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Applications of stable currents and homology groups in CR-warped products of complex space forms

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Abstract: In the present, by using the result Lawson-Simons, we adopt a different technique to show that there is no a stable integral q-currents and a non-trivial homology group in a compact oriented CR-warped product submanifold M^n in a complex space forms space $\mathbb{Q}_c^{2m}(4c)$ by imposing some restrictions on the squared norm of gradient and the Laplacian of warped function. Finally, we show that the same approach can be gotten using Dirichlet energy, positive eigenvalue and Hamiltonian.

Key words: Homology groups, Stable currents, Eigenvalue, Compact CR-warped products, homotopic.

1. Main results with their motivations

The classification of topological obstructions of submanifolds have an important character in global Riemannian geometry. It ought be attractive to realize that the conditions on the main intrinsic and extrinsic curvature invariants effect to the topology of a warped product submanifold of a Riemannian manifold. In this regard, the classical object in geometry and physics is depends on the behavior of the homology groups which are called topological invariants. They provide the important knowledge about topological character of defined manifolds. This type of connection between stable currents and non-trivial integeral homology class in $H_q(M, \mathbb{G})$ is provided by Federer-Fleming's in[9]. If the pinching condition on the second fundamental form of a submanifold M^n is satisfied then a submanifold M^n of $\mathbb{S}^n$ has no stable currents and homology groups are trivial see [14]. The goal of the present article is to construct the vanishing homology theorems along the warped product submanifolds with holomorphic constant sectional curvature at most one and zero. In this background, the following theorem obtained previously in [14, 27], will be useful

Theorem 1.1 *[14, 16, 27] Let N^n is a compact submanifold of dimension n in a space form $\widetilde{N}(c)$ such that the constant curvature c is non-negative. If σ denotes the second fundamental form of N^n, and satisfies the*

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relation

$$\sum_{r=1}^{q}\sum_{\beta=q+1}^{n}\Big(2||\sigma(e_r,e_s)||^2-g\big(\sigma(e_r,e_r),\sigma(e_s,e_s)\big)\Big)<qpc \tag{1.1}$$

for any positive integers q,p *such that* $q+p=n$. *Then there does not exist a stable* q*-current on* M^n *and* $H_q(M^n,\mathbb{G})=H_{n-q}(M^n,\mathbb{G})=0.$, *where* $\{e_j\}_{1\leq j\leq n}$ *is an orthonormal frame for the tangent space* T_xM^n,

The importance of homotopy theory accentuates on its applications of the low-diemnsion statistical-mechanical systems, singularities in liquid crystal, and phase transition in physics [13], while studying warped products and Differential topology approaches in mathematical physics effectively relevant in general relativity [11, 18–20]. Particularly, the space-time homology is one of the main apparatus for quantum gravity [17, 25]. Some new and interesting results for trivial homology groups and stable currents on submanifolds have been gotten by putting some restrictions on on the second fundamental form (see [15, 16, 22, 23, 23, 24, 26, 27, 29]). This has motivated authors of the study to link the concept of warped product manifold and homotopy-homology theory. In [5, 6], Chen started the investigation of warped product CR-submanifolds of Kahler manifolds, and provided the characterization theorem for CR-warped product which isometrically immerssed in complex space form $\mathbb{Q}_c^m$, complex hyperbolic space $\mathbb{C}H^m(-4)$, and complex projective space $\mathbb{C}P^m(4)$ [see in [5]]. Such work of Chen [5] and the above-mentioned studies have pushed us to extend the study of Lawson and Simon [14] to CR-warped product submanifolds in a complex space form $\mathbb{Q}_c^m(4c)$.

Theorem 1.2 *Assume that* $M^n=N_T^q\times_h N_\perp^p$ *is a compact CR-warped product submanifold in a complex space form* $\mathbb{Q}_c^{2m}(4c)$. *If the inequality is satisfied*

$$(1-p)\|\nabla h\|^2+\|\sigma_\mu\|^2h^2<3qch^2+h\Delta h \tag{1.2}$$

then there is no a stable integral q*-current in* M^n *and*

$$H_q(M^n,\mathbb{G})=H_p(M^n,\mathbb{G})=0.$$

Motivated by Lawson and Simon [8, p.441 Theorem 4]. The second goal of the study is to find some new results on the topology for warped product submanifolds and we replace condition (1.1) in terms of the second fundamental form with one of the warping function. Particularly, by assuming a CR-warped product submanifold in $\mathbb{Q}_c^{2m}(4c)$, and we state the following theorem.

Theorem 1.3 *Assume that* $M^n=N_T^q\times_h N_\perp^p$ *is a compact CR-warped product submanifold in* $\mathbb{Q}_c^{2m}(4c)$. *Then, if the following condition is satisfied*

$$(1-p)\|\nabla h\|^2+\|\sigma_\mu\|^2h^2<3qch^2+h\Delta h \tag{1.3}$$

we have:

- *If* $q+p\neq 3$, *the submanifold* M^{q+p} *is homeomorphic to sphere* $\mathbb{S}^{q+p}$,
- *If* $q+p=3$, M^{q+p} *is homotopic to a sphere* $\mathbb{S}^{q+p}$.

Let a compact Riemannian manifold M and φ be a positive smooth function on M^n, in which $\varphi \in \mathcal{F}(M^n)$, then the *Lagrangian* and the *Dirichlet energy* φ are defined by (see [8]):

$$L_\varphi = \frac{1}{2}||\nabla\varphi||^2. \tag{1.4}$$

$$\mathbb{E}(\varphi) = \frac{1}{2}\int_{M^n} ||\nabla\varphi||^2 dV, \qquad 0 < E(\varphi) < \infty \tag{1.5}$$

Moreover, the Euler-Lagrange equation for the *Lagrangian* (1.5) is that

$$\Delta\varphi = 0. \tag{1.6}$$

Let M^n be a compact oriented Riemannian manifold without boundary, i.e., $\partial M^n = \emptyset$. Then, we provide a strong result using (1.5) and we have constructed the following result.

Theorem 1.4 *Let $M^n = N_T^q \times_h N_\perp^p$ be compact oriented CR-warped product submanifolds of complex space form $\mathbb{Q}_c^{2m}(4c)$. Assume that the following relation is satisfied*

$$\int_{M^n} \|\sigma_\mu\|^2 h^2 dV < 3q\int_{M^n} ch^2 dV + 2p\mathbb{E}(h) \tag{1.7}$$

where $\mathbb{E}(h)$ is the Dirichlet energy of the function h. Then there does not exist a stable integral q-current in M^n and

$$H_q(M^n, \mathbb{G}) = H_p(M^n, \mathbb{G}) = 0.$$

Let M^n be an n-dimensional compact Riemannian manifold and then the Laplacian is a second order quasilinear operator on M^n defined by

$$\Delta\varphi = -div(\nabla\varphi). \tag{1.8}$$

Such a Laplacian has found many applications in mathematics as well as in physics, one can consider the eigenvalue problem of Δ. The corresponding Laplace eigenvalue equation is defined that a real number λ is called eigenvalue if there exists a non-zero function φ satisfying the following equation

$$\Delta\varphi = \lambda\varphi, \quad \text{on} \quad M^n, \tag{1.9}$$

with appropriate boundary conditions. Now we consider a Riemannian manifold M^n without boundary. The first nonzero eigenvalue of Δ, denoted by λ_1, has a variational characterization (cf. [5]):

$$\lambda_1 = \inf\left\{\frac{\int_M ||\nabla\varphi||^2 dV}{\int_M |\varphi|^2 dV} \mid \varphi \in W^{1,2}(M^n)\backslash\{0\}, \int_M \varphi dV = 0\right\}. \tag{1.10}$$

Inspired the above characterization, we give the following result

Theorem 1.5 *Let $\mathbb{Q}_c^{2m}(4c)$ is a complex space forms and $M^n = N_T^q \times_h N_\perp^p$ is compact oriented CR-warped product of $\mathbb{Q}_c^{2m}(4c)$ satisfies the following inequality*

$$\|\sigma_\mu\|^2 < \big(3qc + \lambda_1 p\big) \tag{1.11}$$

where μ_1 is a positive eigenvalue endowed with Laplacian operator (1.9). Then, then M^n has no stable integral q-currents and homology groups with respect to M^n are trivial, i.e., $H_q(M^n, \mathbb{G}) = H_p(M^n, \mathbb{G}) = 0$. In addition, M^{q+p} is homeomorphic to sphere $\mathbb{S}^{q+p}$ when $q+p \neq 3$, and M^{q+p} is homotopic to a sphere $\mathbb{S}^{q+p}$ if $q+p=3$

The significance of our results comes from the new conditions on the Laplacian of the warped function. Such results can be considered as a vanishing homology theorem as warped product submanifolds are having sectional curvature at most one and zero.

2. Preliminaries

An almost complex structure J and a Riemannian metric g on $2m$-dimensional manifold $\widetilde{M}$ such that

$$J^2 = -I \qquad \text{and} \qquad g(JX_1, JX_2) = g(X_1, X_2),$$

$\forall$, $X_1, X_2 \in \mathfrak{X}(T\widetilde{M})$, where the tangent bundle on $\widetilde{M}$ is denoted by $T\widetilde{M}$. Then a mamnifold $\widetilde{M}^{2m}$, together with almost Hermitian structure (J, g) is defined almost Hermitian manifold. If the almost complex structure J hold the parallel condition, i.e., $(\widetilde{\nabla}_{X_1} J)X_2 = 0$, for any $X_1, X_2 \in \mathfrak{X}(T\widetilde{M})$, then $\widetilde{M}$ is called a Kahler manifold according to Yano and Kon [28]. In such case, the almost complex structure J is integrable and the fundamental 2-form is closed. On the other hand, a Kahler manifold $\widetilde{M}^{2m}$ with holomorphic constant sectional curvature c is represented to the complex space form $\mathbb{Q}_c^{2m}(4c)$. In the whole paper, we consider the complex space form $\mathbb{Q}_c^{2m}(4c)$ with curvature tensor $\widetilde{R}$ given by;

$$\widetilde{R}(X_1, X_2, X_3, X_4) = c\Big\{ g(X_2, X_3)g(X_1, X_4) - g(X_2, X_4)g(X_1, X_3) + g(X_1, JX_3)g(JX_2, X_4) - g(X_2, JX_3)g(JX_1, X_4) + 2g(X_1, JX_2)g(JX_3, X_4) \Big\} \tag{2.1}$$

for any $X_1, X_2, X_3, X_4 \in \mathfrak{X}(\mathcal{Q}_c^{2m}(4c))$.

The Gauss and Weingarten formulas for a submanifold M^n of Kaehler manifold $\widetilde{M}^{2m}$ with induced connections ∇ and $\nabla^\perp$ on the tangent bundle TM and the normal bundle $T^\perp M$ of M^n, respectively, is defined as;

$$\widetilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \sigma(X_1, X_2), \tag{2.2}$$

$$\widetilde{\nabla}_{X_1} \xi = -A_\xi X_1 + \nabla^\perp_{X_1} \xi \tag{2.3}$$

$\forall$ $X_1, X_2 \in \mathfrak{X}(TM)$ and $\xi \in \mathfrak{X}(T^\perp M)$. The notation are used in the above formulas, i.e., A and σ are denoted as shape operator and the fundamental form, respectively, with the relation holds $g(h(X_1, X_2), \xi) = g(A_\xi X_1, X_2)$. Now, as usual notations, we have

$$(i)\ JX_1 = PX_1 + FX_1, \quad (ii)\ \ J\xi = t\xi + f\xi. \tag{2.4}$$

The notations $PX_1(t\xi)$ and $FX_1(f\xi)$ are referred as tangential and normal parts of $JX_1(J\xi)$, respectively. If $P = 0$, then M^n is known as totally real submanifold. A holomorphic submanifold M^n is defined as $J(T_xM) \subseteq T_xM$, for each $x \in M^n$. Similarly, M^n is *totally real* if $J(T_xM) \subseteq T^\perp M$, for each $x \in M^n$.

Definition 2.1 *[6] A Riemannian submanifold M^n of Kahler $\widetilde{M}^{2m}$ is referred to as a CR-submanifold if the pair of orthogonal distributions $\mathcal{D}^T$ and $\mathcal{D}^\perp$ exists in which $TM = \mathcal{D}^T \oplus \mathcal{D}^\perp$, where holomorphic $\mathcal{D}^T$ and totally real distributions are defined as $J(\mathcal{D}^T) \subseteq \mathcal{D}^T$, and $\mathcal{D}^\perp$ is $J\mathcal{D}^\perp \subseteq (T^\perp M)$, respectively.*

If $\dim(\mathcal{D}^T) = d_1$ and $\dim(\mathcal{D}^\perp) = d_2$ in a CR-submanifold M^n of $\widetilde{M}^{2m}$, then M^n totally real if $d_1 = 0$ and M^n is holomorphic if $d_2 = 0$. It is said to be a proper contact CR-submanifold if neither $d_1 = 0$ nor $d_2 = 0$.The normal bundle $T^\perp M$ can be split as

$$T^\perp M = J\mathcal{D}^\perp \oplus \mu$$

where, μ is a *holomorphic* subspace along J of $T^\perp M$. The Gauss equation for a submanifold M^n is given by

$$\begin{aligned}\widetilde{R}\big(X_1, X_2, X_3, X_4\big) =& R(X_1, X_2, X_3, X_4) + g\big(\sigma(X_1, X_3), \sigma(X_2, X_4)\big)\\ &- g\big(\sigma(X_1, X_4), \sigma(X_2, X_3)\big)\end{aligned} \tag{2.5}$$

$\forall\ X_1, X_2, X_3, X_4 \in \mathfrak{X}(TM)$. The mean curvature H on M^n is defined by

$$H = \frac{1}{n} trace(\sigma) = \frac{1}{n}\sum_{j=1}^{n} \sigma\big(e_j, e_j\big) \tag{2.6}$$

where $\{e_1, e_2, \cdots e_n\}$ is a orthonormal basis for the tangent space TM and $n = \dim M$. In addition, we set

$$\sigma_{ji}^r = g(\sigma(e_j, e_i), e_r),\ and\ ||\sigma||^2 = \sum_{j,i=1}^{n} g\big(\sigma(e_j, e_i), \sigma(e_j, e_i)\big). \tag{2.7}$$

According to Bishop and O'Neill [4], the product manifold $M^n = N_1^q \times N_2^p$ is the warped product manifold $N_1^q \times_h N_2^p$ associated with the Riemannian metric $g = g_1 + h^2 g_2$ such that h be a differentiable function defined on N_1^q. Here, h is referred to as a warping function on M^n. Consequently, we have the following lemma.

Lemma 2.1 *[4] Let $M^n = N_1^q \times_h N_2^p$ be a warped product manifold, then following properties are well defines*

(i) $\nabla_{Z_2} X_1 = \nabla_{X_1} Z_2 = \frac{(X_1 h)}{h} Z_2$,

(ii) $\nabla_{Z_1} Z_2 = \nabla'_{Z_1} Z_2 - g(Z_1, Z_2)\nabla \ln h$

$\forall\ X_1 \in \mathfrak{X}(TN_1)$ and $Z_1, Z_2 \in \mathfrak{X}(TN_2)$, where ∇ and ∇' denote Levi-Civitas connections on M^n and N_2, respectively. Further, the gradient of $\ln h$ is denoted by $\nabla \ln h$ and defined by

$$g(\nabla \ln h, X_1) = X_1(\ln h). \tag{2.8}$$

Therefore, utilizing the first property of the above lemma, we get

$$\mathcal{R}(X_1, X_2)Z = \frac{\mathcal{H}^h(X_1, Z_1)}{h} X_2 \tag{2.9}$$

where $\mathcal{H}^h$ is a Hessian tensor of h.

Remark 2.1 *Trivial or simply a Riemannian product manifold is just a warped product manifold $M^n = N_1^q \times_h N_2^p$ with a constant warping function h along N_1^q.*

Remark 2.2 *Let $M^n = N_1^q \times_h N_2^p$ be a warped product manifold. Then N_1^q is totally geodesic and N_2^p is totally umbilical submanifold of M^n, respectively.*

One of the interesting result for warped product submanifold proved Chen in [7] is the following.

$$\sum_{i=1}^{q}\sum_{j=1}^{p} K(e_i \wedge e_j) = \frac{p\Delta h}{h}. \tag{2.10}$$

From the above, we get

$$\frac{\Delta h}{h} = \Delta(\ln h) - ||\nabla(\ln h)||^2. \tag{2.11}$$

The warped product submanifold is said to be CR-warped product if its factors are totally real and holomorphic submanifolds. In the setting of Kahler manifold according to Chen in [5], we have two cases of CR-warped product, *(i)* $N_{\perp}^{p} \times_f N_T^q$, *and* *(ii)* $N_T^q \times_f N_{\perp}^p$. For case (i), Chen [5] shown that there does not exist any non-trivial warped product CR-submanifold $M^n = N_{\perp}^p \times_f N_T^q$. In the same paper, he proved that there are many CR-warped product of the type $M^n = N_T^q \times_h N_{\perp}^p$ and proved the following results

$$g(\sigma(Z_1, JX_1), JZ_2) =(X_1 \ln h) g(Z_1, Z_2), \tag{2.12}$$

$$g(\sigma(X_1, X_2), JZ_1) =0 \tag{2.13}$$

$\forall\ X_1, X_2 \in \mathfrak{X}(TN_T)$ and $Z_1, Z_2 \in \mathfrak{X}(TN_{\perp})$.

Proof of Theorem 1.2

Let $M^n = N_T^q \times_h N_{\perp}^p$ be an $n = q + p$-dimensional CR-warped product submanifold with $\dim(N_T^q) = q = 2r$ and $\dim(N_{\perp}^p) = p = s$ such that $N_{\perp}^p$ and N_T^q are integral manifolds of $\mathcal{D}^{\perp}$ and $\mathcal{D}$, respectively. Thus, we consider the $\{e_1, e_2, \cdots e_r, e_{r+1} = Je_1, \cdots e_{2r} = Je_r\}$ and $\{e_{2r+1} = e_1^*, \cdots e_{2r+s} = e_p^*\}$ to be orthonormal basis of TN_T and $TN_{\perp}$, respectively. Thus the orthonormal basis of the normal subbundles $J\mathcal{D}^{\perp}$ and μ are $\{e_{n+1} = \bar{e}_1, \cdots e_{n+s} = \bar{e}_{\beta}, \}$ and $\{e_{n+s+1}, \cdots e_m\}$, respectively. Then, using (2.5), we get

$$\sum_{r=1}^{q}\sum_{s=1}^{p} g\big(R(e_r, e_s)e_r, e_s\big) = \sum_{r=1}^{q}\sum_{s=1}^{p} g\big(\widetilde{R}(e_r, e_s)e_r, e_s\big) + ||\sigma(e_r, e_s)||^2 - \sum_{r=1}^{q}\sum_{s=1}^{p} g\big(\sigma(e_s, e_s), \sigma(e_r, e_r)\big). \tag{2.14}$$

Add $||\sigma(e_r, e_s)||^2$ to both sides, one obtains

$$\sum_{r=1}^{q}\sum_{s=1}^{p} g\big(R(e_r, e_s)e_r, e_s\big) + ||\sigma(e_r, e_s)||^2 = \sum_{r=1}^{q}\sum_{s=1}^{p} g\big(\widetilde{R}(e_r, e_s)e_r, e_s\big) - \sum_{r=1}^{q}\sum_{s=1}^{p} g\big(\sigma(e_s, e_s), \sigma(e_r, e_r)\big) + 2||\sigma(e_r, e_s)||^2. \tag{2.15}$$

The orthonormal frames $\{e_r\}_{1 \leq r \leq q}$ and $\{e_s\}_{1 \leq s \leq p}$ of N_T^q and N_{θ}^p, respectively in (2.9), we derive

$$R(e_r, e_s)e_r = \frac{e_s}{h}\mathcal{H}^h(e_r, e_r).$$

For the basis $\{e_s\}_{1\leq s\leq p}$ of tangent space TM which is orthonormal, we derive

$$\sum_{r=1}^{q}\sum_{s=1}^{p} g\big(R(e_r,e_s)e_r,e_s\big)=\frac{p}{h}\sum_{r=1}^{q} g\big(\nabla_{e_r}\nabla h,e_r\big). \tag{2.16}$$

Thus from Eq (2.15) and (2.16), we derive

$$\begin{aligned}\sum_{r=1}^{q}\sum_{s=1}^{p}\Big(2||\sigma(e_r,e_s)||^2-g\big(h(e_s,e_s),\sigma(e_r,e_r)\big)\Big)+\sum_{r=1}^{q}\sum_{s=1}^{p} g\big(\widetilde{R}(e_r,e_s)e_r,e_s\big)\\ =\frac{p}{h}\sum_{r=1}^{q}\sum_{s=1}^{p} g\big(\nabla_{e_r}\nabla h,e_r\big)+||\sigma(e_r,e_s)||^2.\end{aligned} \tag{2.17}$$

Now, we compute the Laplacian of h

$$\begin{aligned}\Delta h&=-\sum_{i=1}^{n} g\big(\nabla_{e_i} gradh,e_i\big)\\ &=-\sum_{r=1}^{q} g\big(\nabla_{e_r} gradh,e_r\big)-\sum_{s=1}^{p} g\big(\nabla_{e_s} gradh,e_s\big).\end{aligned}$$

Using Eq (2.11), we find that

$$\frac{1}{h}\sum_{r=1}^{q} g\big(\nabla_{e_r} gradh,e_r\big)=-\Delta(\ln h)+(1-p)||\nabla\ln h||^2. \tag{2.18}$$

Therefore, utilizing (2.17) and (2.18), we get

$$\begin{aligned}\sum_{r=1}^{q}\sum_{s=1}^{p}\Big(2||\sigma(e_r,e_s)||^2-g\big(\sigma(e_s,e_s),\sigma(e_r,e_r)\big)\Big)+\sum_{r=1}^{q}\sum_{s=1}^{p} g\big(\widetilde{R}(e_r,e_s)e_r,e_s\big)\\ =p(1-p)||\nabla(\ln h)||^2\\ -p\Delta(\ln h)+\sum_{r=1}^{q}\sum_{s=1}^{p}||\sigma(e_r,e_s)||^2.\end{aligned} \tag{2.19}$$

Now, set $X=e_r$ and $Z=e_s$ for $1\leq r\leq q$ and $1\leq s\leq p$, respectively. Follows the (2.7), we defined as

$$\sum_{r=1}^{q}\sum_{s=1}^{p}||\sigma(e_r,e_s)||^2=\sum_{r=1}^{q}\sum_{s=1}^{p} g\big(\sigma(e_r,e_s^*),e_s\big)^2.$$

The right hand side of the above equation is μ-component and $J\mathcal{D}^\perp$-component. Taking summation over the vector fields on N_T^q and $N_\perp^p$, using the adopted frame for orthonormal vector fields and then utilizing (2.12) , we obtain

$$\sum_{r=1}^{q}\sum_{s=1}^{p}||\sigma(e_r,e_s)||^2=p||\nabla\ln h||^2+\|\sigma_\mu\|^2. \tag{2.20}$$

Using (2.19) and (2.20), we find that

$$-p\Delta(\ln f)+p(1-p)||\nabla(\ln f)||^2+p||\nabla(\ln h)||^2+\|\sigma_\mu\|^2$$
$$=\sum_{r=1}^{q}\sum_{s=1}^{p}\Big(2||\sigma(e_r,e_s)||^2-g\big(\sigma(e_s,e_s),\sigma(e_r,e_r)\big)\Big)$$
$$+\sum_{r=1}^{q}\sum_{s=1}^{p}g\big(\widetilde{R}(e_r,e_s)e_r,e_s\big). \tag{2.21}$$

By symmetry of curvature tensor R, we get

$$\sum_{r=1}^{q}\sum_{s=1}^{p}g\big(\widetilde{R}(e_r,e_s)e_r,e_s\big)=\sum_{r=1}^{q}\sum_{s=1}^{p}\widetilde{K}\big(e_r\wedge e_s\big). \tag{2.22}$$

Hence for a warped product submanifold, we get

$$\sum_{r=1}^{q}\sum_{s=1}^{p}\widetilde{K}\big(e_r\wedge e_s\big)=\sum_{r=1}^{q}\sum_{s=1}^{p}\Big\{g(e_s,e_r)g(e_r,e_s)-g(e_r,e_r)g(e_s,e_s)$$
$$-g(Je_r,e_s)g(Je_s,e_s)+g(Je_s,e_r)g(Je_r,e_s)$$
$$+2g(e_r,Je_s)g(Je_r,e_s)\Big\}$$

which implies that

$$\sum_{r=1}^{q}\sum_{s=1}^{p}\widetilde{K}\big(e_\alpha\wedge e_\beta\big)=-qpc. \tag{2.23}$$

Therefore, from (2.21), (2.22) and (2.23), we get

$$\sum_{r=1}^{q}\sum_{s=1}^{p}\Big(2||\sigma(e_r,e_s)||^2-g\big(\sigma(e_s,e_s),\sigma(e_r,e_r)\big)\Big)=-p\Delta(\ln h)+p(1-p)||\nabla(\ln h)||^2$$
$$+p||\nabla(\ln h)||^2+\|\sigma_\mu\|^2+qpc. \tag{2.24}$$

As our supposition (1.2) holds if and only if the inequality holds from (2.24)

$$\sum_{r=1}^{q}\sum_{s=1}^{p}\Big(2||\sigma(e_r,e_s)||^2-g\big(\sigma(e_s,e_s),\sigma(e_r,e_r)\big)\Big)<qpc'. \tag{2.25}$$

Applying Theorem 1.1 for complex space form $\mathbb{Q}^{2m}(c)$ of constant holomorphic sectional curvature $c'=4c$, would complete the proof of Theorem 1.2.

Proof of Theorem 1.4

Using the divergence theorem ([28]) on a compact manifold possibly without boundary, we get that $\int_{M^n}(\Delta h)dV = 0,$. Therefore, we have:

$$0 = \int_{M^n} \Delta\Big(\frac{h^2}{2}\Big)dV = -\int_{M^n} div\Big(\nabla\Big(\frac{h^2}{2}\Big)\Big)dV$$
$$= -\int_{M^n} div(h\nabla h)dV = -\int_{M^n} g(\nabla h, \nabla h)dV + \int_{M^n} h\Delta h dV$$

which implies that

$$\int_{M^n} h\Delta h dV = \int_{M^n} \|\nabla h\|^2 dV. \tag{2.26}$$

If the inequality (1.7) holds and from (1.4), we get

$$\int_{M^n} \|\sigma_\mu\|^2 h^2 dV < 3qc\int_{M^n} h^2 dV + p\int_{M^n} \|\nabla h\|^2 dV. \tag{2.27}$$

Adding and subtracting the Dirichlet energy terms, we get

$$(1-p)\int_{M^n} \|\nabla h\|^2 dV + \int_{M^n} \|\sigma_\mu\|^2 h^2 dV < 3q\int_{M^n} ch^2 dV + \int_{M^n} \|\nabla h\|^2 dV.$$

Utilizing the (2.26) into above equation, we find that

$$(1-p)\int_{M^n} \|\nabla h\|^2 dV + \int_{M^n} \|\sigma_\mu\|^2 h^2 dV < 3q\int_{M^n} ch^2 dV + \int_{M^n} h\Delta f dV,$$

which implies the following

$$(1-p)\|\nabla h\|^2 + \|\sigma_\mu\|^2 h^2 < 3qch^2 dV + h\Delta h. \tag{2.28}$$

Using Theorem 1.2, we get desired result (1.7). Hence, the proof is completed.

Proof of Theorem 1.5

From the hypothesis of theorem, the inequality (1.11) is satisfied, then

$$\|\sigma_\mu\|^2 < 3qc + p\lambda_1.$$

First multiplying h^2 in both side in the above equation and then taking integration along the volume element, we have

$$\int_{M^n} \|\sigma_\mu\|^2 h^2 dV < 3qc\int_{M^n} h^2 dV + p\lambda_1\int_{M^n} h^2 dV.$$

Using the property (1.10) for the eigenvalue λ_1 in proceeding equation, we arrive at

$$\int_{M^n} \|\sigma_\mu\|^2 h^2 dV < 3qc\int_{M^n} h^2 dV + p\int_{M^n} \|\nabla h\|^2 dV. \tag{2.29}$$

Adding the term $\int_{M^n} \|\nabla h\|^2 dV$ and using (2.26), we reached

$$(1-p)\int_{M^n} \|\nabla h\|^2 dV + \int_{M^n} \|\sigma_\mu\|^2 h^2 dV < 3qc\int_{M^n} h^2 dV + \int_{M^n} h\Delta h dV$$

which equivalent to the following

$$(1-p)\|\nabla h\|^2 + \|\sigma_\mu\|^2 h^2 < 3qch^2 + h\Delta h \tag{2.30}$$

Therefore, the results follows from Theorem 1.1. The proof is completed.

3. Classifications

The Hessian tensor of any positive smooth function $\varphi \in C^\infty(M^n)$ is defined as

$$\Delta\varphi = -trace\mathcal{H}^\varphi. \tag{3.1}$$

Thus, the above relation and Theorem 1.2 and 1.3 are verifying the new pinching condition on the Hessian tensor of the warping function as in following form.

Corollary 3.1 *Suppose that $M^n = N_T^q \times_h N_\perp^p$ is a compact oriented CR-warped product submanifolds of a complex space form $\mathbb{Q}_c^{2m}(4c)$. If the relation*

$$(1-p)\|\nabla h\|^2 + \|\sigma_\mu\|^2 h^2 + htrace\mathcal{H}^h < 3qch^2, \tag{3.2}$$

holds, then M^n has no stable integral q-currents and has trivial homology groups, i.e.,

$$H_q(M^n, \mathbb{G}) = H_p(M^n, \mathbb{G}) = 0.$$

Analogously, for topological sphere theorem in the terms of *Dirichlet energy* function, we get

Corollary 3.2 *Let $\mathbb{Q}_c^{2m}(4c)$ is a complex space form and $M^n = N_T^q \times_h N_\perp^p$ is compact oriented CR-warped product of $\mathbb{Q}_c^{2m}(4c)$ satisfies the following*

$$\int_{M^n} \|\sigma_\mu\|^2 h^2 dV < 3q\int_{M^n} ch^2 dV + 2p\mathbb{E}(h) \tag{3.3}$$

holds, then

- *if $q+p \neq 3$, then M^{q+p} is homeomorphic to sphere $\mathbb{S}^{q+p}$,*
- *if $q+p = 3$, M^{q+p} is homotopic to a sphere $\mathbb{S}^{q+p}$.*

Similarly, for Lagrangian satisfied the Euler-Lagrange equation, we have

Corollary 3.3 *Let $\mathbb{Q}_c^{2m}(4c)$ is a complex space form and $M^n = N_T^q \times_h N_\perp^p$ is compact oriented CR-warped product of $\mathbb{Q}_c^{2m}(4c)$ with the warping function satisfies the Euler-Language equation and the following inequality*

$$L_h < \frac{1}{2(1-p)}\left\{h^2\left(3qc - \|\sigma_\mu\|^2\right)\right\} \tag{3.4}$$

where L_h is the Lagrangian defined in the equation (1.5). Then, then M^n has no stable integral q-currents and it is homology groups are trivial, i.e., $H_q(M^n, \mathbb{G}) = H_p(M^n, \mathbb{G}) = 0$. In addition, M^{q+p} is homeomorphic to sphere $\mathbb{S}^{q+p}$ when $q+p \neq 3$, and M^{q+p} is homotopic to a sphere $\mathbb{S}^{q+p}$ if $q+p = 3$.

Proof We can deduce the result by using (1.2), (1.5) and utilizing Euler-Lagrange equation condition (1.6). □

The Hamiltonian at any point $x \in M^n$ in the local orthonormal basis can be given as (see [8]) :

$$H(p, x) = \frac{1}{2} \sum_{j=1}^{n} p(e_j)^2. \tag{3.5}$$

Put $p = d\varphi$, where d is the differential operator, then it leads to:

$$H(d\varphi, x) = \frac{1}{2} \sum_{j=1}^{n} d\varphi(e_j)^2 = \frac{1}{2} \sum_{j=1}^{n} e_j(\varphi)^2 = \frac{1}{2} ||\nabla\varphi||^2. \tag{3.6}$$

From (3.6) and (1.2), we derive a new result as follows

Corollary 3.4 *Let $\mathbb{Q}_c^{2m}(4c)$ is a complex space form and $M^n = N_T^q \times_h N_\perp^p$ is compact oriented CR-warped product of $\mathbb{Q}_c^{2m}(4c)$ the following relation hold*

$$H\big(dh, x\big) < \frac{h}{2(1-p)} \left\{ \Delta h + h\left(3qc - \|\sigma_\mu\|^2\right) \right\}, \tag{3.7}$$

where $H(dh, x)$ is the Hamiltonian of h. Then, then M^n has no stable integral q-currents and it is homology groups are trivial, i.e., $H_q(M^n, \mathbb{G}) = H_p(M^n, \mathbb{G}) = 0$.

Proof Using (3.6) in (1.2), we get desired result. □

Corollary 3.5 *Let $\mathbb{Q}_c^{2m}(4c)$ is a complex space form and $M^n = N_T^q \times_h N_\perp^p$ is compact oriented CR-warped product of $\mathbb{Q}_c^{2m}(4c)$, and the following condition hold*

$$H\big(df, x\big) < \frac{h}{2(1-p)} \left\{ \Delta h + h\left(3qc - \|\sigma_\mu\|^2\right) \right\}. \tag{3.8}$$

Then M^n is homeomorphic to sphere $\mathbb{S}^n$ when $q+p \neq 3$ and if $q+p = 3$, then M^{q+p} is homotopic to a sphere $\mathbb{S}^n$.

4. Conclusion Remark

There are many applications of the singularity structure at liquid crystals, at statistical mechanics considering low dimensions, as well as at physical phase transitions ([13]). additionally, the General relativity includes warped product manifolds as the kind of space-times. The two famous warped product spaces are the general case of Robertson- Walker space-times and the standard static space-times[22, 25]. General relativity relies essentially on differential topological methods specifically at mathematical physics. In particular, how quantum gravity is using the space-time homology ([22]). As this study joining warped product manifold and homotopy-homology theory, its results should be beneficial for the physical applications.

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