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# Generalized Stević-Sharma type operators from Hardy spaces into $n$th weighted type spaces

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**Abstract:** In this paper, some characterizations for boundedness, essential norm and compactness of generalized Stević-Sharma type operators from Hardy spaces into $n$th weighted type spaces are given.

**Key words:** essential norm, generalized Stević-Sharma type operators, $n$th weighted type spaces, Hardy spaces.

## 1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. Every positive and continuous function on $\mathbb{D}$ is called a weight. Suppose that $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\mu$ be a weight. The $n$th weighted type space $\mathcal{W}_\mu^{(n)}(\mathbb{D}) = \mathcal{W}_\mu^{(n)}$ consists of all analytic functions on $\mathbb{D}$ for which the following statement is finite

$$b_{\mathcal{W}_\mu^{(n)}}(f) := \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)|.$$

The above statement is just a semi norm and $\mathcal{W}_\mu^{(n)}$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{W}_\mu^{(n)}} = \sum_{i=0}^{n-1} |\, f^{(i)}(0) \,| + b_{\mathcal{W}_\mu^{(n)}}(f),$$

see for example [1, 9, 10]. Let $\alpha > 0$. Then $\mathcal{W}^{(1)}_{(1-|z|^2)^\alpha} = \mathcal{B}^\alpha$ (Bloch type space), $\mathcal{W}^{(2)}_{(1-|z|^2)^\alpha} = \mathcal{Z}^\alpha$ (Zygmund type space) and $\mathcal{W}^{(1)}_{(1-|z|^2)\log\frac{2}{1-|z|^2}}$ coincides with the logarithmic Bloch space $\mathcal{B}_{\log}$. Also $\mathcal{W}_\mu^{(0)} = H_\mu$ (weighted type space), $\mathcal{W}_\mu^{(1)} = \mathcal{B}\mu$ (weighted Bloch space) and $\mathcal{W}_\mu^{(2)} = \mathcal{Z}_\mu$ (weighted Zygmund space). For more information about Bloch type spaces or Zygmund type spaces see [8, 15, 16].

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For $0<p<\infty$ a function $f\in H(\mathbb{D})$ belong to the Hardy space $H^p$ if

$$\|f\|_{H^p}=\sup_{0<r<1}\Big(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\Big)^{\frac{1}{p}}<\infty.$$

where $0<p<\infty$. If $1\le p<\infty$, $H^p$ is a Banach space and if $0<p<1$, $H^p$ is nonlocally convex topological vector space and in this case it is a complete metric space (see [4]).

For Banach spaces X and Y and a continuous linear operator $T:X\to Y$, the essential norm is the distance of $T$ from the space of all compact operators, that is

$$\|T\|_e=\inf\{\|T-K\|:K:X\to Y \text{ is compact}\}.$$

$T$ is compact if and only if $\|T\|_e=0$.

Let $u,v\in H(\mathbb{D})$ and $\varphi\in S(\mathbb{D})$, the set of all analytic self-maps of $\mathbb{D}$. The Stević-Sharma type operator is defined as follows

$$T_{u,v,\varphi}f(z)=u(z)f(\varphi(z))+v(z)f'(\varphi(z)),\quad f\in H(\mathbb{D}),\quad z\in\mathbb{D}.$$

Indeed $T_{u,v,\varphi}=uC_\varphi+vC_\varphi D$ where $D$ is the differentiation operator and $C_\varphi$ is composition operator. More information about this operator can be found in [7, 11, 12].

From the above definition we generalize the Stević-Sharma type operator. Let $m\in\mathbb{N}$, $u,v\in H(\mathbb{D})$ and $\varphi\in S(\mathbb{D})$. We denote the generalized Stević-Sharma type operator with $T^m_{u,v,\varphi}$ and define it as follows

$$T^m_{u,v,\varphi}f(z)=(uC_\varphi f)(z)+(D^m_{\varphi,v}f)(z)\qquad f\in H(\mathbb{D}),\quad z\in\mathbb{D},$$

where $D^m_{\varphi,u}$ is the generalized weighted composition operator. When $v=0$, then $T^m_{u,0,\varphi}=uC_\varphi$ is the well-known weighted composition operator. If $u=0$, then $T^m_{0,v,\varphi}=D^m_{\varphi,v}$ and for $m=1$, $T^m_{u,v,\varphi}$ is Stević-Sharma type operator.

For $n,k\in\mathbb{N}_0$ and $k\le n$, the partial Bell polynomials are triangulares

$$B_{n,k}(x_1,x_2,...,x_{n-k+1})=\sum\frac{n!}{\prod_{t=1}^{n-k+1}j_t!}\prod_{t=1}^{n-k+1}(\frac{x_t}{t!})^{j_t}.$$

In the above equation we take the sum over all sequences $j_1,j_2,...,j_{n-k+1}$ of nonnegative integers with the following properties

$$\sum_{t=1}^{n-k+1}j_t=k \quad\text{and}\quad \sum_{t=1}^{n-k+1}tj_t=n.$$

See [3, pp 134].

In this paper, first we obtain some characterizations for boundedness of operator $T^m_{u,v,\varphi}:H^p\to\mathcal{W}^{(n)}_\mu$. Then estimations for the essential norm of these operators are given. Finally some equivalence conditions for compactness of generalized Stević-Sharma type operators from Hardy spaces into $n$th weighted type spaces are presented. As some applications, we get some characterizations for boundedness, essential norm and compactness of (generalized) weighted composition operators from the Hardy spaces into $n$th weighted type spaces.

By $A\succeq B$ we mean there exists a constant $C$ such that $A\ge CB$ and $A\approx B$ means that $A\succeq B\succeq A$.

## 2. Preliminaries

This section is devoted to giving some lemmas we use in the next sections.

**Lemma 2.1 ([16], Propositions 7 and 8)** *Let* $\alpha > 0$ *and* $H_\alpha^\infty = \mathcal{W}^{(0)}_{(1-|z|^2)^\alpha}$. *Then* $H_\alpha^\infty = \mathcal{B}^{\alpha+1}$. *Moreover, for any* $f \in \mathcal{B}^\alpha$ *and* $n \in \mathbb{N}$,

$$\|f\|_{\mathcal{B}^\alpha} \approx \sum_{i=0}^{n} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n} |f^{(n+1)}(z)|.$$

**Lemma 2.2 ([5], Lemma 2.1)** *Let* $\alpha > 0$. *The sequence* $\{j^{\alpha-1} z^j\}_1^\infty$ *is bounded in* $\mathcal{B}_0^\alpha$ *and*

$$\lim_{j\to\infty} j^{\alpha-1} \|z^j\|_{\mathcal{B}^\alpha} = \left(\frac{2\alpha}{e}\right)^\alpha.$$

**Lemma 2.3** ([4]) *Let* $0 < p < \infty$, $n \in \mathbb{N}_0$ *and* $f \in H^p$. *Then*

$$|f^{(n)}(z)| \preceq \frac{\|f\|_{H^p}}{(1-|z|^2)^{\frac{1}{p}+n}}, \qquad z \in \mathbb{D}.$$

Let $u \in H(\mathbb{D})$, $i$ and $n$ be integer numbers. For simplicity in calculation, we set

$$I^{n,u}_{i,\varphi}(z) = \begin{cases} \sum_{l=i}^{n} \binom{n}{l} u^{(n-l)}(z) B_{l,i}(\varphi'(z), ..., \varphi^{(l-i+1)}(z)) & 0 \le i \le n \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases}$$

The proof of next lemma resembles to the proof of Lemma 4 [10], therefore it is omitted.

**Lemma 2.4** *Let* $f, u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ *and* $m, n \in \mathbb{N}_0$. *If* $T^m_{u,v,\varphi} = uC_\varphi + D^m_{\varphi,v}$, *then*

$$\left(T^m_{u,v,\varphi} f\right)^{(n)}(z) = \sum_{i=0}^{m+n} f^{(i)}(\varphi(z)) (I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z).$$

For any $a \in \mathbb{D}$ and $j \in \mathbb{N}$, set

$$f_{j,a}(z) = \frac{(1-|a|^2)^j}{(1-\overline{a}z)^{\frac{1}{p}+j}}. \tag{2.1}$$

One can see that $f_{j,a} \in H^p$, for each $j \in \mathbb{N}$, $\sup_{a\in\mathbb{D}} \|f_{j,a}\|_{H^p} < \infty$ and $f_{j,a}$ converges to $0$ as $|a| \to 1$.

**Lemma 2.5** *Let* $m, n \in \mathbb{N}$ *such that* $n \ge m$. *For any* $0 \ne a \in \mathbb{D}$ *and* $i \in \{0, 1, ..., m+n\}$, *there exists a function* $g_{i,a} \in H^p$ *such that*

$$g^{(k)}_{i,a}(a) = \frac{\overline{a}^k \delta_{ik}}{(1-|a|^2)^{\frac{1}{p}+k}},$$

*where* $\delta_{ik}$ *is Kronecker delta. If* $i \in \{0, 1, ..., m-1\}$, *then* $g_{i,a} \in span\{f_{1,a}, ..., f_{m,a}\}$ *and for* $i \in \{m, ..., n\}$, $g_{i,a} \in span\{f_{m+1,a}, ..., f_{n+1,a}\}$ *also when* $i \in \{n+1, ..., m+n\}$

$$g_{i,a} \in span\{f_{n+2,a}, ..., f_{m+n+1,a}\}.$$

**Proof** For any fixed $0 \neq a \in \mathbb{D}$ and coefficients $c_1, ..., c_{m+n+1}$ we set

$$e_{1,a,c_1,...,c_m}(z) = \sum_{j=1}^{m} c_j f_{j,a}(z),$$

$$e_{2,a,c_{m+1},...,c_{n+1}}(z) = \sum_{j=1}^{n-m+1} \frac{c_{j+m}}{\prod_{t=0}^{m-1}(m+j+\frac{1}{p}+t)} f_{j+m,a}(z)$$

$$e_{3,a,c_{n+2},...,c_{m+n+1}}(z) = \sum_{j=1}^{m} \frac{c_{j+1+n}}{\prod_{t=0}^{n}(n+1+j+\frac{1}{p}+t)} f_{j+1+n,a}(z),$$

where $f_{j,a}$ are defined in (2.1). For each $i \in \{0, 1, ..., m+n\}$ the system of linear equations

$$\begin{aligned}
e_{1,a,c_1,...,c_m}(a) &= \frac{1}{(1-|a|^2)^{\frac{1}{p}}} \sum_{j=1}^{m} c_j = \frac{\delta_{i0}}{(1-|a|^2)^{\frac{1}{p}}} \\
&\vdots \\
e^{(m-1)}_{1,a,c_1,...,c_m}(a) &= \frac{\overline{a}^{m-1}}{(1-|a|^2)^{m-1+\frac{1}{p}}} \sum_{j=1}^{m} c_j \prod_{t=0}^{m-2}(j+\frac{1}{p}+t) = \frac{\overline{a}^{m-1}\delta_{i(m-1)}}{(1-|a|^2)^{m-1+\frac{1}{p}}} \\
e^{(m)}_{2,a,c_{m+1},...,c_{n+1}}(a) &= \frac{\overline{a}^{m}}{(1-|a|^2)^{m+\frac{1}{p}}} \sum_{j=1}^{n-m+1} c_{j+m} = \frac{\overline{a}^{m}\delta_{im}}{(1-|a|^2)^{m+\frac{1}{p}}} \\
&\vdots \\
e^{(n)}_{2,a,c_{m+1},...,c_{n+1}}(a) &= \frac{\overline{a}^{n}}{(1-|a|^2)^{n+\frac{1}{p}}} \sum_{j=1}^{n-m+1} c_{j+m} \prod_{t=m}^{n-1}(m+j+\frac{1}{p}+t) = \frac{\overline{a}^{n}\delta_{in}}{(1-|a|^2)^{n+\frac{1}{p}}} \\
e^{(n+1)}_{3,a,c_{n+2},...,c_{m+n+1}}(a) &= \frac{\overline{a}^{n+1}}{(1-|a|^2)^{n+1+\frac{1}{p}}} \sum_{j=1}^{m} c_{j+1+m} = \frac{\overline{a}^{n+1}\delta_{i(n+1)}}{(1-|a|^2)^{n+1+\frac{1}{p}}} \\
&\vdots \\
e^{(m+n)}_{3,a,c_{n+2},...,c_{m+n+1}}(a) &= \quad \dots \quad = \frac{\overline{a}^{m+n}\delta_{i(m+n)}}{(1-|a|^2)^{m+n+\frac{1}{p}}}
\end{aligned}$$

has a unique solution [9, Lemma 2.3] which is independent of the choice of $a$ and therefore it can be shown by $(c^i_1, c^i_2, ..., c^i_{m+n+1})$. Now we set

$$g_{i,a}(z) = e_{1,a,c^i_1,...,c^i_m}(z) + e_{2,a,c^i_{m+1},...,c^i_{n+1}}(z) + e_{3,a,c^i_{n+2},...,c^i_{m+n+1}}(z).$$

□

The proof of the following lemma is similar to the proof of the previous lemma so it is omitted.

**Lemma 2.6** *Let $m, n \in \mathbb{N}$ such that $n < m$. For any $0 \neq a \in \mathbb{D}$ and $i \in \{0, ..., n\} \cup \{m, ..., m+n\}$, there exists a function $g_{i,a} \in H^p$ such that*

$$g^{(k)}_{i,a}(a) = \frac{\overline{a}^k \delta_{ik}}{(1-|a|^2)^{\frac{1}{p}+k}}.$$

*Also for* $i \in \{0,1,...,n\}$ *then* $g_{i,a} \in span\{f_{1,a},...,f_{n+1,a}\}$ *and when* $i \in \{m, m+1, ..., m+n\}$

$$g_{i,a} \in span\{f_{m+1,a},...,f_{m+n+1,a}\}.$$

In Sections 3 and 4, $m,n \in \mathbb{N}$, $0<p<\infty$, $u,v \in H(\mathbb{D})$, $\mu$ is a weight and $\varphi \in S(\mathbb{D})$.

## 3. Boundedness

In this section, we give some necessary and sufficient conditions for the generalized Stević-Sharma type operators to be bounded.

**Theorem 3.1** *Let* $u \in \mathcal{W}_\mu^{(n)}$. *If* $n \geq m$, *then the following statements are equivalent.*

*(i) The operator* $T^m_{u,v,\varphi} : H^p \to \mathcal{W}_\mu^{(n)}$ *is bounded.*

*(ii) If* $p_j(z)=z^j$, *then* $\sup_{j\geq 1} j^{\frac{1}{p}} \|T^m_{u,v,\varphi} p_j\|_{\mathcal{W}_\mu^{(n)}} < \infty$ .

*(iii) For each* $i \in \{0,1,...,m+n\}$, $\sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty$ *and* $\sup_{z\in\mathbb{D}} \mu(z)\left|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\right| < \infty$.

*(iv) For each* $i \in \{0,1,...,m+n\}$, $\sup_{z\in\mathbb{D}} \frac{\mu(z)\left|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-m,\varphi})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}} < \infty$.

**Proof** $(i) \Rightarrow (iii)$ For $i \in \{0,1,...,m+n\}$, $sup_{a\in\mathbb{D}}\|f_{i+1,a}\|_{H^p} < \infty$, so

$$\sup_{a\in\mathbb{D}} \| T^m_{u,v,\varphi} f_{i+1,a} \|_{\mathcal{W}_\mu^{(n)}} \leq \| T^m_{u,v,\varphi} \|_{H^p \to \mathcal{W}_\mu^{(n)}} \sup_{a\in\mathbb{D}} \|f_{i+1,a}\|_{H^p} < \infty.$$

Applying the operator $T^m_{u,v,\varphi}$ to $p_j(z)=z^j$ for $j=0,1,...,m+n$ respectively and using Lemma 2.4, we obtain the other part of $(iii)$.

$(iii) \Rightarrow (iv)$ For any $i \in \{0,1,...,m+n\}$ and $\varphi(a) \neq 0$, by using Lemmas 2.4 and 2.5, we obtain

$$\frac{\mu(a) \mid \varphi(a) \mid^i \mid (I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(a) \mid}{(1- \mid \varphi(a) \mid^2)^{i+\frac{1}{p}}} \leq \sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} g_{i,\varphi(a)}\|_{\mathcal{W}_\mu^{(n)}} \leq \sum_{j=1}^{m+n} \mid c^i_j \mid \sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{j,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty.$$

From the previous inequality,

$$\sup_{|\varphi(a)|>\frac{1}{2}} \frac{\mu(a) \mid (I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(a) \mid}{(1- \mid \varphi(a) \mid^2)^{i+\frac{1}{p}}} \preceq \sum_{j=1}^{m+n} \mid c^i_j \mid \sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{j,a}\|_{\mathcal{W}_\mu^{(n)}} < \infty,$$

and from $(iii)$, we get

$$\sup_{|\varphi(a)|\leq\frac{1}{2}} \frac{\mu(a) \mid (I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(a) \mid}{(1- \mid \varphi(a) \mid^2)^{i+\frac{1}{p}}} \preceq \sup_{|\varphi(a)|\leq\frac{1}{2}} \mu(a) \mid (I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(a) \mid < \infty.$$

Hence from last inequalities, we get $(iv)$.

$(iv) \Rightarrow (i)$ For any $f \in H^p$, by using Lemmas 2.4 and 2.3, we have

$$\mu(z)\Big|(T^m_{u,v,\varphi}f)^{(n)}(z)\Big| \leq \mu(z)\Big|\sum_{i=0}^{m+n} f^{(i)}(\varphi(z))(I^{n,u}_{i,\varphi}(z) + I^{n,v}_{i-m,\varphi}(z))\Big|$$

$$\leq \|f\|_{H^p} \sum_{i=0}^{m+n} \sup_{z\in\mathbb{D}} \frac{\mu(z)\Big|I^{n,u}_{i,\varphi}(z) + I^{n,v}_{i-m,\varphi}(z)\Big|}{(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i}}. \tag{3.1}$$

Also for each $k < n$

$$\Big|(T^m_{u,v,\varphi}f)^{(k)}(0)\Big| \leq \|f\|_{H^p} \sum_{i=0}^{m+k} |(I^{k,u}_{i,\varphi} + I^{k,v}_{i-m,\varphi})(0)|. \tag{3.2}$$

Hence, from (3.1) and (3.2), the operator $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ is bounded.

$(ii) \Rightarrow (iii)$ For each $i \in \{0, ..., m+n\}$ and $a \in \mathbb{D}$

$$f_{i+1,a}(z) = (1-|a|^2)^{i+1} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{p}+i+1+j)}{j!\Gamma(\frac{1}{p}+i+1)} \bar{a}^j z^j.$$

So,

$$\|T^m_{u,v,\varphi}f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} \preceq (1-|a|^2)^{i+1} \sum_{j=0}^{\infty} j^{i+\frac{1}{p}} |\bar{a}|^j \|T^m_{u,v,\varphi}p_j\|_{\mathcal{W}^{(n)}_\mu} \preceq 2^{i+1} \max\{\|u\|_{\mathcal{W}^{(n)}_\mu}, \sup_{j\geq 1} j^{\frac{1}{p}} \|T^m_{u,v,\varphi}p_j\|_{\mathcal{W}^{(n)}_\mu}\}.$$

Therefore, $\sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi}f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} < \infty$. The proof of other part is similar to the proof $(i) \Rightarrow (iii)$.

$(iv) \Rightarrow (ii)$ Let $p_j(z) = z^j (j \geq n)$. By using Lemmas 2.1, 2.2 and 2.4, we get

$$j^{\frac{1}{p}}\mu(z)|(T^m_{u,v,\varphi}p_j)^{(n)}(z)| \leq \mu(z) \sum_{i=0}^{m+n} j^{\frac{1}{p}}(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i} \frac{j!}{(j-i)!} \times \frac{|\varphi(z)|^{j-i}|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)|}{(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i}}$$

$$\leq \sup_j j^{\frac{1}{p}} \|z^j\|_{\mathcal{B}^{\frac{1}{p}+1}} \sum_{i=0}^{m+n} \frac{|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)|}{(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i}}$$

$$\preceq \Big(\frac{2(\frac{1}{p}+1)}{e}\Big)^{\frac{1}{p}+1} \sum_{i=0}^{m+n} \frac{|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)|}{(1-\mid\varphi(z)\mid^2)^{\frac{1}{p}+i}}. \tag{3.3}$$

For any $k < n$, we have

$$j^{\frac{1}{p}}|(T^m_{u,v,\varphi}p_j)^{(k)}(0)| \preceq \Big(\frac{2(\frac{1}{p}+1)}{e}\Big)^{\frac{1}{p}+1} \sum_{i=0}^{m+k} j^{\frac{1}{p}} \frac{|(I^{k,u}_{i,\varphi} + I^{k,v}_{i-m,\varphi})(0)|}{(1-|\varphi(0)|^2)^{\frac{1}{p}+i}}. \tag{3.4}$$

From (3.3) and (3.4), we obtain $(ii)$. The proof is completed. □

In the same way as in the proof of Theorem 3.1 we can prove the following theorem, just use Lemma 2.6 instead of Lemma 2.5.

**Theorem 3.2** *Let $u \in \mathcal{W}^{(n)}_\mu$. If $n < m$, then the following statements are equivalent.*

*(i) The operator* $T^m_{u,v,\varphi}: H^p \to \mathcal{W}^{(n)}_\mu$ *is bounded.*

*(ii) If* $p_j(z) = z^j$, *then* $\sup_{j\geq 1} j^{\frac{1}{p}} \|T^m_{u,v,\varphi,u} p_j\|_{\mathcal{W}^{(n)}_\mu} < \infty$.

*(iii) For each* $i \in \{0,1,...,n\} \cup \{m,...,m+n\}$, $\sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} < \infty$ *and*

$$\sup_{z\in\mathbb{D}} \mu(z)\Big|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\Big| < \infty.$$

*(iv) For each* $i \in \{0,1,...,n\} \cup \{m,...,m+n\}$, $\sup_{z\in\mathbb{D}} \frac{\mu(z)\Big|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\Big|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}} < \infty$.

## 4. Essential norm

In this section, we obtain some estimates for the essential norm of generalized Stević-Sharma type operators from Hardy spaces into $n$th weighted type spaces. Then we give some equivalence conditions for compactness of such operators.

**Theorem 4.1** *Let* $T^m_{u,v,\varphi}: H^p \to \mathcal{W}^{(n)}_\mu$ *is bounded. If* $n \geq m$, *then*

$$\|T^m_{u,v,\varphi}\|_e \approx \max\{A_i\}^{m+n}_{i=0} \approx \max\{B_i\}^{m+n}_{i=0},$$

*where*

$$A_i = \limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu}, \quad B_i = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\Big|I^{n,u}_{i,\varphi}(z) + I^{n,v}_{i-m,\varphi}(z)\Big|}{(1- \mid \varphi(z) \mid^2)^{\frac{1}{p}+i}}.$$

**Proof** For all $i \in \{0,...,m+n\}$, $\sup_{a\in\mathbb{D}} \|f_{i+1,a}\|_{H^p} < \infty$ and $f_{i+1,a}$ converge to $0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Using Lemma 2.10 [13], for any compact operator $K$ from $H^p$ into $\mathcal{W}^{(n)}_\mu$, we get

$$\lim_{|a|\to 1} \|K f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} = 0.$$

Thus, for any $i \in \{0,...,m+n\}$

$$\begin{aligned}\|T^m_{u,v,\varphi} - K\|_{H^p\to\mathcal{W}^{(n)}_\mu} &\succeq \limsup_{|a|\to 1} \|(T^m_{u,v,\varphi} - K) f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} \\ &\geq \limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} - \limsup_{|a|\to 1} \|K f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} = A_i.\end{aligned}$$

So,

$$\|T^m_{u,v,\varphi}\|_e = \inf_K \|T^m_{u,v,\varphi} - K\|_{H^p\to\mathcal{W}^{(n)}_\mu} \succeq \max\{A_i\}^{m+n}_{i=0}.$$

Now, we prove that

$$\max\{B_i\}^{m+n}_{i=0} \preceq \|T^m_{u,v,\varphi}\|_e. \tag{4.1}$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Since $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ is bounded, using Lemmas 2.4 and 2.5 for any compact operator $K : H^p \to \mathcal{W}^{(n)}_\mu$ and $i \in \{0, ..., m+n\}$, we obtain

$$\begin{aligned}\|T^m_{u,v,\varphi} - K\|_{H^p\to\mathcal{W}^{(n)}_\mu} &\succeq \limsup_{j\to\infty} \|T^m_{u,v,\varphi}(g_{i,\varphi(z_j)})\|_{\mathcal{W}^{(n)}_\mu} - \limsup_{j\to\infty} \|K(g_{i,\varphi(z_j)})\|_{\mathcal{W}^{(n)}_\mu}\\ &\succeq \limsup_{j\to\infty} \frac{\mu(z_j) \mid \varphi(z_j) \mid^i \mid I^{n,u}_{i,\varphi}(z_j) + I^{n,v}_{i-m,\varphi}(z_j) \mid}{(1- \mid \varphi(z_j) \mid^2)^{i+\frac{1}{p}}} = B_i.\end{aligned}$$

From the last inequality, we get (4.1).

For each $0 < r < 1$ we consider the compact operator $K_r$ on $H^p$ given by $K_r f(z) = f_r(z) = f(rz)$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Since $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$ then for any positive integer $j$, the operator $T^m_{u,v,\varphi} K_{r_j} : H^p \to \mathcal{W}^{(n)}_\mu$ is compact. So

$$\|T^m_{u,v,\varphi}\|_e \leq \limsup_{j\to\infty} \|T^m_{u,v,\varphi} - T^m_{u,v,\varphi} K_{r_j}\|. \tag{4.2}$$

Hence, it is sufficient to prove that

$$\limsup_{j\to\infty} \|T^m_{u,v,\varphi} - T^m_{u,v,\varphi} K_{r_j}\| \preceq \min\{\max\{A_i\}_{i=0}^{m+n}, \max\{B_i\}_{i=0}^{m+n}\}.$$

For any $f \in H^p$ such that $\|f\|_{H^p} \leq 1$,

$$\begin{aligned}&\|(T^m_{u,v,\varphi} - T^m_{u,v,\varphi} K_{r_j})f\|_{\mathcal{W}^{(n)}_\mu} \leq\\ &\sum_{t=0}^{n-1} \underbrace{\left| \left(T^m_{u,v,\varphi}(f - f_{r_j})\right)^{(t)}(0) \right|}_{S_t} + \underbrace{\sup_{|\varphi(z)|\leq r_N} \mu(z) \left| \sum_{k=0}^{m+n} (f - f_{r_j})^{(k)}(\varphi(z)) \left(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi}\right)(z) \right|}_{H_1}\\ &+ \underbrace{\sup_{|\varphi(z)|>r_N} \mu(z) \left| \sum_{k=0}^{m+n} (f - f_{r_j})^{(k)}(\varphi(z)) \left(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi}\right)(z) \right|}_{H_2},\end{aligned} \tag{4.3}$$

where $N \in \mathbb{N}$ such that $r_j \geq \frac{2}{3}$ for all $j \geq N$. Since $(f - f_{r_j})^{(s)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, for any nonnegative integer $s$, then Theorem 3.1 implies that

$$\limsup_{j\to\infty} H_1 = \limsup_{j\to\infty} S_t = 0 \quad (t = 0, ..., n-1). \tag{4.4}$$

Also

$$\begin{aligned}H_2 \leq &\sum_{k=0}^{m+n} \underbrace{\sup_{|\varphi(z)|>r_N} \mu(z)|f^{(k)}(\varphi(z))||(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}_{M_{2,k}}\\ &+ \sum_{k=0}^{m+n} \underbrace{\sup_{|\varphi(z)|>r_N} \mu(z)|r_j^k f^{(k)}(r_j\varphi(z))||(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}_{N_{2,k}}.\end{aligned} \tag{4.5}$$

For $M_{2,k}$, $k \in \{0, \ldots, m+n\}$, from Lemmas 2.3, 2.4 and 2.5, we get

$$M_{2,k} = \sup_{|\varphi(z)|>r_N} \mu(z) \frac{(1-|\varphi(z)|^2)^{k+\frac{1}{p}} |f^{(k)}(\varphi(z))|}{|\varphi(z)|^k} \times \frac{|\varphi(z)|^k |(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}{(1-|\varphi(z)|^2)^{k+\frac{1}{p}}}$$

$$\preceq \|f\|_{H^p} \sup_{|\varphi(z)|>r_N} \|T^m_{u,v,\varphi} g_{k,\varphi(z)}\|_{\mathcal{W}^{(n)}_\mu} \preceq \sum_{j=0}^{m+n} |c^k_{j+1}| \sup_{|a|>r_N} \|T^m_{u,v,\varphi} f_{j+1,a}\|_{\mathcal{W}^{(n)}_\mu}$$

As $N \to \infty$, we obtain

$$\limsup_{j\to\infty} M_{2,k} \preceq \sum_{i=0}^{m+n} \underbrace{\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu}}_{A_i} \preceq \max\{A_i\}_{i=0}^{m+n} \quad \text{and} \limsup_{j\to\infty} M_{2,k} \preceq B_k. \tag{4.6}$$

Similarly, we get

$$\limsup_{j\to\infty} N_{2,k} \preceq \sum_{i=0}^{m+n} \underbrace{\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu}}_{A_i} \preceq \max\{A_i\}_{i=0}^{m+n} \quad \text{and} \limsup_{j\to\infty} N_{2,k} \preceq B_k. \tag{4.7}$$

Thus, by using (4.3), (4.4), (4.5), (4.6) and (4.7), we obtain

$$\limsup_{j\to\infty} \|T^m_{u,v,\varphi} - T^m_{u,v,\varphi} K_{r_j}\|_{H^p \to \mathcal{W}^{(n)}_\mu} = \limsup_{j\to\infty} \sup_{\|f\|_{H^p}\le 1} \|(T^m_{u,v,\varphi} - T^m_{u,v,\varphi} K_{r_j}) f\|_{\mathcal{W}^{(n)}_\mu} \preceq \max\{A_i\}_{i=0}^{m+n}$$

and

$$\limsup_{j\to\infty} \|T^m_{u,v,\varphi} - T^m_{u,v,\varphi} K_{r_j}\|_{H^p \to \mathcal{W}^{(n)}_\mu} \preceq \max\{B_i\}_{i=0}^{m+n}.$$

Hence, from (4.2),

$$\|T^m_{u,v,\varphi}\|_e \preceq \min\{\max\{A_i\}_{i=0}^{m+n}, \max\{B_i\}_{i=0}^{m+n}\}.$$

The proof is completed. □

The proof of the next theorem is similar to the proof of Theorem 4.1, except that we use Lemma 2.6 instead of Lemma 2.5.

**Theorem 4.2** *Let $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ is bounded. If $n < m$, then*

$$\|T^m_{u,v,\varphi}\|_e \approx \max\{\{A_i\}_{i=0}^{n} \cup \{A_i\}_{i=m}^{m+n}\} \approx \max\{\{B_i\}_{i=0}^{n} \cup \{B_i\}_{i=m}^{m+n}\},$$

*where*

$$A_i = \limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu}, \quad B_i = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}}$$

*and $i \in \{0, 1, \ldots n\} \cup \{m, \ldots, m+n\}$.*

**Theorem 4.3** *Let $T^m_{u,v,\varphi}: H^p \to \mathcal{W}^{(n)}_\mu$ is bounded. If $n \geq m$, then the following statements are equivalent.*

*(i) The operator $T^m_{u,v,\varphi}: H^p \to \mathcal{W}^{(n)}_\mu$ is compact.*

*(ii) If $p_j(z) = z^j$ then, $\lim_{j\to\infty} j^{\frac{1}{p}} \|T^m_{u,v,\varphi,u} p_j\|_{\mathcal{W}^{(n)}_\mu} = 0$.*

*(iii) For each $i \in \{0,1,...,m+n\}$, $\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} = 0$.*

*(iv) For each $i \in \{0,1,...,m+n\}$ $\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}} = 0$.*

**Proof** By using Theorem 4.1, $(i), (iii)$ and $(iv)$ are equivalent.

$(ii) \Rightarrow (iii)$ For any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $k \geq N$,

$$k^{\frac{1}{p}} \|T^m_{u,v,\varphi} p_k\|_{\mathcal{W}^{(n)}_\mu} < \epsilon.$$

For any $j \in \{0,1,...,m+n\}$

$$f_{j+1,a}(z) = (1-|a|^2)^{j+1}\Big(\sum_{k=0}^{N-1} + \sum_{k=N}^{\infty}\Big) \frac{\Gamma(\frac{1}{p}+j+1+k)}{k!\Gamma(\frac{1}{p}+j+1)} \overline{a}^k z^k.$$

So,

$$\|T^m_{u,v,\varphi} f_{j+1,a}\|_{\mathcal{W}^{(n)}_\mu} \leq 2\max\{\|u\varphi^k\|_{\mathcal{W}^{(n)}_\mu}\}_{k=0}^{N-1} (1-|a|^2)^j (1-|a|^N) \frac{\Gamma(\frac{1}{p}+N+j)}{N!\Gamma(\frac{1}{p}+j)} + 2^{j+1}\epsilon.$$

Hence, $\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{j+1,a}\|_{\mathcal{W}^{(n)}_\mu} \leq 2^{j+1}\epsilon$. Since $\epsilon$ is arbitrary, we obtain

$$\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{j+1,a}\|_{\mathcal{W}^{(n)}_\mu} = 0.$$

$(iv) \Rightarrow (ii)$ For any given $\epsilon > 0$ and $k \in \{0,1,...,m+n\}$ there exists a positive constant $\delta$ such that $\delta < |\varphi(z)| < 1$,

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+k}} < \epsilon. \tag{4.8}$$

Let $p_j(z) = z^j (j \geq n)$. By using Lemma 2.4, we have

$$\begin{aligned} & j^{\frac{1}{p}} \|T^m_{u,v,\varphi} p_j\|_{\mathcal{W}^{(n)}_\mu} \leq \\ & \sum_{t=0}^{n-1} \underbrace{j^{\frac{1}{p}} \left|\left(T^m_{u,v,\varphi} p_j\right)^{(t)}(0)\right|}_{S_t} + \underbrace{\sup_{|\varphi(z)|\leq r_N} \mu(z) \sum_{k=0}^{m+n} j^{\frac{1}{p}} \frac{j!}{(j-k)!} |\varphi(z)|^{j-k} |(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}_{H_1} \\ & + \underbrace{\sup_{|\varphi(z)|> r_N} \mu(z) \sum_{k=0}^{m+n} j^{\frac{1}{p}} (1-|\varphi(z)|^2)^{\frac{1}{p}+k} \frac{j!}{(j-k)!} |\varphi(z)|^{j-k} \frac{|(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+k}}}_{H_2}. \end{aligned} \tag{4.9}$$

From Theorem 3.1, it is obvious that

$$\limsup_{j\to\infty} H_1 = \limsup_{j\to\infty} S_t = 0 \quad (t = 0, ..., n-1). \tag{4.10}$$

By using Lemma 2.1, 2.2 and (4.8), we obtain

$$H_2 \preceq \Big(\frac{2(\frac{1}{p}+1)}{e}\Big)^{\frac{1}{p}+1} \sum_{k=0}^{m-1} \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|(I^{n,u}_{k,\varphi} + I^{n,v}_{k-m,\varphi})(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+k}} \preceq \epsilon$$

which implies that

$$\limsup_{j\to\infty} H_2 = 0. \tag{4.11}$$

From (4.9), (4.10) and (4.11), we get $(ii)$. The proof is completed. □

Using the same method as in the proof of Theorem 4.3 we can get the following theorem.

**Theorem 4.4** *Let $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ is bounded. If $n < m$, then the following statements are equivalent.*

*(i) The operator $T^m_{u,v,\varphi} : H^p \to \mathcal{W}^{(n)}_\mu$ is compact.*

*(ii) If $p_j(z) = z^j$ then $\lim_{j\to\infty} j^{\frac{1}{p}} \|T^m_{u,v,\varphi,u} p_j\|_{\mathcal{W}^{(n)}_\mu} = 0$.*

*(iii) For each $i \in \{0, 1, ...n\} \cup \{m, ..., m+n\}$, $\limsup_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^{(n)}_\mu} = 0$.*

*(iv) For each $i \in \{0, 1, ...n\} \cup \{m, ..., m+n\}$, $\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-m,\varphi})(z)\right|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i}} = 0$.*

**Remark 4.5** *By putting $v \equiv 0$ in Theorems 3.1, 3.2, 4.1, 4.2, 4.3 and 4.4, we obtain some characterizations for boundedness, the essential norm and compactness of operator $uC_\varphi : H^p \to \mathcal{W}^{(n)}_\mu$ (see [2]).*

**Remark 4.6** *By setting $u \equiv 0$ in Theorems 3.1, 3.2, 4.1, 4.2, 4.3 and 4.4, we get some characterizations for boundedness, the essential norm and compactness of generalized weighted composition operator $D^m_{\varphi,u} : H^p \to \mathcal{W}^{(n)}_\mu$ (see [6]).*

**Remark 4.7** *By taking $m = 1$ in Theorems 3.1, 3.2, 4.1, 4.2, 4.3 and 4.4, we find some characterizations for boundedness, the essential norm and compactness of Stević-Sharma type operators from Hardy space into $n$th weighted type spaces (see [14]).*

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