - 1\right)^{-1}$, and necessary arrangements are made, then it is obtained that

$$x_{i,i+s} = -\frac{1}{2}\left[\sum_{l=1}^{s-1}\left(\sum_{m=0}^{l} x_{i,i+m}x_{i+m,i+l}\right)x_{i+l,i+s} + \sum_{m=1}^{s-1} x_{i,i+m}x_{i+m,i+s}x_{i+s,i+s}\right].$$

3 This proves the item (iii) of the theorem.

Now, suppose that $x_{ii} \neq x_{i+s,i+s}$. It is clear that $x_{ii}^2 + x_{ii}x_{i+s,i+s} + x_{i+s,i+s}^2 - 1 = 0$ since $x_{ii} \in \{-1,0,1\}$. From this, considering the equality (2.3) yields

$$\sum_{l=1}^{s-1}\left(\sum_{m=0}^{l} x_{i,i+m}x_{i+m,i+l}\right)x_{i+l,i+s} + \sum_{m=1}^{s-1} x_{i,i+m}x_{i+m,i+s}x_{i+s,i+s} = 0.$$

4 Now, we define a submatrix of the matrix X as

$$X_{(i)}^{(s)} = \begin{pmatrix} x_{ii} & x_{i,i+1} & \cdots & x_{i,i+s} \\ & x_{i+1,i+1} & \cdots & x_{i+1,i+s} \\ & & \ddots & \vdots \\ & & & x_{i+s,i+s} \end{pmatrix}.$$

This matrix can be written via block matrices as follows:

$$X_{(i)}^{(s)} = \begin{pmatrix} x_{ii} & K & x_{i,i+s} \\ \mathbf{0} & L & M \\ 0 & \mathbf{0} & x_{i+s,i+s} \end{pmatrix}, \tag{2.4}$$

where

$K = \begin{pmatrix} x_{i,i+1} & \cdots & x_{i,i+s-1} \end{pmatrix}, \quad L = \begin{pmatrix} x_{i+1,i+1} & x_{i+1,i+2} & \cdots & x_{i+1,i+s-1} \\ 0 & x_{i+2,i+2} & \cdots & x_{i+2,i+s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{i+s-1,i+s-1} \end{pmatrix},$

and $M = \begin{pmatrix} x_{i+1,i+s} \\ x_{i+2,i+s} \\ \vdots \\ x_{i+s-1,i+s} \end{pmatrix}$. The matrix $X_{(i)}^{(s)}$ is tripotent because of the tripotency of the matrix X. Thus, considering the facts that $x_{ii}^3 = x_{ii}$ and $x_{i+s,i+s}^3 = x_{i+s,i+s}$, it is seen that the matrices $X_{(i)}^{(s-1)} = \begin{pmatrix} x_{ii} & K \\ \mathbf{0} & L \end{pmatrix}$ and $X_{(i+1)}^{(s-1)} = \begin{pmatrix} L & M \\ \mathbf{0} & x_{i+s,i+s} \end{pmatrix}$ are tripotent, too.

The tripotency of the matrices $X_{(i)}^{(s-1)}$ and $X_{(i+1)}^{(s-1)}$ lead to the equalities

$$x_{ii}^2 K + x_{ii} KL + KL^2 = K \quad \text{and} \quad L^3 = L,$$

and

$$L^2 M + LM x_{i+s,i+s} + M x_{i+s,i+s}^2 = M \quad \text{and} \quad L^3 = L,$$

respectively. Thus, we get

$$(X_{(i)}^{(s)})^3 = \begin{pmatrix} x_{ii} & K & \omega \\ \mathbf{0} & L & M \\ 0 & \mathbf{0} & x_{i+s,i+s} \end{pmatrix},$$

where

$$\omega = \left(x_{ii}^2 + x_{ii} x_{i+s,i+s} + x_{i+s,i+s}^2\right) x_{i,i+s} + KM(x_{ii} + x_{i+s,i+s}) + KLM.$$

Recall that $x_{ii} \neq x_{i+s,i+s}$. So, from (2.3), we obtain

$$\begin{aligned} &KM(x_{ii} + x_{i+s,i+s}) + KLM = \\ &\sum_{l=1}^{s-1} \left(\sum_{m=0}^{l} x_{i,i+m} x_{i+m,i+l} \right) x_{i+l,i+s} + \left(\sum_{m=1}^{s-1} x_{i,i+m} x_{i+m,i+s} \right) x_{i+s,i+s} = 0. \end{aligned}$$

Thus, we have $\omega = x_{i,i+s}$. Therefore, $X_{(i)}^{(s)}$ is tripotent regardless of the value of $x_{i,i+s}$. Consequently, the desired result in (iv) is obtained.

Now, let's prove the sufficiency part of the theorem.

Assume that the matrix X satisfies the items (i), (ii), (iii), and (iv) of the theorem. We will now show that the matrix X is tripotent. For this, we will apply induction on s. It can be easily seen that all the matrices $X_{(i)}^{(2)}$ are tripotent.

Suppose that the matrices $X_{(i)}^{(2)}, X_{(i)}^{(3)}, \ldots X_{(i)}^{(s-1)}$ are tripotent for all i. We must show that the matrix $X_{(i)}^{(s)}$ is also tripotent for all i. From (2.4), we obtain

$$(X_{(i)}^{(s)})^3 =$$

$$\begin{pmatrix} x_{ii}^3 & x_{ii}^2 K + (x_{ii}K + KL)\,L & x_{ii}^2 x_{i,i+s} + (x_{ii}K + KL)\,M + (x_{ii}x_{i,i+s} + KM + x_{i,i+s}x_{i+s,i+s})\,x_{i+s,i+s} \\ 0 & L^3 & L^2 M + (LM + Mx_{i+s,i+s})\,x_{i+s,i+s} \\ 0 & 0 & x_{i+s,i+s}^3 \end{pmatrix}.$$

Since $(X_{(i)}^{(s-1)})^3 = X_{(i)}^{(s-1)}$ by the induction hypothesis, we get

$$\begin{pmatrix} x_{ii}^3 & x_{ii}^2 K + (x_{ii}K + KL)\,L \\ \mathbf{0} & \beta^3 \end{pmatrix} = \begin{pmatrix} x_{ii} & K \\ \mathbf{0} & L \end{pmatrix}. \tag{2.5}$$

Similarly, from the equality $(X_{(i+1)}^{(s-1)})^3 = X_{(i+1)}^{(s-1)}$, it can be written

$$\begin{pmatrix} L^3 & L^2 M + (LM + x_{i+s,i+s}M)\,x_{i+s,i+s} \\ \mathbf{0} & x_{i+s,i+s}^3 \end{pmatrix} = \begin{pmatrix} L & M \\ \mathbf{0} & x_{i+s,i+s} \end{pmatrix}. \tag{2.6}$$

From the matrix equalities (2.5) and (2.6), we get

$$x_{ii}^2 K + (x_{ii}K + KL)\,L = K \quad \text{and} \quad L^2 M + (LM + x_{i+s,i+s}M)\,x_{i+s,i+s} = M. \tag{2.7}$$

Also, from (2.5) or (2.6), it is clear that

$$L^3 = L. \tag{2.8}$$

If we postmultiply the first equality of (2.7) by M, and premultiply the second equality of (2.7) by K, then we obtain

$$x_{ii}^2 KM + x_{ii}KLM + KL^2M = KM \quad \text{and} \quad KL^2M + KLMx_{i+s,i+s} + x_{i+s,i+s}^2 KM = KM.$$

So, we get

$$KL^2M = \left(1 - x_{ii}^2\right) KM - x_{ii}KLM = \left(1 - x_{i+s,i+s}^2\right) KM - x_{i+s,i+s}KLM. \tag{2.9}$$

If $x_{ii} \neq x_{i+s,i+s}$, then it is clear that

$$x_{ii}^2 + x_{ii}x_{i+s,i+s} + x_{i+s,i+s}^2 = 1. \tag{2.10}$$

Also, when $x_{ii} \neq x_{i+s,i+s}$, from the second and third equalities of (2.9), we have

$$(x_{ii} + x_{i+s,i+s})\,KM + KLM = 0. \tag{2.11}$$

On the other hand, recall that ω which is the $(1,3)$-block of the matrix $(X_{(i)}^{(s)})^3$ is $(x_{ii} + x_{i+s,i+s})\,KM + KLM + \left(x_{ii}^2 + x_{ii}x_{i+s,i+s} + x_{i+s,i+s}^2\right) x_{i,i+s}$. Considering (2.10) and (2.11) in this statement, we get

$$\omega = x_{i,i+s}. \tag{2.12}$$

If $x_{ii}=x_{i+s,i+s}$, then from each of the parts (ii) and (iii) of theorem, it is obtained that $(x_{ii}+x_{i+s,i+s})KM+KLM+\left(x_{ii}^2+x_{ii}x_{i+s,i+s}+x_{i+s,i+s}^2\right)x_{i,i+s}=x_{i,i+s}$. Consequently, (2.12) is satisfied again. Thus, by considering (2.7), (2.8), and (2.12), it is seen that $(X_{(i)}^{(s)})^3=X_{(i)}^{(s)}$. So, the proof is completed. □

It is noteworthy that if a matrix X is a lower triangular matrix, then the matrix X^T which is the transpose of the matrix X is an upper triangular matrix. Therefore, in case that the matrix X is lower triangular matrix, the statement of Theorem 2.1 becomes as follows:

Theorem 2.2 *Let $\mathfrak{R}$ be a ring whose tripotents are only -1, 0, and 1 with identity 1. Then, $X\in\mathcal{T}^L(\mathfrak{R})$ is tripotent if and only if the entries of X have the following structure:*

(i) $x_{ii}\in\{-1,0,1\}$ *for all* i.

(ii) For $i>j$, *if* $x_{ii}=x_{jj}=0$, *then*

$$x_{ij}=\begin{cases}0 & , \quad i=j+1;\\ \sum\limits_{l=j+1}^{i-1}\left(\sum\limits_{m=1}^{l-j}x_{j+m,j}x_{l,j+m}\right)x_{il}, & i>j+1.\end{cases}$$

(iii) For $i>j$, *if* $x_{ii}=x_{jj}\in\{-1,1\}$, *then*

$$x_{ij}=\begin{cases}0 & ,i=j+1;\\ -\frac{1}{2}\left(\sum\limits_{l=j+1}^{i-1}\left(\sum\limits_{m=0}^{l-j}x_{j+m,j}x_{l,j+m}\right)x_{il}+\sum\limits_{m=j+1}^{i-1}x_{mj}x_{im}x_{ii}\right) & ,i>j+1.\end{cases}$$

(iv) For $i>j$, *if* $x_{ii}\neq x_{jj}$, *then* x_{ij} *is arbitrary.*

3. Number of Tripotent Matrices in Triangular Matrix Rings

In this section, we will give a result that determines the number of tripotents in the matrix ring $\mathcal{T}_n^U(\mathfrak{R})$ when $\mathfrak{R}$ is finite, where n is a positive integer.

Theorem 3.1 *Let $\mathfrak{R}$ be a finite ring whose tripotents are only -1, 0, and 1 with identity 1. Then, the number of tripotents in the matrix ring $\mathcal{T}_n^U(\mathfrak{R})$ with n being a positive integer is*

$$\mathcal{N}(n,\mathfrak{R})=\sum_{s_1=0}^{n}\sum_{s_2=0}^{n-s_1}\binom{n}{s_1}\binom{n-s_1}{s_2}|\mathfrak{R}|^{s_1(n-s_1)+s_2(n-s_1-s_2)}.$$

Proof According to Theorem 2.1, the number of upper triangular tripotents searched depends on the pairs of main diagonal entries satisfying $x_{ii}\neq x_{jj}$. To calculate these probabilities, let's make the following observations.

We consider the vector $d=\begin{bmatrix}d_1\\ d_2\\ \vdots\\ d_n\end{bmatrix}$ where $d_i\in\{-1,0,1\}$ for each i. Let Δ be the number of the pairs (d_i,d_j) with $i<j$ and $d_i\neq d_j$. By the nature of the vector d, there are two column vectors f and g such that $d=f-g$ and $\langle f,g\rangle=0$ with $f_i,g_i\in\{0,1\}$ for all i.

Now, let us denote the number of the pairs (f_i, f_j) with $i < j$ and $f_i \neq f_j$ by Δ_1. Let Δ_2 be the number of the pairs (g_i, g_j) such that $g_i \neq g_j$ with $f_i, f_j \neq 1$ (or equivalently, $(f_i, g_i) \neq (1, 0)$ and $(f_j, g_j) \neq (1, 0)$) and $i < j$. It is clear that $\Delta = \Delta_1 + \Delta_2$.

Now, let F be an $n \times n$ matrix whose i, j-entry is

$$f_i (1 - f_j) + (1 - f_i) f_j = \begin{cases} 1, & f_i \neq f_j \\ 0, & f_i = f_j \end{cases}. \tag{3.1}$$

The term in the left of (3.1) states that the matrix F can be written as

$$F = f (e - f)^T + (e - f) f^T,$$

where e is n-vector whose all entries are 1. We know that $e^T F e$ is the sum of the entries in F. Also, $s_1 := e^T f$ is the sum of the entries in f that gives the number of the entries 1 in d. Moreover, it is easy to see that $n = e^T e$. Since Δ_1 is the half of the sum of the entries in F, it is obtained that

$$\begin{aligned} \Delta_1 = \frac{e^T F e}{2} &= \frac{e^T \left[f (e - f)^T + (e - f) f^T \right] e}{2} \\ &= \frac{\left(e^T f\right) \left[(e - f)^T e \right] + \left[e^T (e - f) \right] \left(f^T e \right)}{2} = s_1 (n - s_1). \end{aligned}$$

Now, let G be an $n \times n$ matrix whose i, j-entry is

$$g_i (1 - f_j - g_j) + (1 - f_i - g_i) g_j = \begin{cases} 1, & g_i \neq f_j + g_j, (f_i, g_i) \neq (1, 0) \text{ and } (f_j, g_j) \neq (1, 0) \\ 0, & g_i = f_j + g_j, (f_i, g_i) \neq (1, 0) \text{ and } (f_j, g_j) \neq (1, 0) \\ 0, & (f_i, g_i) = (1, 0) \text{ or } (f_j, g_j) = (1, 0). \end{cases} \tag{3.2}$$

The left-hand side of (3.2) shows that the matrix G can be written as

$$G = g(e - f - g)^T + (e - f - g) g^T.$$

Now, suppose that $s_2 := e^T g$. This gives the number of the entries -1 in d. Also, $e^T G e$ is the sum of the entries in G, and $\Delta_2 = \frac{e^T G e}{2}$. Thus, we obtain

$$\begin{aligned} \Delta_2 &= \tfrac{e^T G e}{2} \\ &= \tfrac{e^T \left[g (e - f - g)^T + (e - f - g) g^T \right] e}{2} \\ &= \tfrac{e^T g \left[(e - f - g)^T e \right] + \left[e^T (e - f - g) g^T e \right]}{2} \\ &= \tfrac{s_2 \left[e^T e - f^T e - g^T e \right] + \left[e^T e - e^T f - e^T g \right] s_2}{2} \\ &= \tfrac{s_2 [n - s_1 - s_2] + [n - s_1 - s_2] s_2}{2} \\ &= s_2 (n - s_1 - s_2). \end{aligned}$$

Since $\Delta = \Delta_1 + \Delta_2$, we get $\Delta = s_1 (n - s_1) + s_2 (n - s_1 - s_2)$. Δ is independent of the order of -1, 0, and 1 in d. So, each arrangement of s_1 ones, s_2 minus ones, and $n - s_1 - s_2$ zeros on the main diagonal leads to $|\mathfrak{R}|^{s_1 (n - s_1) + s_2 (n - s_1 - s_2)}$ possible upper triangular tripotent matrices. Since there are $\begin{pmatrix} n \\ s_1 \end{pmatrix} \begin{pmatrix} n - s_1 \\ s_2 \end{pmatrix}$

possibilities to choose such an arrangement, the number of $n \times n$ upper triangular tripotent matrices, whose main diagonal entries consist of -1, 0, and 1, is as expressed in Theorem 3.1.

□

Let's close this section by giving some examples and remarks.

Example 3.2 *All matrices in* $\mathcal{T}_2^U(\mathfrak{R})$ *satisfying Theorem 2.1 have one of the following forms with* $a \in \mathfrak{R}$ *being an arbitrary element:*

$$\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & a \\ 0 & \mp1 \end{pmatrix}$$

Example 3.3 *All tripotent matrices in* $\mathcal{T}_3^U(\mathfrak{R})$ *satisfying Theorem 2.1 have one of the following forms with* $a, b, c \in \mathfrak{R}$ *being arbitrary elements:*

$$\begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & a & b \\ 0 & \mp1 & c \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & \mp1 & c \\ 0 & 0 & \pm1 \end{pmatrix},$$

$$\begin{pmatrix} \pm1 & a & b \\ 0 & 0 & c \\ 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 & b \\ 0 & \pm1 & c \\ 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} \mp1 & a & b \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & \pm1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & a & \pm ac \\ 0 & \pm1 & c \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 & b \\ 0 & \pm1 & c \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix},$$

$$\begin{pmatrix} \pm1 & a & \mp ac \\ 0 & 0 & c \\ 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1 & a & \frac{\mp ac}{2} \\ 0 & \mp1 & c \\ 0 & 0 & \pm1 \end{pmatrix}.$$

Example 3.4 *All tripotent matrices in* $\mathcal{T}_4^U(\mathfrak{R})$ *satisfying Theorem 2.1 are as in the following forms with* $a, b, c, d \in \mathfrak{R}$ *being arbitrary elements:*

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 \\ 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 & a & b \\ 0 & \pm1 & c & d \\ 0 & 0 & \mp1 & e \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \pm1 & 0 & a & b \\ 0 & \pm1 & c & d \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} \pm1 & a & \mp\frac{ac}{2} & b \\ 0 & \mp1 & c & \pm\frac{cd}{2} \\ 0 & 0 & \pm1 & d \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} \mp1 & 0 & a & \pm\frac{ac}{2} \\ 0 & \mp1 & b & \pm\frac{bc}{2} \\ 0 & 0 & \pm1 & c \\ 0 & 0 & 0 & \mp1 \end{pmatrix},$$

$$\begin{pmatrix} \mp1 & a & \pm ab & \pm ac \\ 0 & 0 & b & c \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 & a & b \\ 0 & \pm1 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mp1 & a & \pm ac & b \\ 0 & 0 & c & \mp cd \\ 0 & 0 & \mp1 & d \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \mp1 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} \mp1 & a & b & c \\ 0 & \pm1 & d & \mp de \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1 & a & \mp\frac{ac}{2} & b \\ 0 & \mp1 & c & d \\ 0 & 0 & \pm1 & e \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \mp1 & a & b & c \\ 0 & \pm1 & 0 & d \\ 0 & 0 & \pm1 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \pm1 \end{pmatrix},$$

$$\begin{pmatrix} \mp1 & 0 & 0 & a \\ 0 & \mp1 & 0 & b \\ 0 & 0 & \mp1 & c \\ 0 & 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1 & a & b & c \\ 0 & \mp1 & 0 & 0 \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} 0 & a & b & c \\ 0 & \mp1 & 0 & 0 \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \mp1 \end{pmatrix},$$

$$\begin{pmatrix} \mp1 & 0 & 0 & a \\ 0 & \mp1 & 0 & b \\ 0 & 0 & \mp1 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mp1 & a & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & \mp1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & a & \pm ac & b \\ 0 & \pm1 & c & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} \mp1 & 0 & a & b \\ 0 & \mp1 & c & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1 & a & b & c \\ 0 & \mp1 & d & e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & a & b & c \\ 0 & \pm1 & d & e \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} 0 & a & \pm ac & b \\ 0 & \pm1 & c & \mp cd \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & a & b & ac \mp bd \\ 0 & \mp1 & 0 & c \\ 0 & 0 & \mp1 & d \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & a & b & ace \mp ad \pm be \\ 0 & \mp1 & c & d \\ 0 & 0 & \pm1 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & a & b & \mp\left(\frac{ac+bd}{2}\right) \\ 0 & \mp1 & 0 & c \\ 0 & 0 & \mp1 & d \\ 0 & 0 & 0 & \pm1 \end{pmatrix},$$

$$\begin{pmatrix} \pm1 & 0 & a & \mp ac \\ 0 & \pm1 & b & \mp bc \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} \mp1 & a & b & \pm(ac+bd) \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & \mp1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & a & b & c \\ 0 & \mp1 & 0 & d \\ 0 & 0 & \mp1 & e \\ 0 & 0 & 0 & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & a & b & c \\ 0 & \mp1 & d & \pm\frac{ed}{2} \\ 0 & 0 & \pm1 & e \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} \mp1 & a & \pm ac & b \\ 0 & 0 & c & d \\ 0 & 0 & \mp1 & e \\ 0 & 0 & 0 & \pm1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & a & \pm ab & \pm ac \\ 0 & \pm1 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a & \pm ac \\ 0 & 0 & b & \pm bc \\ 0 & 0 & \pm1 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mp1 & a & b & c \\ 0 & 0 & d & \pm de \\ 0 & 0 & \pm1 & e \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \mp1 & a & b & \pm be - \frac{ad+ace}{2} \\ 0 & \pm1 & c & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & \mp1 \end{pmatrix}, \begin{pmatrix} \pm1 & a & b & \mp ad - \frac{ace}{2} \mp \frac{be}{2} \\ 0 & 0 & c & d \\ 0 & 0 & \mp1 & e \\ 0 & 0 & 0 & \pm1 \end{pmatrix},$$

$$\begin{pmatrix} \mp 1 & a & \pm\frac{ab}{2} & \pm\frac{ac}{2} \\ 0 & \pm 1 & b & c \\ 0 & 0 & \mp 1 & 0 \\ 0 & 0 & 0 & \mp 1 \end{pmatrix}$$

Note that all the above examples include all the idempotent matrix examples in Hou's work in [22] and all the involutive matrix examples in Petik et al.'s work in [26]. It is also easy to see that matrices whose entries on the main diagonal belong to the set $\{-1, 0\}$ are skew idempotent. Notice that from the examples above the matrices which are not idempotent or skew idempotent are essentially tripotent matrix examples.

Considering Theorem 2.2, it can be seen that the tripotent matrix examples in the lower triangular matrix rings are also the transpositions of the tripotent matrix examples we have given in the upper triangular matrix rings.

Remark 3.5 *The number of tripotent matrices in the lower triangular matrix rings is exactly the same as in Theorem 3.1, since the number sought depends only on the main diagonal entries and the main diagonal entries do not change because the lower triangular matrices are transpose of the upper triangular matrices. On the other hand, the expression $s_1(n - s_1) + s_2(n - s_1 - s_2)$ in Theorem 3.1 states the number of arbitrary variables in the super diagonals of upper triangular matrices whose main diagonal consists of s_1 ones, s_2 minus ones, and $n - s_1 - s_2$ zeros. It's easy to see that the number of arbitrary variables in the sub diagonals equals exactly the same number when matrices are lower triangular.*

Remark 3.6 *Consider the number of different forms of matrices in given examples. It is seen that there are 9 different forms of 2×2 dimensional upper (lower) triangular tripotents, 27 different forms of 3×3 dimensional upper (lower) triangular tripotents, and 81 different forms of 4×4 dimensional upper (lower) triangular tripotents. Notice that these numbers consist of the numbers 3^2, 3^3, and 3^4. Here, the number 3 in the base indicates the number of elements of the set $\{-1, 0, 1\}$, and the numbers in powers indicate the size of the matrices. Such examples can also be expanded to integers $n > 4$.*

4. Discussions

Triangular matrices have an important place in linear algebra and matrix analysis. These matrices form the basis of matrix decompositions, and many physical problems can be easily solved thanks to matrix decompositions. For example, the solution of a differential equation can be associated with the solution of a system of linear equations, and most systems of linear equations are solved using matrix decompositions (for example, LU, LDU, etc.). These decompositions are useful tricks for many computational reasons. If the matrix of coefficients is lower triangular or upper triangular, linear equations have particularly transparent solutions.

Moreover, tripotent matrices have special importance in digital image encryption (see, for instance, [25]). In addition, it is well known that a real tripotent matrix can be decomposed into the difference of two disjoint idempotent matrices. Statistically, tripotent matrices are useful in determining if a real quadratic form can be decomposed into the difference of two independent chi-square variables (see, for instance, [12]).

Hou's work (this work) is on the rings with trivial idempotents (tripotents). In [26], the author extended this case to general rings. Perhaps, based on that study, trivial case in our current work can be extended to the general rings. Also, the current work can be considered for quadratic, generalized quadratic, cubic matrices (which contain tripotent matrices). Other than all, alternative observations can be made from the facts that a tripotent matrix is expressed as the difference of two disjoint idempotent matrices and that the square of a tripotent matrix is an idempotent matrix.

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