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A note on soft topological spaces - relations to ordinary topological structures

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Abstract: This work contributes to the theory of soft sets. We studied relations of soft set and some soft topological notions to their ordinary (non-soft) counterparts. Among other things, we show that every soft topological space induces an ordinary topological space such that some topological properties remain the same. By using some simple observations, we also present a simple construction allowing us to construct a soft topological space possessing some required properties. This approach is general enough to be applied to many other (soft) topological properties. One of main goals of this note is to show that general topology already contains instruments allowing us to construct soft topological spaces and to study their properties.

Key words: soft set, soft topological space, strong product, product topology, separation axioms, productive properties

1. Introduction

Soft set theory provides a tool allowing us to deal with uncertainty. Since Molodtsov's pioneering paper [13] attracted a lot of attention, soft sets were applied in many areas (e.g. [11, 13, 16], etc.) and also studied from the theoretical point of view. For instance, many papers were especially devoted to study algebra of soft sets and their variants (e.g. [1, 4, 5] etc.) and, very recently, soft and soft fuzzy topological spaces were studied e.g. in [2, 3, 6, 7, 12, 17, 18] and other papers. In the latter papers the authors followed notation and set operations introduced e.g. in [10].

As far as we know, only two relations between soft topological and ordinary topological structures were mentioned. Namely, it was shown that a soft topological space need not be constructed as a standard product of topologies, and it was proved (see Lemma 5.1) that every soft topological space induces several topological spaces (U, τ_e). In this manuscript some closer relations between soft and ordinary topological structures are pointed out. We have chosen one of separations axioms (the T_1 one) to demonstrate our approach, however the ideas are simple enough to be applied to other common (soft) topological properties (see Remark 4.5).

More precisely, in Section 4 we showed that every soft topological space defines an ordinary topological space having "the same" properties. Since this construction does not involve topologies τ_e, so we tried to find a way how to use them in Section 5. Then in Section 6 we take three standard tools from general topology (strong product topology, product topology, subspace topology) to show that they can be successfully used to construct a soft topological space of desired properties.

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The main purpose of this short paper is to stress that there already exist powerful tools and results in general topology for dealing with soft topological spaces and that development of this theory could be also done with the help of standard topological instruments.

The structure of this paper is the following - in Section 2 we introduce some elementary definitions and notions. Then relations between soft and ordinary set operations are discussed in Section 3. As it is mentioned above, in Sections 4, 5 and 6 our main results are discussed. This paper is concluded by presenting some final remarks in Section 7.

2. Basic notions

2.1. Soft sets and soft operations

In this subsection we recall definitions of "soft" notions that recently appeared in several papers (e.g. in [10]). Let U and E denote a *universal* and an *index* set, respectively. Below $\#(E)$ denotes the cardinality of E. In order to avoid trivialities we assume that both U and E are not singletons.

Let $\mathcal{P}(U)$ denote the set of all subsets of U. A basic notion of this paper is the so-called soft set. A *soft set* is a pair (F, A) where $F : A \to \mathcal{P}(U)$ is a map and $A \subseteq E$ is non-empty. A soft set (F, A) is *empty*, resp. *null*, soft set if $F(x) = \emptyset$ for any $x \in A$ - notation $\tilde{\emptyset}$. Analogously for $V \subseteq U$, $\tilde{V}$ denotes a soft set (V, E) defined by $V(x) = V$ for every $x \in E$. The family of soft sets over the universe U and the index set E is denoted by $\mathcal{S}(U, E)$. To simplify the notation (e.g. if the index set E is fixed) we sometimes use $\mathcal{S}(U)$ and F instead of $\mathcal{S}(U, E)$ and (F, E), respectively.

Remark 2.1 *In fact, only sets of the form (F, E) could be considered, since any soft set (F, A) (resp. the function F) could be "extended" to the whole E by $F(x) = \emptyset$ for every $x \in E \setminus A$. (This simplification was already used in [14] and [15].)*

For instance, in order to specify a *complement* (F^c, A^c) of (F, A) we consider an extension of (F, A) to (F, E) as in Remark 2.1. Then, (F^c, A^c) is the complement of the soft set (F, E) where $F^c(x) = U \setminus F(x)$ and $A^c = \{x \in E \,|\, F^c(x) \neq \emptyset\}$. The complement $\tilde{U}$ of $\tilde{\emptyset}$ is the so-called *absolute* soft set over U.

Remark 2.2 *The notation in [10] is slightly confusing in our opinion. The authors specified A^c in a slightly different way (i.e., by using the "not" set of A) according to the linguistic interpretation of the soft set A. But both interpretations are mathematically the same.*

Further, set operations on soft sets are defined pointwise - to distinguish them from ordinary (i.e. non-soft) set operations we use a lower index s. Having in mind Remark 2.1, it suffices to work only with soft sets of the form $(\cdot, E)$. Then, for any $(F, E), (G, E) \in \mathcal{S}(U, E)$, a *soft union* $(G, E) \cup_s (F, E)$ (resp. *soft intersection* $(G, E) \cap_s (F, E)$) is defined as a soft set (H, E) where

$$H(e) = F(e) \cup G(e) \quad \text{(resp. } H(e) = F(e) \cap G(e))$$

for any $e \in E$. Similarly, a *soft difference* $(G, E) \setminus_s (F, E)$ is given by $F(e) \setminus G(e)$ for any $e \in E$.

Remark 2.3 *Let us stress that many authors currently follow notation introduced by Maji et al. in [10] where, in our opinion, definitions of soft notions are unnecessarily complicated. For instance in Definition 2.1 in [10]* *a union of two soft sets (F,A) and G,B is defined as follows:*

$$H(e)=\begin{cases} F(e) & \text{if } e\in A\setminus B,\\ G(e) & \text{if } e\in B\setminus A,\\ F(e)\cup G(e) & \text{if } e\in A\cap B.\end{cases}$$

It is easy to see that it coincides with our definition.

In the rest of this paper we deal with the only index set E. Thus, with respect to Remark 2.1, we can simplify the notation and write $A\cap_s B$ instead of $(A,E)\cap_s(B,E)$ etc. Finally, a point $x\in U$ is an element of a soft set A (notation $x\in_s A$) if $x\in A(e)$ for each $e\in E$. We denote by $x\notin_s A$ if the point $x\in U$ is not an element of the soft set A.

2.2. Soft topological spaces

Below we refer to several basic notions from general topology without defining them again as they belong to standard knowledge of every mathematician. In this subsection we recall definitions of "soft topological" notions that recently appeared in several papers (e.g. in [17]).

Definition 2.4 *A family τ consisting of soft sets over U is a* soft topology *if*

- $\emptyset_s,\tilde{U}\in\tau$,
- *τ is closed under arbitrary soft unions,*
- *τ is closed under finite soft intersections.*

Then (U,τ,E) (resp. (U,τ)) is a soft topological space*.*

Elements of τ are *soft open* subsets of U and their soft complements are *soft closed* sets. It should be mentioned that the notion of *soft closed* set over U is defined by relative complements in [10]. But mathematically, there is no difference between these two notions. Since unions, intersections and complements are, in fact, defined point-wise, de Morgan's laws are valid (see [10]) in $\mathcal{S}(U)$ and hence the family τ' of all soft closed sets satisfies expected properties.

Lemma 2.5 *Let (U,τ,E) be a soft topological space. Then*

1. *$\emptyset_s,\tilde{U}$ are soft closed,*
2. *τ' is closed under finite soft unions,*
3. *τ' is closed under arbitrary soft intersections.*

It is possible to define many "soft topological" notions which are inspired by their ordinary analogues. For the purpose of this article we deal only with one of separation axioms. A soft topological space (U,τ,E) is a *soft T_1-space* if for any $x,y\in U$, $x\neq y$, there exist $A,B\in\tau$ such that

$$x\in_s A,\ y\notin_s A,\ y\in_s B,\ x\notin_s B.$$

3. Soft vs. ordinary operations

Let us recall the standard definition of the notion of the cartesian product of sets. Let E be an index set and $\{X_e\}$ be a system of non-empty sets. Then a *cartesian product* $\prod_{e\in E} X_e$ (shortly *product system*) of the system $\{X_e\}_{e\in E}$ is a system of all maps $x : E \to \bigcup_{e\in E} X_e$ such that $x(e) \in X_e$ for any $e \in E$. Each $x \in \prod_{e\in E} X_e$ (resp. $x \in \prod X_e$ since E is the only index set in this paper) can be written as $(x_e)_{e\in E}$ where $x_e = x(e)$. By $(x)_{e\in E}$ (resp (x)) we denote an element $(x_e)_{e\in E} \in \prod_{e\in E} X_e$ such that $x(e) = x$ for every $e \in E$. For $g \in E$, an element $x_g \in X_g$ is called g*-component* of x and a map $pr_g : \prod X_e \to X_g$, $pr_g(x) = x_g$, is called g*-projection* (resp. g-canonical projection) of the product system $\prod X_e$. The following result is trivial.

Lemma 3.1 *Any subsystem of soft sets from* $\mathcal{S}(U, E)$ *can be viewed as a subsystem of some product system* $\prod_{e\in E} X_e$.

Proof Put $X_e = U$ for every $e \in E$. Then, by the definition of the cartesian product, it is easy to see that every soft set $A \in \mathcal{S}(U, E)$ is uniquely represented as an element from $\mathcal{P}(\prod_{e\in E} X_e)$. □

From this statement it immediately follows that there can be connections between soft set operations (intersections, complements etc.) and ordinary set operations applied to elements of the product system. Respectively, provided that soft sets are considered as elements of the product system, connections between ordinary and soft set operations can be expressed by using well-known properties of ordinary sets. For example, if $E = \{1, 2\}$, it is known that

$$(A_1 \cap B_1) \times (A_2 \cap B_2) = (A_1 \times B_1) \cap (A_2 \times B_2).$$

Consequently, since soft operations are defined pointwise,

$$A \cap B = A \cap_s B \tag{3.1}$$

and this relation between soft and ordinary intersection can be expressed for infinite index set E as well. On the other hand, soft unions and soft complements may provide different results than their ordinary counterparts, i.e. we have (again for $E = \{1, 2\}$)

$$A \cup_s B = (A(1) \cup B(1)) \times (A(2) \cup B(2)) \supseteq (A(1) \times A(2)) \cup (B(1) \times B(2)) = A \cup B$$

and

$$A \setminus_s B = (A(1) \setminus B(1)) \times (A(2) \setminus B(2)) \subseteq (A(1) \times A(2)) \setminus (B(1) \times B(2)) = A \setminus B.$$

The following example demonstrates that the inclusions above can be strict.

Example 3.2 *Let* $E = \{1, 2\}$ *and* $U = \mathbb{R}$*, the set of real numbers. Then, clearly, for* $A(1) = [2, 3]$, $A(2) = [0, 1]$ *and* $B(1) = [0, 1]$, $B(2) = [2, 3]$,

$$A \cup_s B = ([0, 1] \cup [2, 3]) \times ([0, 1] \cup [2, 3]) \supseteq ([2, 3] \times [0, 1]) \cup ([0, 1] \times [2, 3]) = A \cup B,$$

and, for $A(1) = [0, 1] \cup [2, 3]$, $A(2) = [0, 1] \cup [2, 3]$ *and* $B(1) = [0, 1]$, $B(2) = [0, 1]$,

$$A \setminus_s B = ([2, 3]) \times ([2, 3]) \subseteq ([2, 3]) \times ([2, 3]) \cup ([2, 3]) \times ([0, 1]) \cup ([0, 1]) \times ([2, 3]) = A \setminus B.$$

4. Soft vs. ordinary topological spaces

In this section we demonstrate that every soft topology defines a crisp topology on a subspace of the product system which inherits some properties of the soft topology. According to Lemma 3.1, soft sets are identified with subsets of the product system. Before introducing our construction, we must define the *diagonal* of U in an ordinary way

$$\Delta := \left\{ (x_e)_{e\in E} \in \prod_{e\in E} U \mid x_e = x_f \text{ for every } e, f \in E \right\}.$$

4.1. System of neighborhoods

In this subsection we study the neighborhood system of a given soft topology. It is good to say that the neighborhood system carries many important topological properties of the soft topology. For example, the neighborhood system is needed if we want to check whether the soft topological space is soft Hausdorff, soft T_i-space ([17]) etc.

Let E be an index set and τ denotes a soft topology on U. Let us consider a system $\mathcal{N}(U)$ of open neighborhoods of τ, i.e.

$$\mathcal{N}(U) := \{A \in \tau \,|\, \text{there is } x \in U \text{ s.t. } x \in A(e) \text{ for all } e \in E\}. \tag{4.1}$$

For any soft topology τ, the neighborhood system is quite big - the soft union of any pair of non-empty open soft sets belongs to $\mathcal{N}(U)$. However, this system need not be closed under soft intersections - see the next example.

Example 4.1 *Let $E = \{1, 2\}$, $U = \mathbb{R}$ and let A, B be soft neighborhoods defined by*

$$A(1) = [0, 1], \;\; A(2) = [0, 1] \cup [2, 3], \;\; B(1) = [0, 1] \cup [2, 3], \;\; B(2) = [2, 3].$$

Then $(A \cap_s B)(1) = [0, 1]$, $(A \cap_s B)(2) = [2, 3]$ and hence $A \cap_s B \notin \mathcal{N}(U)$.

Thus, the system $\mathcal{N}(U)$ also admits properties which do not occur in ordinary topological spaces.

4.2. Topology induced from soft topology

Namely, consider a soft topological space (U, τ, E) and its neighborhood system $\mathcal{N}(U)$. Below we work with a system

$$\sigma = \{A \cap \Delta \,|\, A \in \mathcal{N}(U)\}\,. \tag{4.2}$$

We are able to show (Lemma 4.3) that σ is a base of some ordinary topology τ_D on the diagonal Δ. To prove this property, the following characterization of the base of a topological space is used.

Lemma 4.2 (e.g. [8, 9]) *A system σ of sets from U form a base of some topology on U if and only if*

(i) σ covers U,

(ii) for all sets $V_1, V_2 \in \sigma$ and any $x \in V_1 \cap V_2$ there exists $W \in \sigma$ such that $x \in W$ and $W \subseteq V_1 \cap V_2$.

To make this proof more legible we recall Lemma 3.1 that any soft set can be seen as a subset of some product system and that soft intersection can be replaced by ordinary intersectino (see (3.1)).

Lemma 4.3 *The system σ defined by (4.2) is a base of some topology on the diagonal Δ.*

Proof According to Lemma 4.2, we must check conditions (i) and (ii) above. The condition (i) is satisfied since $\tilde{U} \in \mathcal{N}(U)$ and $\tilde{U} \cap \Delta = \Delta$. Let us check the condition (ii).

Assume that we have $A, B \in \sigma$ and $(x)_{e\in E} \in A \cap B$. Then, by the definition of σ, there are soft sets $\bar{A}, \bar{B} \in \mathcal{N}(U)$, such that $A = \Delta \cap \bar{A}$ and $B = \Delta \cap \bar{B}$ satisfying

$$x \in_s \bar{A},\ x \in_s \bar{B}.$$

Since τ is closed under finite intersections, there is $\bar{W} = \bar{A} \cap \bar{B}$ in τ. Since soft set operations are defined pointwise, we have

$$x \in W(e) = A(e) \cap B(e), \quad \text{for any } e \in E.$$

Consequently, for $W := \Delta \cap \bar{W}$,

$$(x)_{e\in E} \in (\bar{A} \cap \Delta) \cap (\bar{B} \cap \Delta) = (\bar{A} \cap \bar{B}) \cap \Delta = \bar{W} \cap \Delta = W,$$

which clearly implies that $W := \bar{W} \cap \Delta \in \sigma$, $(x)_{e\in E} \in W$ and $W \subseteq A \cap B$. □

Having in our minds the previous lemma, we can define a *diagonal topological space* (Δ, τ_D) where τ_D is the smallest topology containing the system σ from (4.2). Now we demonstrate that the two topologies possess analogous features. We demonstrate this fact on the notion of (soft) T_1-space.

It is said that (U, τ, E) is a *soft T_1-space* if and only if for any two points $x, y \in U$, $x \neq y$, there are $A, B \in \tau$ for which

$$x \in_s A,\ y \notin_s A, \quad y \in_s B,\ x \notin_s B.$$

Theorem 4.4 *A soft topological space (U, τ, E) is a soft T_1-space if and only if (Δ, τ_D) is a T_1-space.*

Proof Let $(x)_{e\in E}, (y)_{e\in E} \in \Delta$, $(x)_{e\in E} \neq (y)_{e\in E}$, be fixed. Let us assume that (U, τ, E) is a soft T_1-space. This means that for $x, y \in U$, $x \neq y$, there are $A, B \in \tau$ such that

$$x \in_s A,\ y \notin_s A, \quad y \in_s B,\ x \notin_s B.$$

Since A, B also are soft neighborhoods of x and y, these expressions are equivalent to

$$(x)_{e\in E} \in A \cap \Delta,\ (y)_{e\in E} \notin_s A \cap \Delta, \quad (y)_{e\in E} \in_s B \cap \Delta,\ (x)_{e\in E} \notin_s B \cap \Delta,$$

respectively, which implies that (Δ, τ_D) is a T_1-space.

To prove the converse implication, fix any $x, y \in U$, $x \neq y$, and assume that (Δ, τ_D) is a T_1-space. For $(x)_{e\in E}, (y)_{e\in E} \in \Delta$, $(x)_{e\in E} \neq (y)_{e\in E}$, there are $A, B \in \tau_D$ such that $A := \tilde{A} \cap \Delta$ and $B := \tilde{B} \cap \Delta$, and hence

$$(x)_{e\in E} \in A,\ (y)_{e\in E} \notin A, \quad (y)_{e\in E} \in B,\ (x)_{e\in E} \notin B.$$

By Lemma 4.3 we can assume that A and B are elements of the base σ, i.e. there are soft sets $\tilde{A}, \tilde{B} \in \mathcal{N}(U)$ such that

$$(x)_{e\in E} \in \tilde{A} \cap \Delta,\ (y)_{e\in E} \notin_s \tilde{A} \cap \Delta, \quad (y)_{e\in E} \in_s \tilde{B} \cap \Delta,\ (x)_{e\in E} \notin_s \tilde{B} \cap \Delta$$

for $A := \tilde{A} \cap \Delta$ and $B := \tilde{B} \cap \Delta$. But this already proves that (U, τ, E) is a soft T_1-space. □

Remark 4.5 *As a consequence of this theorem one can see that the connection between a soft topology* (U,τ,E) *and the induced topology* (Δ,τ_D) *is very close. We expect that analogies to Theorem 4.4 can be stated for many other topological properties - for instance, for other* T_i*-separation axioms, Hausdorffness, regularity, normality and so on. We do not provide proofs of these facts here since this is not the purpose of this note. We left these facts as hypotheses only.*

5. Reconstructions of soft topologies

Theorem 4.4 showed that there is a close relationship between a given soft topological space (U,τ,E) and some topological space (Δ,τ_D) induced by (U,τ,E). Moreover, since soft set operations are, in fact, ordinary set operations defined pointwise, the next result follows immediately from Definition 2.4 (see also [2, 17]).

Lemma 5.1 *Let* (U,τ,E) *be a soft topological space and* $e\in E$ *be fixed. Put* $\tau_e := \{\pi_e(V) \mid V\in\mathcal{S}(U)\}$*. Then* (U,τ_e) *is a topological space.*

Thus, every soft topological space (U,τ,E) induces $\#(E)$ topological spaces (U,τ_e). And it is easy to see that some topological properties can be inherited from (U,τ,E), see the next lemma for instance.

Lemma 5.2 *If* (U,τ,E) *is a soft* T_1*-space, then* (U,τ_e) *is a* T_1*-space for any* $e\in E$*.*

Proof Obvious. □

It is natural to ask whether there exists a construction opposite to this. When we take $\#(E)$ topological spaces (U,τ_e), is there a mathematical tool in general topology allowing us to reconstruct a soft topological space (U,τ,E)?

Within the next two sections we provide only a partial answer to this question. We study the notion of a strong topology (known also as box topology) and in this section we indicate there exists a diagonal topological space induced by open sets of a strong topology. However, in Example 5.5 one can see that even simple soft topological spaces may induce discrete topological spaces (U,τ_e). Therefore, this notion is not suitable for reconstruction of the original soft topological space (U,τ,E) (The construction is described in the next section.).

The following definition of a strong product of topological spaces is quite familiar. Consider a collection $\{U_e\}_{e\in E}$ of topological spaces. A system σ_s of subsets $\prod_{e\in E} V_e$ forms a basis of a topology on $\prod_{e\in E} U_e$ whenever V_e is open in U_e for every $e\in E$. A topology τ_s generated by σ_s defines a *strong product topology* on $\prod_{e\in E} U_e$. Here we would like to emphasize that the box product is not the categorical product in the category of topological spaces.

Following the fact that the system $\mathcal{S}(U)$ can be seen as a subsystem of $(\prod_{e\in E} U)$, we study the following construction. By using topologies τ_e, we can define

$$D' := \left\{ \Delta \cap \prod_{e\in E} A_e \mid A_e \in \tau_e \right\} \tag{5.1}$$

Then, a topology τ'_D can be defined as the smallest topology on the diagonal Δ containing the system D' (see Lemma 5.3). A pair (Δ,τ'_D) is a *diagonal topological space* induced by the soft topological space (U,τ,E). It is easy to see that this topological space is induced by open sets from strong product topology on $\prod_{e\in E} U$.

Lemma 5.3 *The system D' forms a base of the topology τ'_D.*

Proof It is sufficient to check the two conditions from Lemma 4.2. The first condition is obviously satisfied, since $U \in \tau_e$ for each $e \in E$. To prove the second condition, let us take arbitrary $V_1, V_2 \in D'$ and $(x)_{e\in E} \in V_1 \cap V_2$. From the fact that $(x)_{e\in E} \in V_1$ we have $x \in \pi_e(V_1)$ and $\pi_e(V_1) \in \tau_e$ for each $e \in E$ by the definition of D'. Similarly, for every $e \in E$, $x \in \pi_e(V_2)$ and $\pi_e(V_2) \in \tau_e$. Thus, for any $e \in E$, since $x \in \pi_e(V_1) \cap \pi_e(V_2)$ there exists an element $W_e \in \tau_e$ such that

$$x \in W_e, \quad W_e \subseteq \pi_e(V_1) \cap \pi_e(V_2).$$

Clearly, by choosing $W := \Delta \cap \prod_{e\in E} W_e$ we get $(x)_{e\in E} \in W$ and $W \subseteq V_1 \cap V_2$ for some $W \in D'$. □

We are able to show (Theorem 5.4) that this construction still preserves some properties of the soft topological space (U, τ, E).

Theorem 5.4 *Let (U, τ, E) be a soft topological space, (Δ, τ'_D) be a topological space induced by (U, τ, E). If (U, τ, E) is a soft T_1-space, then (Δ, τ'_D) is a T_1-space.*

Proof It is easy to see that (Δ, τ_D) is weaker than (Δ, τ'_D). Since (Δ, τ_D) is a T_1-space by Lemma 4.4, the result concludes from the fact that each stronger topological space must be a T_1-space as well. □

The converse implication need not be satisfied - see the next example. However, in the next section one can see that if a soft topological space is induced by open sets of strong topological space, the result can be strengthened. The following example demonstrates that a soft topology need not be a soft T_1-space while all the induced topological spaces (U, τ_e) can be T_1-spaces.

Example 5.5 *Let $E = \{1, 2\}$ and $U = \{a, b\}$ and let $\tau = \{\emptyset_s, \tilde{U}, A, \}$ be a soft topology given by $A(2) = \{a\}$, $A(1) = \{b\}$. Clearly, (U, τ, E) is not a soft T_1-space, since $\mathcal{N}(U)$ consists only of $\tilde{U}$ and $\tilde{U}$ contains both a and b. On the other hand, τ_1 and τ_2 are discrete (and hence T_1-) topological spaces.*

Remark 5.6 *It seems that it is a difficult task to establish relations between (U, τ, E) and some diagonal topological space in general. For instance, we were not able to find any "homeomorphism" between these two spaces, i.e. we did not find any bijection which would transform soft set operations to ordinary ones, and which would map soft open sets to open ones. The problem is in the fact that important features of soft topologies are carried by systems of neighborhoods, but these systems need not be closed under soft intersections. Moreover, some soft topologies can be "sparse" in the sense that they contain soft sets (e.g. A) such that $A(e) = \emptyset$ for some $e \in E$ and $A(f) \neq \emptyset$ for some $f \in E$. Projections of such sets appear in τ_e's, but it is hard to "reconstruct" them e.g. as a product of some open sets.*

Even, we were not able to prove the converse of Theorem 5.4 under quite strong assumptions - for $\#(E) < \infty$, for D' (see (5.1)) constructed with the help of projections of soft neighborhoods only, when we assumed that the system of neighborhoods is closed under soft intersections and consists only of soft neighborhoods whose images are non-empty for every $e \in E$ etc.

6. Two standard topological tools

As it has been indicated in the previous section, within this section we would like to show that if we intend to create a soft topology on U, there are well-known standard mathematical tools allowing us to keep relations between the constructed soft topology and topologies representing topologies τ_e above.

Let us recall some notations. We have an index set E and $\#(E)$ topological spaces (U, τ_e). We want to find a construction of a soft topology (U, τ_r, E) which allows us to get a soft topology of desired properties. Probably the simplest idea (see Lemma 3.1) is to define a soft topology τ_r such that every open soft set $A \in \tau_r$ is identified with a set the following form (i.e. with an open set of the strong topology)

$$A = \prod_{e \in E} A_e, \quad \text{where } A_e \in \tau_e. \tag{6.1}$$

Since operations of soft unions and soft intersections are defined pointwise and all τ_e are topologies, the system τ_r is clearly closed under finite soft intersections and arbitrary soft unions. Consequently, we have the following result.

Lemma 6.1 *Let E be an index set and (U, τ_e) be topological spaces. Let τ_r is a system of all soft sets of the form (6.1). Then (U, τ_r, E) is a soft topological space.*

It follows e.g. from Theorem 5.4 that the induced diagonal topological space may carry some properties of the original topological space. However, we are able to prove that in the case of (U, τ_r, E) the relationship can be mutual - for example, we are able to prove the converse of Theorem 5.4. Let us note that topologies τ_e generating (U, τ_r, E) coincide with topologies τ_e induced from (U, τ_r, E) (see Lemma 5.1).

Theorem 6.2 *Let (U, τ_r, E) be a soft topological space, (Δ, τ'_D) be a diagonal topological space induced by (U, τ_r, E). If (Δ, τ'_D) is T_1-space, then (U, τ_r, E) is a soft T_1-space.*

Proof Let the assumptions be fulfilled and $x, y \in U$, $x \neq y$, be fixed. For $(x), (y) \in \Delta$ there are open sets $A, B \in \tau'_D$ such that

$$(x) \in A,\ (y) \notin A,\ (x) \notin B,\ (y) \in B.$$

Clearly, we may assume that A, B are elements of the base of τ'_D, i.e. they are of the form (5.1). Following the notation in (5.1), it follows from the definition of τ_r that there are $\tilde{A}, \tilde{B} \in \tau_r$ defined by

$$\tilde{A} := \prod_{e \in E} A_e, \quad \tilde{B} := \prod_{e \in E} B_e.$$

The statements

$$x \in_s \tilde{A},\ y \notin_s \tilde{A},\ x \notin_s \tilde{B},\ y \in_s \tilde{B}$$

are obvious. □

Although we could not find a direct relation between the soft set and ordinary set operations, respectively (see Remark 5.6), we found some relations between the topological structure of (U, τ_r, E) and some topology on $\prod_{e \in E} U$. Namely, elements of the soft topology (U, τ_r, E) are elements of the base of the strong product space $(\prod_{e \in E} U, \tau_s)$. We get immediately from definitions that every soft open set $A \in \tau_r$ is an open set in τ_s, every soft closed set of (U, τ_r, E) is a closed set in $(\prod_{e \in E} U, \tau_s)$ and so on. It is also easy to see that (Δ, τ'_D) is a topological subspace $(\Delta \cap \prod_{e \in E} U, \tau_s)$ of $(\prod_{e \in E} U, \tau_s)$ with the topology induced from τ_s. We expect that more relations between soft and ordinary topological spaces can be easily proven (see Remark 4.5).

In the rest of this section we deal with one more well-known construction on the product system from general topology. A *product topology* τ_p on $\prod_{e\in E} U_e$ is defined as the weakest topology on $\prod_{e\in E} U_e$ for which all canonical projections π_e are continuous. It is well-known that the base of τ_p consists of sets $V=(V_e)\in\prod_{e\in E} U_e$ such that $V_e=U$ except for a finite number of e's from E.

It is known that $\tau_s=\tau_p$ whenever E is finite, and they differ if E is infinite. It is also obvious that τ_p is weaker than τ_s in general. Thus, for a finite index E, a product topology tool provides us an easy construction allowing to define a soft topology on U with desired properties. This construction can be used for all productive properties, i.e. properties which are preserved by products of topological spaces (see the next lemma). For instance, having in mind that (Δ,τ'_D) is a topological subspace $(\Delta\cap\prod_{e\in E} U,\tau_s)$ of $(\prod_{e\in E} U,\tau_s)$, several results on properties of the topological space related to the soft topology (U,τ_s,E) can be stated. The way how to create a soft T_1 space is demonstrated in the next example.

Example 6.3 *Let E be finite and (U,τ_e) be T_1-topological spaces. By Lemma 6.1 we are able to construct a soft topological space (U,τ_r,E). By Lemma 6.4 we know that $(\prod U,\tau_s)$ is a T_1-space, by Lemma 6.5 we know that $(\Delta\cap\prod U,\tau_s)$ is a T_1-space as well. But the latter coincides with (Δ,τ'_D) generated by (5.1). Thus, Theorem 6.2 claims that (U,τ_r,E) is a soft T_1-space.*

A similar construction can be used to all productive properties (Lemma 6.4), which are kept by a subspace topology on the diagonal (Lemma 6.5).

Lemma 6.4 *Let E be a finite index set and $\{U_e\}_{e\in E}$, $U_e=(U,\tau_e)$, be a system of topological spaces. Then $\mathcal{U}:=(\prod U_e,\tau_p)$ satisfies the following:*

1. *every product $\mathcal{U}$ of T_i-spaces $\{U_e\}_{e\in E}$ is a T_i-space, $i=0,1,2,3$,*
2. *every product $\mathcal{U}$ of Hausdorff spaces $\{U_e\}_{e\in E}$ is Hausdorff,*
3. *every product $\mathcal{U}$ of (completely) regular spaces $\{U_e\}_{e\in E}$ is (completely) regular,*
4. *every product $\mathcal{U}$ of Tychonoff spaces $\{U_e\}_{e\in E}$ is Tychonoff,*
5. *every product $\mathcal{U}$ of compact spaces $\{U_e\}_{e\in E}$ is compact,*
6. *every product $\mathcal{U}$ of (path-)connected spaces $\{U_e\}_{e\in E}$ is a (path-)connected space,*
7. *every product $\mathcal{U}$ of hereditarily disconnected spaces $\{U_e\}_{e\in E}$ is hereditarily disconnected, etc.*

Lemma 6.5 1. *Every topological subspace of T_i space is a T_i space,*

2. *Every topological subspace of a Hausdorff space is a Hausdorff space,*
3. *Every topological subspace of a regular space is a regular space, etc.*

7. Conclusions

Within this paper we have pointed out that some recently developed and studied notions of soft topological spaces can be represented and studied with the help of standard topological spaces (Theorem 4.4). A purpose of this note is that to fully understand the behavior and structure of soft topological spaces, it seems to be useful to develop other mathematical instruments (Theorems 5.4 and 6.2) and this can be done with the help of standard topological notions. We would like to stress that there are known powerful instruments (e.g. strong product topology, product topology on some product systems, subspace topology) allowing us to construct (e.g. Example 6.3) soft topological spaces of required properties and to deal with topologies on spaces of soft sets in a more pleasant way.

Although in constructions were used only for some topological properties (namely for the (soft) T_1 axiom) for demonstration purposes, we believe that this kind of research may significantly enrich the theory of soft topological spaces. We also think that analogous constructions can be used for some generalizations of soft sets - for instance, for fuzzy soft topological spaces mentioned e.g. in [18].

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