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T_4, Urysohn's lemma, and Tietze extension theorem for constant filter convergence spaces

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Abstract: In this paper, we characterize various local forms of T_4 constant filter convergence spaces and investigate the relationships among them as well as showing that the full subcategories of the category of constant filter convergence spaces consisting of local T_4 constant filter convergence spaces that are hereditary. Furthermore, we examine the relationship between local T_4 and general T_4 constant filter convergence spaces. Finally, we present Urysohn's lemma and Tietze extension theorem for constant filter convergence spaces.

Key words: Topological category, T_1 objects, T_4 objects, constant filter convergence spaces

1. Introduction

In 1978, Schwarz [14] introduced the category $\boldsymbol{ConFCO}$ whose objects are constant filter convergence spaces and morphisms are continuous maps, and he showed that $\boldsymbol{ConFCO}$ is isomorphic to the category $\boldsymbol{FILTER}$ whose objects are filter spaces and morphisms are continuous maps. He also showed that it is a bireflective subcategory of $\boldsymbol{FCO}$ whose objects are filter convergence spaces and morphisms are continuous maps. Hence, Schwarz proved that $\boldsymbol{ConFCO}$ is the natural link between $\boldsymbol{FILTER}$ and the category $\boldsymbol{FCO}$.

In 1991, Baran [3] introduced the local T_1 separation property that is used to define the notion of strongly closed subobject of an object of a topological category, which are used in the notions of compactness [8], connectedness [10], and normal objects [3].

In general topology, one of the most important uses of separation properties is theorems such as the Urysohn's lemma and the Tietze extension theorem. In this regard, it is useful to be able to extend these various notions to arbitrary topological categories.

The main goals of this paper are as follows:

(1) to give characterizations each of various forms of local T_4 constant filter convergence spaces,

(2) to investigate the relationships among these various forms as well as the general form of T_4 constant filter convergence space,

(3) to show that the subcategories of local T_4 constant filter convergence spaces are productive and hereditary,

(4) to present Urysohn's Lemma and Tietze Extension Theorem for constant filter convergence spaces.

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2. Preliminaries

Let B be a nonempty set and $F(B)$ be the set of filters on B. A filter $\alpha \in F(B)$ is called proper (resp., improper) iff $\emptyset \notin \alpha$ (resp., $\emptyset \in \alpha$). Let $\alpha, \beta \in F(B)$. We denote by $\alpha \cup \beta$ the smallest filter containing both α and β, i.e.

$$\alpha \cup \beta = \{M \subset B : U \cap V \subset M \text{ for some } U \in \alpha \text{ and } V \in \beta\}.$$

If the map $K : B \to P(F(B))$ satisfies

(1) $[\{x\}] = [x] \in K(x)$ for each $x \in B$, where $[U] = \{V \subset B : U \subset V\}$, $U \subset B$,

(2) if $\alpha \in K(x)$ and $\beta \supset \alpha$, then $\beta \in K(x)$,

then (B, K) is called a filter convergence space [13, 14]. If K is a constant function, then (B, K) is called a constant filter convergence space [13, 14].

Let (B, K) and (C, L) be constant filter convergence spaces. A map $f : (B, K) \to (C, L)$ is called continuous if $f(\alpha) \in L$ for each $\alpha \in K$, where

$$f(\alpha) = \{U \subset C : \exists V \in \alpha \text{ such that } f(V) \subset U\}.$$

Let ***ConFCO*** be the category consisting of all constant filter convergence spaces and continuous maps which is a normalized topological category [1, 14].

Fact 2.1 Let $\{(B_i, K_i), i \in I\}$ in ***ConFCO***, B be a set, and $\{f_i : B \to B_i, i \in I\}$ be a source in ***Set***, the category of sets and functions. $\{f_i : (B, K) \to (B_i, K_i), i \in I\}$ in ***ConFCO*** is an initial lift iff $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ for all $i \in I$ [13].

Fact 2.2 Let $\{(B_i, K_i), i \in I\}$ in ***ConFCO***, B be a set, and $\{f_i : B_i \to B, i \in I\}$ be a sink in ***Set***. An epi sink $\{f_i : (B_i, K_i) \to (B, K), i \in I\}$ in ***ConFCO*** is a final lift iff $\alpha \in K$ implies that there exist $i \in I$ and $\beta_i \in K_i$ such that $f_i(\beta_i) \subset \alpha$ [13].

Lemma 2.1 *([2]) Let $\alpha, \beta \in F(A)$, $\delta \in F(B)$ and $f : A \to B$ be a function. Then,*

(1) $f(\alpha \cap \beta) = f(\alpha) \cap f(\beta)$.

(2) $f(\alpha \cup \beta) \supset f(\alpha) \cup f(\beta)$.

(3) $f^{-1} f \alpha \subset \alpha$.

(4) $\delta \subset f f^{-1} \delta$.

Let $U : \boldsymbol{E} \to \boldsymbol{Set}$ be a topological functor, X be an object of $\boldsymbol{E}$ with $p \in U(X) = B$, F be a nonempty subset of B, and X/F be the final lift of the epi U-sink

$$q : U(X) \to B/F = (B \backslash F) \cup \{*\},$$

where q is the identity on B/F and identifying F with a point $*$ [3].

Lemma 2.2 *([2, 6]) Let B be a set, $\emptyset \neq F \subset B$, $\alpha, \beta, \sigma \in F(B)$, and $q : B \to B/F$ be the identification map.*

(1) *For $a \notin F$, $q\alpha \subset [a]$ iff $\alpha \subset [a]$.*

(2) *$q\alpha \subset [*]$ iff $\alpha \cup [F]$ is proper.*

(3) *If $\alpha \cup [F]$ is not proper, then $q\sigma \subset q\alpha$ iff $\sigma \subset \alpha$.*

(4) *If $\alpha \cup [F]$ is proper, then $q\sigma \subset q\alpha$ iff $\sigma \cup [F]$ is proper and $\sigma \cap [F] \subset \alpha$.*

(5) *$q\beta \cup q\alpha$ is proper iff $\beta \cup \alpha$ is proper or $\beta \cup [F]$ and $\alpha \cup [F]$ are proper.*

3. Local T_4 constant filter convergence spaces

Let B be a set, $p \in B$ and the wedge at p is two disjoint copies of B identified at p and is denoted by $B \bigvee_p B$ [3]. A point x in $B \bigvee_p B$ will be denoted by x_1 (resp., x_2) if x is in the first (resp., second) component of $B \bigvee_p B$. Note that $p_1 = p_2$.

Define $S_p : B \bigvee_p B \to B^2$ by

$$S_p(x_i) = \begin{cases} (x, x) & \text{if } i = 1 \\ (p, x) & \text{if } i = 2 \end{cases},$$

and $\triangledown_p : B \bigvee_p B \to B$ by

$$\triangledown_p(x_i) = x$$

for $i = 1, 2$ [3].

The infinite wedge $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p, where $B^\infty = B \times B \times ...$ is the countable cartesian product of B [5].

Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by

$$A_p^\infty(x_i) = (p, ..., p, x, p, p, ...)$$

where x_i is in the i-th component of $\vee_p^\infty B$ and $\triangledown_p^\infty : \vee_p^\infty B \longrightarrow B$ by

$$\triangledown_p^\infty(x_i) = x$$

for all $i \in I$, where I is the index set $\{i : x_i$ is in the i-th component of the infinite wedge $\vee_p^\infty B\}$ [3].

Definition 3.1 *([3]) Let $U : \boldsymbol{E} \to \boldsymbol{Set}$ be a topological functor, X be an object of $\boldsymbol{E}$ with $p \in U(X) = B$, F be a nonempty subset of B, and X/F be the final lift of the epi U-sink $q : U(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where q is the identification map defined above.*

(1) *If the initial lift of the U-source $S_p : B \bigvee_p B \to U(X^2) = B^2$ and $\triangledown_p : B \bigvee_p B \to U(D(B)) = B$ is discrete, then X is called T_1 at p, where D is the discrete functor, a left adjoint of U.*

(2) *If the initial lift of the U-source*

$$\{A_p^{\infty} : \vee_p^{\infty} B \longrightarrow U(X^{\infty}) = B^{\infty} \quad and \quad \nabla_p^{\infty} : \vee_p^{\infty} B \longrightarrow U(D(B)) = B\}$$

is discrete, then $\{p\}$ is said to be closed.

(3) *If $\{*\}$ is closed in X/F, then $F \subset X$ is said to be closed.*

(4) *If X/F is T_1 at $*$, then F is said to be strongly closed.*

(5) *If X is T_1 at p and X/F is $\overline{T}_3$ at $*$ for all closed (resp., strongly closed) F in $U(X)$ containing p, then X is called $\overline{T}_4$ (resp., $S\overline{T}_4$) at p.*

(6) *If X is T_1 at p and X/F is T_3' at $*$ for all closed (resp., strongly closed) F in $U(X)$ containing p, then X is called T_4' (resp., ST_4') at p.*

Note that if (B,τ) is a topological space and $p \in B$, then by Theorem 2.1 of [4], all of various local T_4 structures in Definition 3.1 are equivalent. The notion of closedness coincides with the usual closedness and if (B,τ) is T_1, by Theorem 2.2.16 of [3], the notions of closedness and strongly closedness coincide.

Theorem 3.2 *([5]) Let $(B,K) \in$ **ConFCO**, $p \in B$, and $\emptyset \neq F \subset B$. Then,*

(1) *$\{p\}$ is closed iff $[x] \cap [p] \notin K$ for all $x \in B$ with $x \neq p$.*

(2) *(B,K) is T_1 at p iff for any $x \in B$ with $x \neq p$, $[x] \cap [p] \notin K$.*

(3) *The following are equivalent.*

(a) *F is strongly closed.*

(b) *F is closed.*

(c) *$\alpha \not\subset [a]$ or $\alpha \cup [F]$ is improper for every proper filter $\alpha \in K$ and $a \in B$ with $a \notin F$.*

Theorem 3.3 *([11]) Let (B,K) be a constant filter convergence space with $p \in B$. Then:*

(1) *The following are equivalent.*

(a) *(B,K) is ST_3' at p.*

(b) *(B,K) is T_3' at p.*

(c) *$[x] \cap [p] \notin K$ for all $x \in B$ with $x \neq p$ and $K_p = \{[p]\}$ for $p \notin F$ where F is a nonempty closed subset of B.*

(2) *The following are equivalent.*

(a) *(B,K) is $S\overline{T}_3$ at p.*

(b) (B,K) *is* $\overline{T}_3$ *at* p.

(c) *The following conditions are satisfied:*

(i) For any $x \in B$ *with* $x \neq p$, $[x] \cap [p] \notin K$.

(ii) If $\alpha, \beta \in K_p$, *then* $\alpha \cap \beta \in K_p$, *where* $K_p = \{\alpha : \alpha \subset [p]$ *and* $\alpha \in K\}$.

(iii) For any $\alpha \in K_p$, $\beta \in K$ *and nonempty closed subset* F *of* b *is closed with* $p \notin F$, *if* $\alpha \cup \beta$ *is proper or* $\beta \cup [F]$ *and* $\alpha \cup [F]$ *are proper, then* $\beta \cap [p] \in K$.

Theorem 3.4 *Let* (B,K) *be a constant filter convergence space with* $p \in B$. *The following are equivalent.*

(1) (B,K) *is* T_4' *at* p.

(2) (B,K) *is* ST_4' *at* p.

(3) $[x] \cap [p] \notin K$ *for all* $x \in B$ *with* $x \neq p$ *and for any nonempty disjoint closed subset* F_1 *of* B *with* $p \in F_1$ *and for any proper filter* $\alpha \in K$, *if* $\alpha \cup [F_1]$ *is proper, then* $F_1 \in \alpha$.

Proof By Theorem 3.3 and Definition 3.1, (B,K) is ST_4' at p iff (B,K) is T_4' at p. Hence, $(1) \Leftrightarrow (2)$.

Suppose (B,K) is T_4' at p. By Definition 3.1, in particular, (B,K) is T_1 at p and by Theorem 3.3, $[x] \cap [p] \notin K$ for all $x \in B$ with $x \neq p$.

Suppose F_1 is nonempty disjoint closed subset of B with $p \in F_1$, $\alpha \in K$ and $\alpha \cup [F_1]$ is proper. $\alpha \in K$ implies $q\alpha \in K'$, where K' is the final structure on B/F_1 induced by the map $q : B \to B/F_1$.

By Lemma 2.2(2), $\alpha \cup [F_1]$ is proper implies $q\alpha \subset [*]$. Since $(B/F_1, K')$ is T_3' at $*$, by Theorem 3.3, $q\alpha \in K_*'$, i.e. $q\alpha = [*]$ and by Lemma 2.1,

$$\alpha \supset q^{-1}q(\alpha) = q^{-1}[*] = [F_1]$$

and consequently $F_1 \in \alpha$.

Suppose that the conditions hold and $x \in B$ with $x \neq p$ and by the assumption, $[x] \cap [p] \notin K$, by Theorem 3.2, (B,K) is T_1 at p.

Next, we show that $(B/F_1, K')$ is T_3' at $*$, where F_1 is a closed subset of B with $p \in F_1$ and K' is the final structure on B/F_1 induced by the map $q : B \to B/F_1$. Let $a \in B/F_1$ with $a \neq *$. If $[a] \cap [*] \in K'$, then by Fact 2.2, there exists $\alpha \in K$ such that $\alpha \subset [a] \cap [*]$.

By Lemma 2.1(1),

$$[a] \cap [*] = q([a] \cap [F_1])$$

and by Lemma 2.2,

$$\alpha \cap [F_1] \subset [a] \cap [F_1]$$

and

$$\alpha \cup [F_1]$$

is proper since $([a] \cap [F_1]) \cup [F_1] = [F_1]$ is proper.

Note that $\alpha \subset [a]$, $a \notin F_1$ and $\alpha \cup [F_1]$ is proper, a contradiction to the fact that F_1 is closed. Hence, $[a] \cap [*] \notin K'$ for all $a \in B/F_1$ with $a \neq *$ and by Theorem 3.2, $(B/F_1, K')$ is T_1 at $*$.

Finally, we need to show that $K'_* = \{[*]\}$ for $* \in F_2$, where F_2 is a nonempty closed subset of B/F_1. Let $\beta \in K'$ and $\beta \subset [*]$. By Fact 2.2, there exists $\alpha \in K$ such that $q\alpha \subset \beta$ and by Lemma 2.2, $\alpha \cup [F_1]$ is proper. Since $x \notin F_2$, $F_1 \cap F_2 = \emptyset$ and by the assumption $F_1 \in \alpha$. Hence, $* = q(F_1) \in q\alpha$ i.e. $[*] = q\alpha \subset \beta$ implies $\beta = [*]$. As a result, by Theorem 3.3, $(B/F_1, K')$ is T'_3 at $*$ and by Definition 3.1, (B, K) is T'_4 at p. □

Theorem 3.5 *Let (B, K) be a constant filter convergence space with $p \in B$. (B, K) is $\overline{T}_4$ at p iff the following are satisfied.*

(1) *For all $x \in B$ with $x \neq p$, $[x] \cap [p] \notin K$.*

(2) *If $\alpha, \beta \in K$ with $\alpha \cup [F_1]$ and $\beta \cup [F_1]$ are proper for any closed subset F_1 of B containing p, then there exists $\delta \in K$, such that $\delta \cap [F_1] \subset \alpha \cap \beta$ and $\delta \cup [F_1]$ is proper.*

(3) *Suppose that $\alpha \cup [F_1]$ is proper for any closed subset F_1 of B with $p \in F_1$ and F_2 for any nonempty closed subset of B disjoint from F_1 and for any $\alpha, \beta \in K$. If $\alpha \cup \beta$ is proper or $\beta \cup [F_1]$ is proper or $\alpha \cup [F_2]$ and $\beta \cup [F_2]$ are proper, then there exists $\delta \in K$, such that $\delta \cup [F_1]$ is proper, $\delta \cap [F_1] \subset \alpha \cap \beta$ or $\delta \cap [F_1] \subset \beta \cap [F_1]$.*

Proof Suppose (B, K) is $\overline{T}_4$ at p. By Definition 3.1, (B, K) is T_1 at p and by Theorem 3.2, $[x] \cap [p] \notin K$ for all $x \in B$ with $x \neq p$. This shows that (1) holds.

Suppose $\alpha, \beta \in K$ with $\alpha \cup [F_1]$ and $\beta \cup [F_1]$ are proper for any closed subset F_1 of B containing p. Then $q\alpha, q\beta \in K'$, where K' is the final structure on B/F_1 induced by the map $q : B \to B/F_1$ and by Lemma 2.2, $q\alpha \subset [*]$ and $q\beta \subset [*]$. Since $(B/F_1, K')$ is $\overline{T}_3$ at $*$, by Theorem 3.3, $q\alpha \cap q\beta \in K'_*$ and by Fact 2.2, there exists $\delta \in K$ such that

$$q\delta \subset q\alpha \cap q\beta = q(\alpha \cap \beta).$$

Since $q(\alpha \cap \beta) \subset [*]$, by Lemma 2.2, $(\alpha \cap \beta) \cup [F_1]$ is proper and by Lemma 2.2, $\delta \cap [F_1] \subset \alpha \cap \beta$ and $\delta \cup [F_1]$ is proper. This shows that (2) holds.

Suppose $\alpha, \beta \in K$, $\alpha \cup [F_1]$ is proper for any closed subset F_1 of B with $p \in F_1$ and F_2 is any nonempty closed subset of B disjoint from F_1. Note that $q\alpha, q\beta \in K'$ and by Lemma 2.2, $q\alpha \subset [*]$ and $* \notin F_2$. Suppose $\alpha \cup \beta$ is proper. Then by Lemma 2.2, $q\alpha \cup q\beta$ is proper and by Theorem 3.3, $q\beta \cap [*] \in K'$ and by Fact 2.2, there exists $\delta \in K$ such that

$$q\delta \subset q\beta \cap [*] = q\beta \cap q[F_1] = q(\beta \cap [F_1]).$$

Since $(\beta \cap F_1) \cup [F_1] = [F_1]$ is proper, by Lemma 2.2, $\delta \cap [F_1] \subset \beta \cap [F_1]$ and $\delta \cup [F_1]$ is proper.

Suppose $\alpha \cup [F_2]$ and $\beta \cup [F_2]$ are proper. By Lemma 2.1,

$$q\alpha \cup [qF_2] = q\alpha \cup [F_2]$$

and $q\beta \cup [qF_2] = q\beta \cup [F_2]$ are proper. By Theorem 3.3, $q\beta \cap [*] \in K'$ and by Fact 2.2, there exists $\delta \in K$ such that

$$q\delta \subset q\beta \cap [*] = q(\beta \cap [F_1]).$$

Since $(\beta \cap [F_1]) \cup [F_1] = [F_1]$ is proper, by Lemma 2.2, $\delta \cap [F_1] \subset \beta \cap [F_1]$ and $\delta \cup [F_1]$ is proper. Suppose $\alpha \cup [F_1]$ and $\beta \cup [F_1]$ are proper. Note that $q\alpha, q\beta \in K'$ and by Lemma 2.2, $q\alpha \subset [*]$ and $q\beta \subset [*]$. By Lemma 2.1, $q\alpha \cap q\beta \in K'_*$ and by Fact 2.2, there exists $\delta \in K$, such that $q\delta \subset q\alpha \cap q\beta = q(\alpha \cap \beta)$. Since $(\alpha \cap \beta) \cup [F_1]$ is proper, by Lemma 2.2, $\delta \cap [F_1] \subset \alpha \cap \beta$ and $\delta \cup [F_1]$ is proper.

Suppose that the conditions hold and $x \in B$ with $x \neq p$. By Part (1), $[x] \cap [p] \notin K$ and by Theorem 3.2, (B, K) is T_1 at p. We show that $(B/F_1, K')$ is $\overline{T}_3$ at $*$ for every closed subset F_1 of B containing p, where K' is the final structure on B/F_1 induced by the map $q : B \to B/F_1$. Suppose $a \in B/F_1$ and $a \neq *$. If $[a] \cap [*] \in K'$, then by Fact 2.2, there exists $\alpha \in K$, such that $q\alpha \subset [a] \cap [*]$. It follows that $q\alpha \subset [*]$. By Lemma 2.2, $\alpha \cup [F_1]$ is proper and $\alpha \subset [a]$, $a \notin F_1$, a contradiction since F_1 is closed. Hence, $[a] \cap [*] \notin K'$ for all $a \in B/F_1$ with $a \neq *$ and by Theorem 3.2, $(B/F_1, K')$ is T_1 at $*$.

Suppose $\alpha, \beta \in K'_*$, then by Fact 2.2 and Lemma 2.2, there exist $\alpha_1, \alpha_2 \in K$ such that $q\alpha_1 \subset \alpha$, $q\alpha_2 \subset \beta$ with $\alpha_1 \cup [F_1]$ and $\alpha_2 \cup [F_1]$ are proper. By condition (2), there exists $\delta \in K$ such that $\delta \cap [F_1] \subset \alpha_1 \cap \alpha_2$ and $\delta \cup [F_1]$ is proper. It follows by Lemma 2.1 that

$$q\delta = q(\delta \cap [F_1]) = q\delta \cap [*] \subset q(\alpha_1 \cap \alpha_2) = q\alpha_1 \cap q\alpha_2 \subset \alpha \cap \beta$$

which implies $\alpha \cap \beta \in K'_*$.

Suppose $\alpha \in K'_*$, $\beta \in K'$, and F_2 is any nonempty closed subset of B/F_1 with $* \notin F_2$ such that $\alpha \cup \beta$ is proper or $\alpha \cup [F_2]$ and $\beta \cup [F_2]$ are proper. By Fact 2.2 and Lemma 2.2, there exist $\alpha_1, \alpha_2 \in K$ such that $q\alpha_1 \subset \alpha$, $q\alpha_2 \subset \beta$ and $\alpha_1 \cup [F_1]$ is proper. Suppose $\alpha \cup \beta$ is proper. Then, $q\alpha_1 \cup q\alpha_2$ is proper and by Lemma 2.2, $\alpha_1 \cup \alpha_2$ is proper or $\alpha_1 \cup [F_1]$ and $\alpha_2 \cup [F_1]$ are proper, and by(3), there exists $\delta \in K$ such that $\delta \cup [F_1]$ is proper and

$$\delta \cap [F_1] \subset \alpha_1 \cap \alpha_2$$

or

$$\delta \cap [F_1] \subset \alpha_2 \cap [F_1].$$

Let's apply q to each extension. By Lemmas 2.1 and 2.2,

$$q\delta = q\delta \cap [*] = q\delta \cap q[F_1] = q(\delta \cap [F_1]) \subset q(\alpha_1 \cap \alpha_2) = q\alpha_1 \cap q\alpha_2 \subset [*] \cap q\alpha_2$$

$$q\delta = q\delta \cap [*] = q\delta \cap q[F_1] = q(\delta \cap [F_1]) \subset q([F_1] \cap \alpha_2) = q[F_1] \cap q\alpha_2 \subset [*] \cap q\alpha_2.$$

Since $q\delta \in K'$, $[*] \cap q\alpha_2 \in K'$ and consequently, $[*] \cap \beta \in K'$.

Suppose $\alpha \cup [F_2]$ and $\beta \cup [F_2]$ are proper. By Lemma 2.1

$$q\alpha_1 \cup [F_2] = q\alpha_1 \cup q[F_2]$$

and

$$q\alpha_2 \cup [F_2] = q\alpha_2 \cup q[F_2]$$

are proper and by Lemma 2.2, $\alpha_1 \cup [F_2]$ and $\alpha_2 \cup [F_2]$ are proper since $[F_1] \cup [F_2]$ is improper. By (3), there exists $\delta \in K$ such that $\delta \cup [F_1]$ is proper and

$$\delta \cap [F_1] \subset \alpha_1 \cap \alpha_2$$

or

$$\delta \cap [F_1] \subset \alpha_2 \cap [F_1].$$

By Lemmas 2.1 and 2.2, we have

$$q\delta = q\delta \cap [*] = q\delta \cap q[F_1] = q(\delta \cap [F_1]) \subset q(\alpha_1 \cap \alpha_2) = q\alpha_1 \cap q\alpha_2 \subset [*] \cap q\alpha_2$$

$$q\delta = q\delta \cap [*] = q\delta \cap q[F_1] = q(\delta \cap [F_1]) \subset q([F_1] \cap \alpha_2) = q[F_1] \cap q\alpha_2 \subset [*] \cap q\alpha_2.$$

Since $q\delta \in K'$, $[*] \cap q\alpha_2 \in K'$ and consequently, $[*] \cap \beta \in K'$. Hence, by Theorem 3.3, $(B/F_1, K')$ is $\overline{T}_3$ at $*$ and by Definition 3.1, (B,K) is $\overline{T}_4$ at p.

□

Remark 3.6 **(1)** *In* ***Top****, the category of topological spaces and continuous functions, by Theorem 2.2.14 of [3],* T_4' *at* p*,* $\overline{T}_4$ *at* p*,* ST_4' *at* p*, and* $S\overline{T}_4$ *at* p *are equivalent. Also, in* ***Top****, the notion of closedness coincides with the usual closedness and if* (B,τ) *is* T_1*, by Theorem 2.2.16 of [3], the notions of closedness and strongly closedness coincide.*

Let $\boldsymbol{T_4Top}$ *be the full subcategory of* ***Top*** *consisting of all local* T_4 *topological spaces. Then the categories* $\boldsymbol{T_4'Top}$*,* $\boldsymbol{\overline{T}_4Top}$*,* $\boldsymbol{ST_4'Top}$*, and* $\boldsymbol{S\overline{T}_4Top}$ *are isomorphic.*

(2) *For the category* ***ConFCO****, by Theorems 3.4 and 3.5,* T_4' *at* p *(resp.,* $\overline{T}_4$ *at* p*) and* ST_4' *at* p *(resp.,* $S\overline{T}_4$ *at* p*) are equivalent and* T_4' *at* p *implies* $\overline{T}_4$ *at* p*. Moreover, in* ***ConFCO****, by Theorem 3.2, the notions of closedness and strongly closedness coincide.*

(3) *Let* $\boldsymbol{T_4ConFCO}$ *be the full subcategory of* ***ConFCO*** *whose objects are the local* T_4 *constant filter convergence spaces, where* $T_4 = T_4'$*,* ST_4'*,* $\overline{T}_4$ *or* $S\overline{T}_4$*. Theorems 3.4 and 3.5 yield the following.*

(a) $\boldsymbol{T_4'ConFCO}$ *and* $\boldsymbol{ST_4'ConFCO}$ *are isomorphic categories.*

(b) $\boldsymbol{\overline{T}_4ConFCO}$ *and* $\boldsymbol{S\overline{T}_4ConFCO}$ *are isomorphic categories.*

Theorem 3.7 **(1)** *Let* (A,L) *be a constant filter convergence space. If* $M \subset N$ *and* $N \subset A$ *is closed, then* $M \subset A$ *is closed.*

(2) *Let* (B_i, K_i) *be constant filter convergence spaces for all* $i \in I$ *and* $p = (p_1, p_2, ...) \in B = \prod_{i\in I} B_i$*. Each* $F_i \subset B_i$ *is closed,* $i \in I$*, iff* $F = \prod_{i\in I} F_i \subset B$ *is closed, where* K *is the product structure on* B*.*

Proof **(1)** is proved in [12].

(2) Suppose $F_i \subset B_i$ is closed, for each $i \in I$, for any proper filter $\alpha \in K$, where K is the product structure on B, and $a = (a_1, a_2, ...) \notin F$. There exists $j \in I$ such that $a_j \notin F_j$ and $\pi_j \alpha \in K_j$. Since $F_j \subset B_j$ is closed, by Theorem 3.2, $\pi_j \alpha \not\subset [a_j]$ or $\pi_j \alpha \cup [F_j]$ is improper. If $\pi_j \alpha \cup [\pi_j F = F_j]$ is improper, then, by Theorem 3.2, $\pi_j(\alpha \cup [F])$ is improper and $\pi_j^{-1}\pi_j(\alpha \cup [F])$ is improper. Therefore, by Theorem 3.2, $\alpha \cup [F]$ is improper. If $\pi_j \alpha \not\subset [\pi_j a]$, then $\alpha \not\subset [a]$ and by Theorem 3.2, $F \subset B$ is closed.

Conversely, suppose $F \subset B$ is closed, for each $i \in I$, $\alpha_i \in K_i$ is proper and $a_i \notin F_i$. Let $\alpha = \{U \subset B : U \supset U_1 \times U_2 \times ..., U_i \in \alpha_i\}$ and $a = (a_1, a_2, ...)$. Note that $a \notin F = \prod_{i \in I} F_i$ and by Fact 2.1, $\alpha \in K$ is proper. F is closed implies $\alpha \not\subset [a]$ or $\alpha \cup [F]$ is improper. If $\alpha \not\subset [a]$, then $\pi_i \alpha = \alpha_i \not\subset [a_i]$. (Indeed, if $U \in \pi_i \alpha$, then there exists $W \in \alpha$ such that $U \subset \pi_i W$. $W \in \alpha$ implies $W \supset W_1 \times W_2 \times ...$, $W_i \in \alpha_i$. It follows easily that $\pi_i W \supset W_i$, $\pi_i W \in \alpha_i$. Hence, $\pi_i \alpha \subset \alpha_i$. Conversely, suppose $U_i \in \alpha_i$. Then $\pi^{-1}(U_i) = B_1 \times B_2 \times ... \times B_{i-1} \times U_i \times B_{i+1} \times ... \in \alpha$. If we apply π_i to expression, then $\pi_i \pi_i^{-1} U_i = U_i \in \pi_i \alpha$. Hence, $\alpha_i \subset \pi_i \alpha$. Consequently, $\pi_i \alpha = \alpha_i$).

If $\alpha \cup [F]$ is improper, then by Lemma 2.2, $\pi_i \alpha \cup [F_i]$ is improper. By Theorem 3.2, for each $i \in I$, $F_i \subset B_i$ is closed.

□

Theorem 3.8 *Let (B, K) be a constant filter convergence space with $p \in B$.*

(1) *(B, K) is T_1 at p iff $\{p\}$ is closed.*

(2) *(B, K) is T_1 iff (B, K) is T_1 at p for all $p \in B$.*

(3) *(B, K) is T_1 iff all points of B are closed.*

(4) *(B, K) is $preT_2'$ iff (B, K) is $preT_2'$ at p for all $p \in B$.*

(5) *(B, K) is T_4' iff (B, K) is T_4' at p for all $p \in B$.*

(6) *(B, K) is $\overline{T}_4$ iff (B, K) is $\overline{T}_4$ at p for all $p \in B$.*

Proof **(1)** follows from Theorem 3.2.

(2) follows from Theorem 3.2 and Theorem 2.1 of [7].

(3) follows from Part (1) and Part (2).

(4) If (B, K) is $preT_2'$ [3], then by Theorem 2.8 of [6], (B, K) is $preT_2'$ at p for all $p \in B$ since ***ConFCO*** is normalized.

Conversely, suppose (B, K) is $preT_2'$ at p for all $p \in B$ and $\alpha \in K$ is proper. Note that for every $U \in \alpha$, $[U] \subset [a]$ for some $a \in U$, and $[U] \in K_a$. By Theorem 3.6 of [11], $[a] = [U]$ and consequently, $\alpha = [a]$. Hence, by Theorems 2.1 and 2.7 of [7], (B, K) is $preT_2'$.

(5) Suppose (B,K) is T_4'. Since the category ***ConFCO*** is normalized, by Theorem 2.8 of [6] and by Theorem 2.7 of [7], (B,K) is T_4' at p for all $p \in B$.

Suppose (B,K) is T_4' at p for all $p \in B$. By (2), and by Theorem 3.2, (B,K) is T_1. Suppose α is any proper filter in K, and F_1 and F_2 are any nonempty disjoint closed subsets of B. If $p \in F_1$ (resp., $p \in F_2$) and $\alpha \cup [F_1]$ (resp., $\alpha \cup [F_2]$) is proper, then, by Theorem 3.4, $F_1 \in \alpha$ (resp., $F_2 \in \alpha$).

Suppose $\alpha \cup [F_1]$ is improper and $p \in F_1$. Since $\alpha \cup [F_1] \subset \alpha \cup [p]$ for all $p \in F_1$, it follows that $\alpha \cup [p]$ is proper and consequently, $\alpha \not\subset [p]$ for all $p \in F_1$. Also $\alpha \not\subset [a]$ for all $a \notin F_1$. If $\alpha \subset [a]$ for some $a \notin F_1$, then by Theorem 3.2, $\{a\} \in \alpha$. By Theorem 3.2, a is closed. If $\alpha \not\subset [a]$ for all $a \notin F_1$, α must be improper, a contradiction.

By similar argument, if $\alpha \cup [F_2]$ is improper and $p \in F_2$, then $\alpha = [a]$ for some $a \notin F_2$. Hence, by Theorem 2.4 of [9], (B,K) is T_4'.

(6) If (B,K) is $\overline{T}_4$, by Theorem 2.8 of [6], (B,K) is $\overline{T}_4$ at p for all $p \in B$ since ***ConFCO*** is normalized.

Conversely, suppose (B,K) is $\overline{T}_4$ at p for all $p \in B$. By Definition 3.1, (B,K) is T_1 at p for all $p \in B$ and by Part (2), (B,K) is T_1. Suppose that for any proper filters $\alpha,\beta \in K$ and for any nonempty disjoint closed subsets F_1 and F_2, $\alpha \cup \beta$ is proper or both $\alpha \cup [F_1]$ and $\beta \cup [F_1]$ are proper or both $\alpha \cup [F_2]$ and $\beta \cup [F_2]$ are proper. By Theorem 3.5, since (B,K) is $\overline{T}_4$ at p and there exists $\delta \in K$ such that $\delta \cap [F_1] \subset \alpha \cap \beta$ and $\delta \cup [F_1]$ is proper or $\delta \cap [F_2] \subset \alpha \cap \beta$ and $\delta \cup [F_2]$ is proper. Hence, by Theorem 2.3 of [9], (B,K) is $\overline{T}_4$. □

Theorem 3.9 **(1)** *If a constant filter convergence space (B,K) is $\overline{T}_4$ at p and $M \subset B$ is closed with $p \in M$, then M is $\overline{T}_4$ at p.*

(2) *If $(B = \prod_{i \in I} B_i, K)$, where K is the product structure on B is $\overline{T}_4$ at $p = (p_1, p_2,)$, then for all $i \in I$ and $p_i \in B_i$, (B_i, K_i) is $\overline{T}_4$ at p_i.*

Proof **(1)** Let (B,K) be a constant filter convergence space, K_M be the initial structure on M induced from the inclusion map $i : M \to B$, and $[x] \cap [p] \in K_M$ for some $x \in M$ with $x \neq p$, $p \in M$. By Fact 2.1 and Lemma 2.1,

$$i([x] \cap [p]) = i([x]) \cap i([p]) = [x] \cap [p] \in K,$$

a contradiction since (B,K) is $\overline{T}_4$ at p. Thus, $[x] \cap [p] \notin K_M$ for all $x \in M$ with $x \neq p$ and $p \in M$.

Suppose $\alpha,\beta \in K_M$ with $\alpha \cup [F_1]$ and $\beta \cup [F_1]$ are proper for any nonempty closed subset F_1 of M containing p. Note that $i(\alpha), i(\beta) \in K$ and i is a monomorphism (one to one map) implies

$$i(\alpha \cup [F_1]) = i(\alpha) \cup i([F_1]) \quad \text{and} \quad i(\beta \cup [F_1]) = i(\beta) \cup i([F_1])$$

are proper, where $i(p) = p \in F_1 = i([F_1])$. Since $F_1 \subset M$ and $M \subset B$ are closed, by Theorem 3.7, $F_1 \subset B$ is closed. Since (B,K) is $\overline{T}_4$ at p, by Theorem 3.5, there exists $\delta \in K$ such that $\delta \cap [i(F_1)] \subset i(\alpha) \cap i(\beta)$ and $\delta \cup [i(F_1)]$ is proper.

By Lemma 2.1, $i(i^{-1}(\delta)) \supset \delta$, and $i(i^{-1}(\delta)) \in K$. Since i is an initial lift and $i(i^{-1}(\delta)) \in K$, by Fact 2.1, we get $i^{-1}(\delta) \in K_M$. Since

$$\delta \cap [i(F_1)] \subset i(\alpha) \cap i(\beta) = i(\alpha \cap \beta) \quad \text{and} \quad \delta \cup [i(F_1)]$$

is proper, by Lemma 2.2,

$$i^{-1}((\delta)\cap[F_1]) = i^{-1}((\delta)\cap[i(F_1)]) = i^{-1}(\delta)\cap[F_1] \subset i^{-1}(i(\alpha)\cap i(\beta)) = i^{-1}(i(\alpha\cap\beta)) = \alpha\cap\beta$$

and $i^{-1}(\delta)\cup[F_1]$ is proper.

Suppose $\alpha,\beta\in K_M$, $\alpha\cup[F_1]$ is proper, for any closed subset F with $p\in F_1$ and F_2 is a nonempty closed subset of M disjoint from F_1 with $\alpha\cup\beta$ is proper or $\beta\cup[F_1]$ is proper or $\alpha\cup[F_2]$ and $\beta\cup[F_2]$ are proper.

It follows that $i(\alpha), i(\beta)\in K$, $i(\alpha)\cup[F_1] = i(\alpha)\cup[i(F_1)]$ is proper, $i(p) = p\in i(F_1) = F_1$ and F_2 is a nonempty (strongly) closed subset of B. Hence,

$$i(\alpha)\cup i(\beta) = i(\alpha\cup\beta)$$

is proper or

$$i(\beta)\cup i([F_1]) = i(\beta\cup[F_1])$$

is proper or

$$i(\alpha)\cup[i(F_2)] = i(\alpha\cup[F_2]) \quad \text{and} \quad i(\beta)\cup[i(F_2)] = i(\beta\cup[F_2])$$

are proper. Since F_1 and F_2 are closed subsets of M and $M\subset B$ is closed, by Theorem 3.7, $i(F_1)$ and $i(F_2)$ are closed subsets of B.

Since (B,K) is $\overline{T}_4$ at p, by Theorem 3.5, there exists $\delta\in K$ such that

$$\delta\cup[i(F_1)]$$

is proper. Moreover,

$$\delta\cap[i(F_1)]\subset i(\alpha)\cap i(\beta) = i(\alpha\cap\beta)$$

or

$$\delta\cap[i(F_1)]\subset i(\beta)\cap[iF_1] = i([F_1]).$$

It follows that $i^{-1}(\delta)\cup[F_1]$ is proper. Hence,

$$i^{-1}(\delta\cap[F_1]) = i^{-1}(\delta\cap[i(F_1)])\subset i^{-1}(i(\alpha)\cap i(\beta)) = i^{-1}(i(\alpha\cap\beta)) = \alpha\cap\beta$$

or

$$i^{-1}(\delta)\cap[F_1]\subset\beta\cap[F_1].$$

Hence, by Theorem 3.5, (M,K_M) is $\overline{T}_4$ at p.

(2) Suppose that $(B=\prod_{i\in I} B_i, K)$ is $\overline{T}_4$ at p. By Theorem 3.8, each (B_i,K_i) is isomorphic to a closed subspace of (B,K) and by Part (1), (B_i,K_i) is $\overline{T}_4$ at p_i for all $i\in I$. □

Theorem 3.10 **(1)** *If a constant filter convergence space (B,K) is T_4' at p and $M\subset B$ is closed with $p\in M$, then M is T_4' at p.*

(2) *If $(B=\prod_{i\in I} B_i, K)$ is T_4' at $p=(p_1,p_2,)$, where K is the product structure on B, then (B_i,K_i) is T_4' at p_i for all $i\in I$ and $p_i\in B_i$.*

Proof

(1) Let (B, K) be a constant filter convergence space, K_M be the initial structure on M induced from the inclusion map $i : M \to B$. Suppose $F \subset M$ is closed with $p \in F$ and $\alpha \in K_M$ with $\alpha \cup [F]$ is proper. By Theorem 3.7(1), $F \subset B$ is closed, and, by the same argument used in the proof of Theorem 3.9, we get $[x] \cap [p] \notin K_M$ for all $x \in M$ with $p \in M$ and $x \neq p$.

Suppose α is any proper filter in K_M and $\alpha \cup [F]$ is proper, where $p \in F$ and F is closed in M. By Fact 2.1, $i(\alpha) \in K$, $i(p) \in i(F)$, and since i is a monomorphism, $i(\alpha \cup [F]) = i(\alpha) \cup i([F])$ is proper. Since (B, K) is T_4' at p, $i(F) \in i(\alpha)$ and consequently, $i^{-1}(i(F)) = F \in i^{-1}(i(\alpha)) = \alpha$. Hence, by Theorem 3.4, (M, K_M) is T_4' at p.

(2) Suppose that $(B = \prod_{i \in I} B_i, K)$ is T_4' at p. By Theorem 3.8, each (B_i, K_i) is isomorphic to a closed subspace of (B, K) and by Part (1), (B_i, K_i) is T_4' at p_i for all $i \in I$. □

Theorem 3.11 (Tietze extension theorem) *If (B, K) is a T_4' constant filter convergence space and A is closed subset of B, then every morphism $f : (A, K_A) \to (R, L)$, where R is the set of real numbers, L is any constant filter structure on R, and K_A is the initial structure on A induced from the incusion map $i : A \to (B, K)$, has an extension morphism (continuous function) $g : (B, K) \to (R, L)$.*

Proof Define $g : B \to R$ by $g(x) = \begin{cases} f(x), & if \quad x \in A \\ 0, & if \quad x \notin A \end{cases}$. Note that g is an extension of f. We show that g is a morphism (continuous) i.e. if $\alpha \in K$, then $g(\alpha) \in L$.

If $\alpha \in K$ is improper, then $g(\alpha)$ is improper. Suppose α is proper. Since (B, K) is a T_4', by Theorem 2.4 of [9] and by Theorem 3.8, either $A \in \alpha$ or $A^c \in \alpha$ or $\alpha = [a]$ for some $a \in B$.

If $A \in \alpha$, then $i^{-1}(A) \in i^{-1}(\alpha) \in K_A$,

$$f(i^{-1}\alpha) = g(i^{-1}\alpha)$$

and

$$g(i^{-1}(A)) = g(A) = f(A) = f(i^{-1}(A)) \in f(i^{-1}\alpha) \in L$$

since f is a morphism (continuous).

If $A^c \in \alpha$, then $g(A^c) = \{0\} \in g(\alpha)$, i.e. $g(\alpha) = [0] \in L$.

If $\alpha = [a]$ for some $a \in B$, then $g(\alpha) = g([a]) = [g(a)] \in L$. Hence, g is an extension morphism of f. □

Theorem 3.12 (Urysohn's lemma) *If (B, K) is a T_4' constant filter convergence space, F_1 and F_2 are any nonempty disjoint subsets of B, then there exists a morphism $f : (B, K) \to ([0, 1], L)$, where $[0, 1]$ is the unit interval and L is any constant filter structure on $[0, 1]$, such that $f(x) = 0$ if $x \in F_1$ and $f(x) = 1$ if $x \in F_2$.*

Proof Define $f : B \to [0,1]$ by $f(x) = \begin{cases} 0, & if \quad x \in F_1 \\ 1, & if \quad x \notin F_1 \end{cases}$. We show that if $\alpha \in K$, then $f(\alpha) \in L$.

If α is improper, then $f(\alpha)$ is improper. Suppose α is proper, by Theorem 2.4 of [9] and by Theorem 3.8, $F_1 \in \alpha$ or $F_2 \in \alpha$ or $\alpha = [a]$ for some $a \in B$.

If $F_1 \in \alpha$, then $f(F_1) = \{0\} \in f(\alpha)$ and consequently, $f(\alpha) = [0] \in L$.
If $F_2 \in \alpha$, then $f(F_2) = \{1\} \in f(\alpha)$ and consequently, $f(\alpha) = [1] \in L$.

If $\alpha = [a]$ for some $a \in B$, then $f(\alpha) = [f(a)] \in L$. Hence, f is a morphism such that $f(x) = 0$ if $x \in F_1$ and $f(x) = 1$ if $x \in F_2$.

□

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